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Abstract: *Mathematics has a history of being a two-dimensional inscribing practice. We describe the potential evolution in doing, thinking, and learning mathematics with the emergence of a technological innovation that enables real-time 3D virtual and material interactions. Using the 3D drawing pen as a simple and recently available technology, we highlight how it helps re-think long-standing assumptions and dichotomies in mathematics education including, for example, the material distinction between diagram and manipulative, the semiotic distinction between icon and index, and the developmental progression of action-icon-symbol. We then speculate on the future possibilities of the shift in technological infrastructure that 3D pens and similar technology may give rise to.*

Keywords: 3D technologies; 3D pens; Technology and Mathematics

Introduction

From the Ancient Greek sand reckoning through to contemporary sketching on sheets of paper or tablet screens, mathematics has a three-millennium history of being a two-dimensional inscribing practice. The instruments have changed—the finger, the stylus, the pencil, the chalk, the mouse (see Figure 1)—but the surface has remained flat, even when the mathematical objects and relations being inscribed are not. While the flatness has led to interesting mathematical innovation (such as perspective drawing and its relation to projective geometry), we might also ask how it has constrained mathematical thinking, or even how it has affected the way students learn, the tasks they are offered and the concepts at play. In a more general sense, Shaffer and Kaput (1999) describe the cognitive evolution associated with the development of external symbolic representations (such as paper), which enabled mathematical information to be stored, and the ensuing cognitive evolution of the virtual culture, in which these symbolic representations can be *processed*. In a similar way, we are interested in how the technological infrastructure that supports a new kind of external symbolic representation might also occasion cognition shifts.

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In light of the long-standing technological stasis of paper-and-pencil, coupled to a recognition that technologies affect the doing, teaching and learning of mathematics, our aim in this paper is to investigate some of the potential consequences of emerging technologies that make it possible to draw in three dimensions and, furthermore, to interact with what is drawn as a 3D object (this latter point is important, so we provide a brief example here: it is possible to draw a planar shape, such as a triangle, with the 3D drawing pen, but once the shape has been drawn, it can then be touched, lifted off the page and manipulated in 3D space).

As a web search will confirm, the use of 3D imaging technologies is forecast to grow exponentially over the next several years. It is fair to assume that these technologies will soon be dominating everyday experiences, and will eventually also be populating mathematical classrooms. While we recognise that there are potentially more powerful 3D tools on the horizon, such as virtual reality headsets and three-dimensional cameras, we focus on one type of 3D printing technology here because it is readily available and easily adopted, and also because it has so much in common with the regular 2D drawing devices that have become essential features of the mathematics classroom[2].

Our goal is twofold. First we use the "3D drawing pen" (or 3D pen) as a means to interrogate some of the cognitive and pedagogical assumptions that have been made in mathematics education that are, at least in part, based on technologies of the past. Our second goal is to draw on our observations of students using 3D drawing pens in order to speculate about the 3D enactive revolution we see coming. We have chosen the 3D drawing pen as an intermediary, constrained technology that might be misleading in its similarity to the millennia-old 2D drawing devices, but that we argue provides insights into the amplifying of possibilities that shifting from 2D to 3D might afford. In other words, our interest is not in providing evidence of particular changes in student thinking nor to offer suggestions for classroom use; rather, it is in helping us think about the current natures and potential evolutions of mathematical practice and its associated pedagogies.

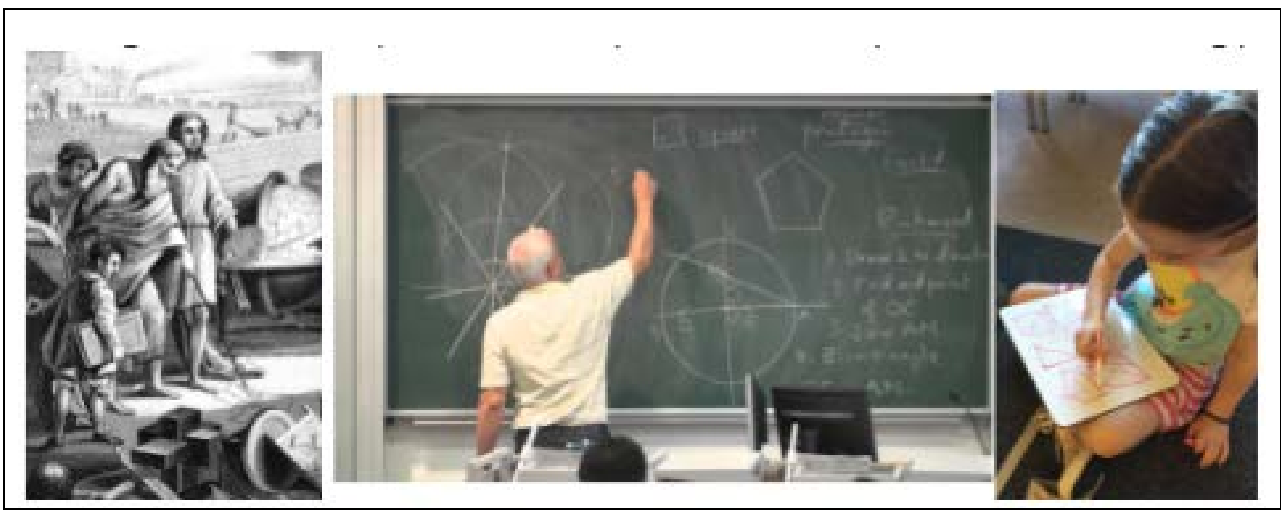

Figure 1. Examples of mathematical inscribing on flat surfaces of different kinds.

[2] This wasn't always the case, of course. As Kidwell et al. (2008) document, the tools of the mathematics classroom have changed greatly since the 1800s.

A new way of drawing

The 3D drawing pen is a handheld device that operates on the same principle as some 3D printers. It extrudes small, flattened strings of molten thermoplastic (ABS or PLA) and forms a volume of "ink" as the material hardens immediately after extrusion from the nozzle. As the pen moves along with the hand holding it, a 3D drawing is created at once, either on a surface or in the air (Figure 2). The process of drawing quickly transforms into a made object, that is, a concrete object that can be manipulated. (This process can only be approximated with regular 2D pens by first drawing and then cutting out what has been drawn, and perhaps then folding or bending what has been cut out in order to produce a 3D object.) 3D drawing frees the hand—as well as that which the hand makes—from the flat constraints of paper-and-pencil.

Besides the ease of creating and visualising 3D objects generated through the moving hands, 3D drawing changes the experience of drawing 2D figures. A diagram that would have been drawn with paper-and-pencil, and then stayed dormant on the page, can be drawn with the 3D pen and can then become a physical object that can be held, moved and turned. As such, a 3D drawing can have a dual nature, both as a diagram and as a physical manipulative. Each of these artefacts—diagram and manipulative—have been the subject of study in mathematics education research, but we have become interested in their interplay in the context 3D drawing, which we think may have important pedagogical, theoretical and mathematical implications for teaching and learning.

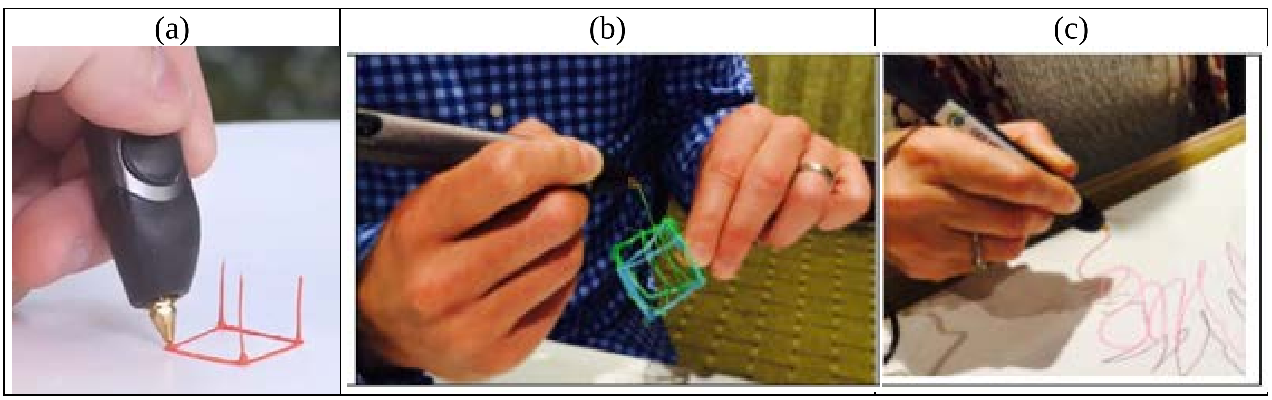

Figure 2. (a) Drawing a cube with a 3D Drawing Pen. (b) 3D drawing "in the air." (c) A spiral.

In order to appreciate this particular interplay, it might be useful to consider the shifts in experience afforded by other technologies that have been widely studied in mathematics education. Consider the advent of dynamic geometry environments (DGEs), over thirty years ago, in which it became possible to drag the vertex of a triangle on the screen. At first, many saw this as mathematically heretical, since geometric points were taken to have fixed locations on the plane. Now, dynamism has become a widespread feature of mathematics education technologies. And, as a consequence, the dimension of time that comes into play in the act of dragging has not only become mathematically acceptable, but also pedagogically significant in enabling learners to experience mathematical variance and invariance. It is a technology that has changed the way we think about mathematical concepts (and triangle is not a three-sided polygon that moves on the plane) and of mathematics learning.

In writing this article, we are engaging in the kind of speculation that would have been happening thirty years ago when DGEs first came out; except, we are doing so for a different kind of technology, one that we think can also shed light on long-standing assumptions about the nature of mathematics and of mathematics learning.

Our interest in 3D drawing stems from joint research on spatial reasoning and its importance in mathematics thinking and learning (see Davis et al., 2015), particularly in relation to drawing (McGarvey et al., 2015). This research led us to focus on the role of gestures and diagrams in mathematics thinking and learning, each of which is an established area of research in the mathematics education literature. More recently, researchers have begun to investigate how gesturing and diagramming might also be related (see de Freitas & Sinclair, 2014). In terms of the latter, our opening paragraph underscores the way in which diagramming in mathematics has long been a 2D activity. The very act of drawing, even in 2D, involves a certain movement of the hand that can become associated with certain ways of thinking and communicating mathematics. In this sense, there is a tight, generative and significant connection between the way the body moves (in this case, especially the hand), the technological device used to create marks on the page and the mathematics itself. For example, if you are asked to describe a triangle, you might gesture-draw it in the air, evoking the way it is drawn on paper; or, when asked to multiply two-digit numbers, you might use your finger to write out the numbers in a column just as you might have done on paper. When asked to describe a parabola, you are likely to use your index finger to trace the graph of a quadratic function in the air, moving your finger in much the same way you would move your pencil. The gesture you used has likely emerged from the act of drawing, thereby pointing to an important connection between drawing and gesturing.

But the connection can go in the other way as well, as when the movement of your hand in the air generates certain marks on the page: a gesture of the forearm held up to show the steepness of a hill can become an oblique line on the page. Thus, the boundary between gesturing and diagramming can melt away. According to the philosopher and historian of mathematics, Gilles Châtelet (2000), this interplay between gestures and diagrams is at heart of mathematical invention and crucial in helping shed light on how embodied, material actions can evolve into a formal mathematical discourse. In other words, changing the way one diagrams can not only give rise to different gestures, but can also changes how gestures are captured "mid flight", to use Châtelet's imagery, in the act of drawing.

In our context, the three-dimensional nature of gestures "in the air" is even closer to the potential surface of a 3D pen, thereby further disturbing the boundary between gesture and diagram. We are interested in what sorts of learning and thinking possibilities that arise when there is a material record of one's gestural history, something that has not been possible in the past. It is this detail that most intrigues us around the mathematical and pedagogical possibilities of 3D pens. As discussed in greater detail in the next section, we suspect that the sort of artefacts that are generated by drawing in space—that is, that are left behind as physical, tangible objects after the hand has moved in the air—are phenomenologically distinct from other physical, tangible objects that currently dot the "manipulative" landscape of mathematics classrooms, such as nets of solids.

If the use of 3D pens raises questions about the assumed distinction between drawing and gesturing, then it also complicates another set of distinctions formulated in Peircean semiotics, in which signs (icons, indices and symbols) differ in terms of the nature of the relationships between the signifying sign and the signified (Peirce, 1994). The cube being drawn in Figure 2a can be seen as an icon, which operates according to likeness and resemblance between signifier and signified. Seen as an icon, however, it becomes a representation of a Platonic object (the cube) that is simply being copied, more or less faithfully. Iconic gestures are also described in terms of their resemblance with events or objects. The word *cube* is a symbol in that it has an arbitrary relationship with that to which it refers. But what if we think of the 3D drawing as an index, which has a more material link between signifier and signified. Indeed, unlike icons and symbols, indexical signs are bound to the context in important ways as they "show something about things, on account of their being physically connected with them" (Peirce, 1998, p. 5).

The canonical example used by Peirce is that of smoke billowing from a chimney, which indicates that there is a fire in the fireplace: in this case, the smoke indexes the fire. An index "refers to its object not so much because of any similarity or analogy with it, (…) because it is in dynamical (including spatial) connection both with the individual object, on the one hand, and with the senses or memory of the person for whom it serves as a sign, on the other" (Peirce, 1932, 2.305). The gesturing hand that leaves a trace of congealed wax can therefore be seen as indexing a cube, producing a temporal and spatial record of cube-making. The final 3D drawing can thus be seen as an indexical sign that refers to the prior movement of the 3D pen, including the way in which the vertical edges rise up from the base square in a gravity-defying manner. More crudely yet, the process of making the cube might be visible also in the clumps of wax that are formed as one edge meets another, or by the thickness of an edge, which may depend on the speed at which it was made. As Sinclair and de Freitas (2014) have argued,

> This latter indexical dimension is usually not emphasized in the semiotic study of mathematical meaning making, since we tend to focus on the completed trace and dislocate it from the labour that produced it. Such habits of focus have resulted in our neglect of how the activity of the body and various other material encounters factor in mathematical activity. (p. 356)

The above discussion highlights three ways in which 3D drawing blurs long-standing material and semiotic distinctions in mathematics education, one between diagrams and manipulatives, another between drawing and gesturing and a third among icons, indices and symbols. We suggest that there are important philosophical undertones to these distinctions to which 3D drawing helps draw attention, including ontological assumptions about the status of mathematical objects (as being concrete or abstract, physical or symbolic, etc.).

Rephrased in terms of Bruner's (1966) categories of conceptual representations, 3D drawing spans and blends enactive (action-based), iconic (image-based) and symbolic (character-based) instantiations. In the process, the processes and products of 3D drawing sidestep a commonplace "split" in opinion on the nature and role of manipulatives in mathematics class. To explain, manipulatives are most often seen and used as concrete instantiations of concepts, evidenced by Wikipedia's "Manipulative (mathematics education)" entry:

> Mathematical manipulatives are frequently used in the first step of teaching mathematical concepts, that of *concrete representation*. The second and third steps are representational and abstract, respectively. (emphasis added; accessed 2017 September 11).

Considered against the backdrop of the above discussion, in the space of school mathematics, this focus on *manipulatives as concrete representations* seems to have eclipsed a more fundamental reason to incorporate artefacts into mathematics learning. As articulated by Piaget (1954), and since elaborated by many mathematics education researchers with interests in the bodily basis of understanding (see de Freitas & Sinclair, 2014), the main reason for using manipulatives is neither to concretize a concept nor to excavate the ideas built into objects, but *to move*. That is, one's senses of shape, quantity, proportion and so on have more to do with *structured acts of moving* than with *acts of moving structures.* From this perspective, the main purpose of a manipulative is not to (re)present mathematical concepts, but to mould the learner's motions, in the process occasioning opportunities for learners to expand and interweave their repertoires of mathematically relevant structures. Summing up, as it operates across Bruner's action–icon–symbol triad, 3D drawing addresses and situates the tendencies to categorise, prioritise, and sequence different sorts of experience and representation.

As with any new technology, it can be very difficult to convey the nature and newness of the technology in the two-dimensional, alphanumeric and static medium of a journal article. We will thus, in the next section, provide some specific examples of what it looks like to use 3D drawing in a mathematics education context and how such drawing can become differently intermingled with mathematical concepts. Our intention is to exemplify some of the more theoretical claims we have been making thus far about the potential impact of taking mathematics "off the page." We then consider broader implications of the use of 3D drawing, particularly in terms of shifting students' perceptions of school mathematics from being about computing to being about modelling.

Initial encounters and speculations

We have begun experimenting with the use of 3D pens in a variety of contexts, including elementary and high school classrooms. We provide examples of the types of tasks and activities that we have seen as a way to exemplify some of the interesting issues at play in the move from 2D to 3D drawing. Our first two examples are related to functions and calculus. Our third example involves grade 3 students exploring the concept of triangle and congruence. We then turn to a more unstructured experimentation that involved ten mathematics educators new to 3D drawing. These examples will enable us to speculate further on futural directions in mathematics education, in anticipation of the arrival of more real-time, 3D technologies.

Drawing functions

In one of our pilot lessons, we explored drawing functions in the Cartesian plane with 3D pens in a high school calculus classroom. During the lesson, students were asked to draw the graphs of basic functions, such as $y = x^2$, as well as a "line" in 3D. We saw this task as being 2D in nature, in that students drew parabolas and lines; however, the third dimension came into play when they began to manipulate the line so that it would "just touch" the curve at one point. The students used their fingers to manipulate the "tangent line" by moving it along the curve and observed the change of its slope at different points of tangency (re-animating the etymological roots of tangent—from the latin, to touch). We found that this task offered a *physical instantiation* of tangent to a curve, in that student could feel the idea of local linearity by the sense of touch (Figure 3a–c). The students were later asked to pick up the parabola and *translate* or *reflect* it when working with functions such as $y = x^2 + 1$ or $y = -x^2$, which allowed them to explore the relationships between the graphs of derivative functions.

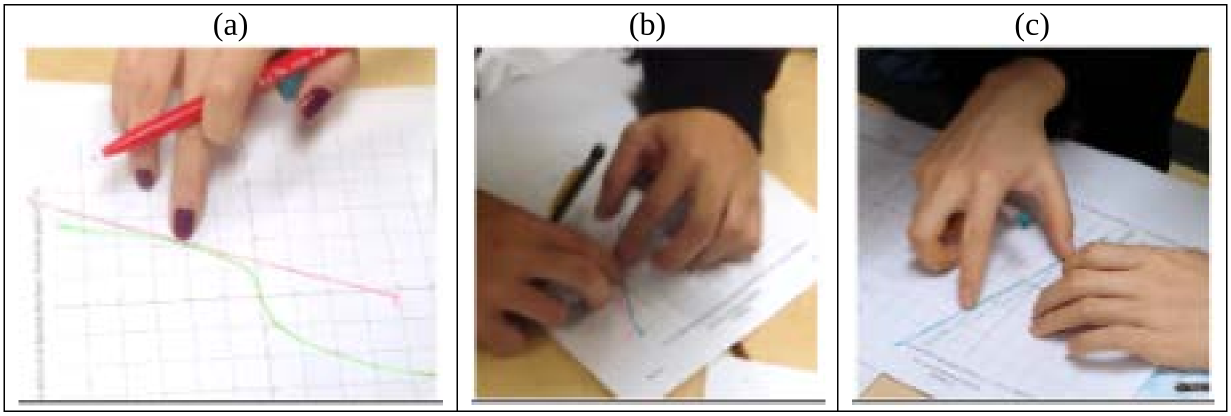

Figure 3. Physically feeling tangent to a curve with (a) one or (b & c) two fingers.

The act of drawing a tangent line with a 3D pen was similar to diagramming on a piece of paper. However, in picking up the tangent line, the diagram became a manipulative: the temporal and material idea of function as a *process*, was encapsulated into one graph that could be picked up, and manipulated. The tangent line thus was dually objectified both in the mathematical sense of Gray and Tall (1994) and in the physical sense. Even if the task was 2D in nature, in that students drew functions in two variables, the 3D drawing came into play with the moving and touching of the tangent lines, which gave rise to new gesture-diagram interaction, including the two-handed gestures of pushing the line on the curve (Figure 3b) and moving the line along the curve with two fingers (Figure 3c). In terms of the semiotic nature of this situation, the line is both indexing the tangent and operating as an icon for it. The tangent line as an indexical sign preserves the process of drawing it and therefore the physical connection to the mathematical object.

Drawing in space

In another pilot calculus lesson, we asked students to draw the "solids by revolution" before they learned to solve for their volume using definite integrals. This lesson was designed to support visualisation of the structure, including the cross sections, of solids generated by revolution of a function about an axis with the aid of 3D pens that enabled one to "draw in space." The students employed various 3D drawing strategies, which are worth describing because of the interplay between the drawing process and the solid formed—the gesture-diagram interaction—that was facilitated in the process. For example, when asked by the classroom teacher to visualise the solid formed by revolving a curve about the *x*-axis, the students invented a strategy that made use of the "*x*-axis" as a manipulative and the action of spinning the axis. Having drawn a curve and the coordinate axes with a 3D pen, they picked up the drawing from paper, hold the two ends of the *x*-axis and began rotating it physically and rapidly (Figures 4a–c). Upon spinning the axis to form a virtual solid, this student also interacted with the diagram with their hands by tracing the drawings and the imagined solid in the air (Figure 4d).

In a different example which would generate a paraboloid, a student placed the 3D pen at the tip of one arm of the parabola and rotated the parabola gently while holding the 3D pen still, which resulted in a curved line in space. When the parabola had been turned by one full rotation, the curved line had formed a circle. In both strategies, the students physically performed the rotation of their drawings. Moreover, their hand movements of diagramming and manipulating with their drawings were the very movements that gave rise to formal mathematics (Châtelet, 2000). Interestingly, this process of generating the solids required two hands rather than one—a breakthrough from drawing and diagramming as a one-handed activity. These examples not only highlight mathematical thinking as an embodied activity but also the intricate interaction among gesture, diagram and mathematical thinking that is mobilised in a 3D drawing environment.

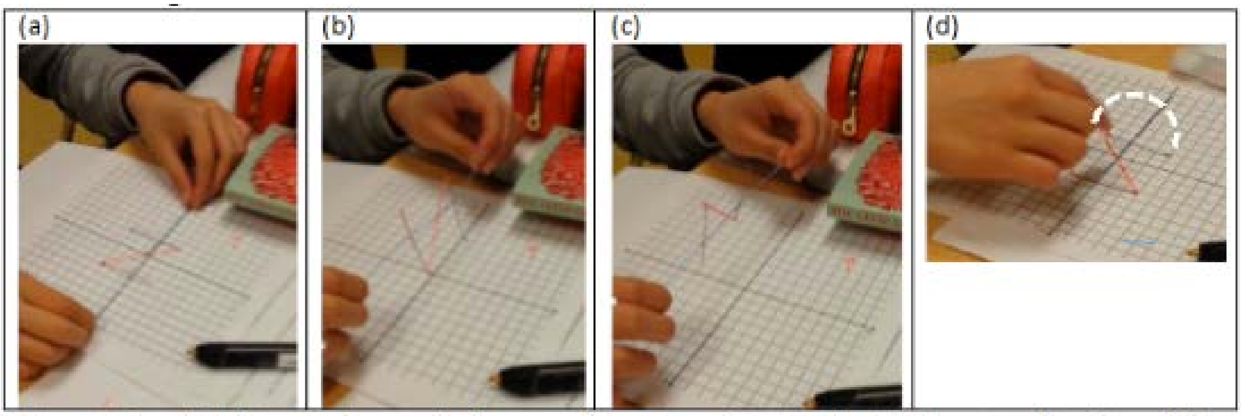

Figure 4. (a–c) Picking up the graph drawn and rotating the axis physically to visualize the solid formed. (d) Gesturing a semi-circle above the diagram.

Drawing and comparing triangles

As their introduction to using 3D pens, Grade 3 students were asked to work in pairs to draw three triangles that had been printed on a sheet of paper (the three triangles were scalene, and each oriented differently on the page so that the students could not immediately see that they were congruent). Some of the students had to try several times before they were successful in connecting the edges together. Once they had drawn their triangles, they were asked to pick them up and compare them (Figure 5a). When each pair had created three triangles, they formed a foursome with another pair and tried to arrange their triangles together so that they would not overlap or leave holes (the teacher pointed to other examples of tessellations in the classroom). The students could thus rotate and flip their six triangles and use them to compose new shapes (for example, two triangles could be put together to form a parallelogram, or they could all be fit together to create a hexagon (Figure 5b). The triangles they had drawn thus became manipulatives, so that they effectively moves from the 1d objects that make up a triangle (the sides) to the 2D shape—thereby engaging in the kind of dimensional composition and decomposition that Duval (2005) sees as significant in thinking geometrically. Further, instead of plastic manipulatives that are most frequently equilateral or right-angled triangles, the 3D drawn triangles were scalene, which meant that trying to fit them together required attending to the lengths of each side of the triangle and trying to match them together. Again, as with the tangent line example, in terms of semiotics, the 3D triangle is both indexing the triangle and operating as an icon for it.

As a collective activity, the teacher asked the students how they would compare the triangles they had made. Since they had used differently coloured wax, colour was one characteristic that was mentioned. One student, who was dangling all of the six triangles from his group on his finger suggested that that the triangles were all the same. Another student put one triangle on top of the over (Figure 5c) and announced that her classmate was right, they were all the same. When asked by the teacher how they could tell that the triangles were all the same, another student, who had also overlapped a stack of triangles, explained that each side of one triangle matched the side of another triangle. The overlapping strategy thus became available as a means to determine whether two triangles are congruent, something that would be much harder to accomplish on a static paper surface.

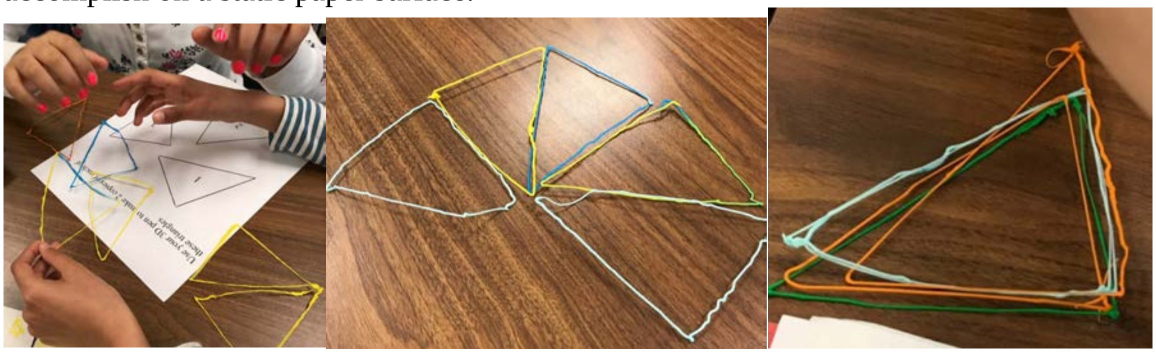

Figure 5. (a) 3D drawn triangles. (b) Tessellating triangles. (c) Stacking triangles.

Other possibilities

To further examine other possible places where 3D drawing might be useful in school mathematics, we invited a group of mathematics educators to try using the 3D pens. We were intrigued by what they would draw when using this technology for the first time. We were also curious about probing into any insights about mathematical topics and tasks that 3D drawing would complement. In addition to drawing geometrical objects such as cubes, spirals, and other 3D figures, we found that they also made use of the third dimension of their drawings for more creative objects (Figure 6a–d), such as:

1. explored non-Euclidean geometries on apples and other curved surfaces (e.g., drawing a triangle on a sphere),
2. experimented with flexible, dynamic forms in which parts move in relation to one another (e.g., drawing a swing),
3. constructed sculptures in 3D (i.e., without relying on the rules of perspective drawing), and finally
4. created sets of 2D shapes on grid paper, thereby giving them a tangibility (e.g., drawing and then picking up a square, a triangle, a circle etc.).

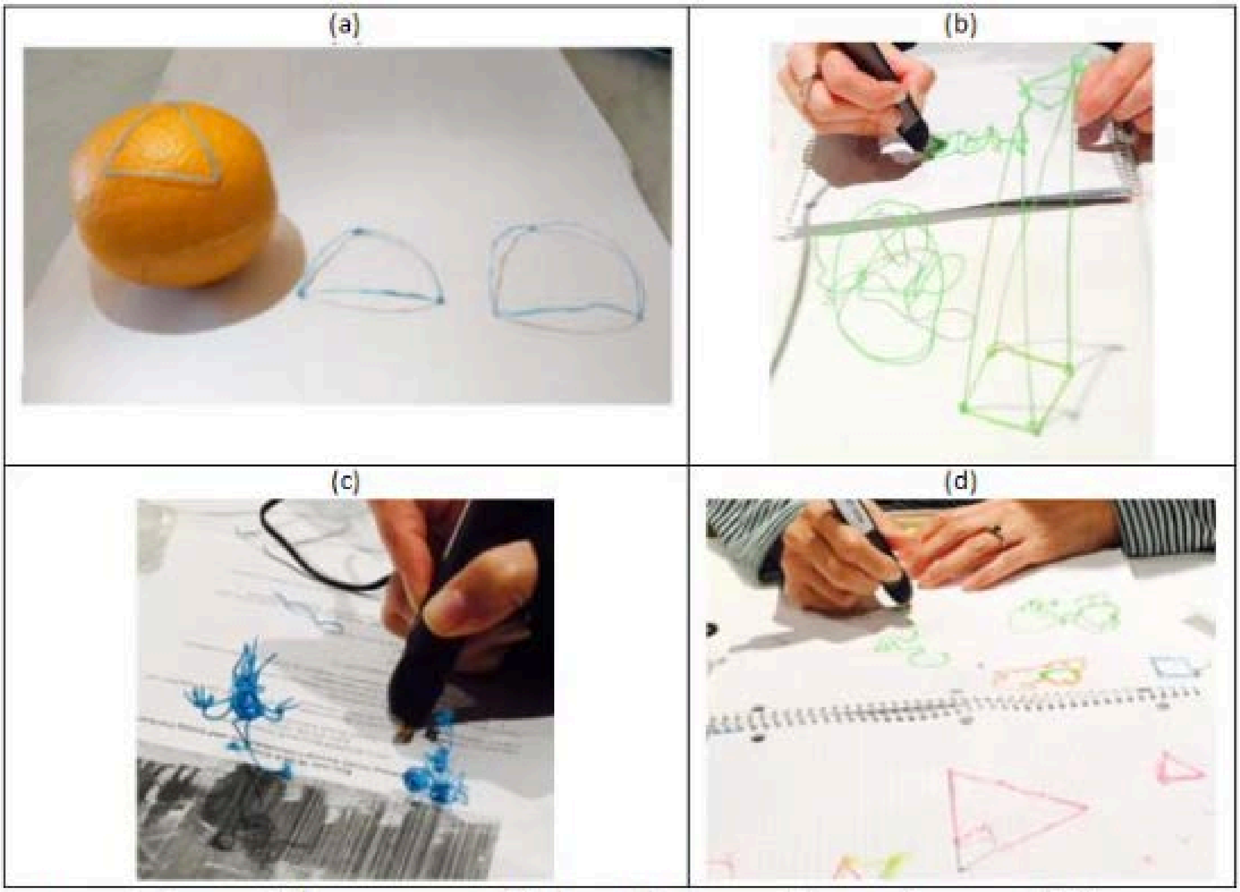

Figure 6. (a) non-Euclidean geometry, (b) objects that move, (c) 3D sculptures, and (d) 2D shapes drawn by mathematics educators.

While some of these drawings might not be immediately perceived as mathematical, they do widen the scope of what is possible in terms of drawing, thinking and expressing in 3D. In particular, being able to construct 3D sculptures and to experiment with objects that move might provide some interesting connection to STEM learning. We speculate that these connections associated with constructing or engineering, as well as the prospect of working with non-Euclidean geometry, may give rise to new topics in the K–12 mathematics curriculum.

As informed by these initial encounters, we speculate that there may be two ways in which 3D drawing, if taken up, may impact teaching and learning mathematical topics. First, we recognise that our subjects, including the mathematics educators, found it quite powerful even to produce 3D drawings that were flat. They were attracted to the tangibility of their creations—the ability to pick up and interact with the drawings physically even if they were 2D in nature. Thus, the physical and tactile interactions of drawing, touching and turning a 2D figure may in itself change the learning of geometric shapes and transformations. Moreover, 3D pens may also reduce the need to rely on numerical and algebraic approaches to certain topics in 2D geometry, such as angles, congruence and trigonometry (see also Ng & Sinclair, 2014). For example, one can compare angles and line segments by superimposing one object on another or by the sense of touch. As seen in Figure 7a and 7b, while the angles drawn illustrate particular examples, they also maintain a sense of generality since they did not rely on numerical measurements.

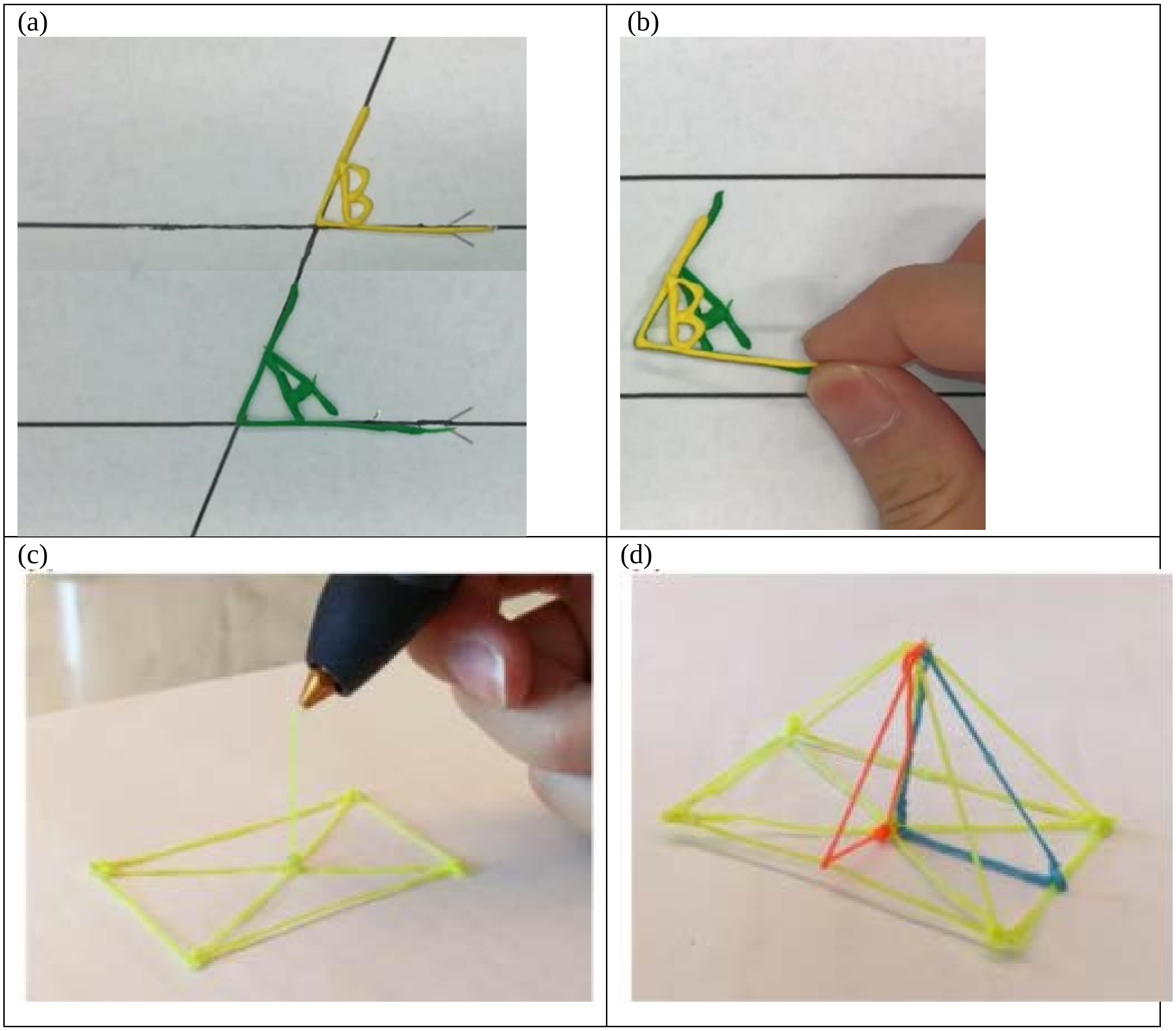

Figure 7 (a–b) Comparing angles formed by two parallel lines and a transversal. (c–d) Drawing a rectangular pyramid with a 3D pen.

Secondly, when drawing 3D figures, one does not need to rely on the rules of perspective drawing. As opposed to drawing flat diagrams, 3D drawing requires reconstructing the 3D solids in space, through which particular features, such as perpendicularity, parallelism, height, and relationships between faces and vertices can be observed. The long tradition of 2D drawing has given rise to major curriculum topic and assessment tasks around the interpretation of 3D objects on 2D surfaces such as nets and techniques of perspective drawing. The availability of 3D pens could greatly change the nature of these topics. For example, the drawing process of a pyramid as shown in Figures 7c and 7d makes it possible to observe the relationship between the height and the diagonals of the base of the pyramid as well as the three different right triangular plane that are perpendicular to the rectangular base of a pyramid.

In summary, we discussed examples of a range of mathematical topics spanning the current elementary and secondary school curricula that 3D drawing could complement, illustrating how 3D drawings may enhance the learning of *2D shape* recognition and transformation in early grades, *shapes and space* in the elementary school level, as well as *functions* and *calculus* in the secondary level. We are intrigued by the possibilities of these tasks, particularly the "drawing in space" kind, which seem to offer more significantly novel opportunities for teaching and learning school mathematics. As a result, though, we anticipate that these latter tasks will be both more difficult to design and to integrate into current classroom practices.

Locating 3D drawing technologies in the broader picture of school mathematics

We suspect that most mathematics educators would greet 3D drawing technologies as potentially powerful supports to the development and extension of shape-associated concepts–in much the same way we did, as revealed in our examples. Our own interest in these technologies extends further however. We see in them another tool to help transform persistent popular beliefs on what mathematics is all about.

One of the most common laments that we encounter in our interactions with other members of the mathematics education research community is that the subject matter is too often reduced to memorization of facts, application of rules, and manipulation of symbols – in a word, *computation*. While it is easy to trace the origins of this perspective to the modern school's originating obligation to prepare citizens capable of dealing with the numerate demands of a newly industrialized society, it is much less clear why the associated number-focused, procedure-laden, and manipulation-heavy conception of school mathematics has been so resilient—not just despite efforts to reform it, but in the face of cultural transformations that render much of its contents so ill-fitted to contemporary needs and possibilities.

We wonder whether the sorts of activities and artefacts that become possible through 3D pens might serve as exemplars in ongoing efforts to reform perceptions of school mathematics. In particular, with regard to possible conceptions of what "doing mathematics" is all about, our observation is that assumptions that "math is computation" quickly give way to notions that are more towards "math is modelling." This is a perspective that many mathematicians have, as became evident for one of the author who recently visited the mathematical models museum at the Institut Henri Poincaré in Paris, which gathers over a 100 years worth of mathematical models made of paper, wood, wire and clay. In a book describing some of the collection, Villani and Uzan (2017) write about how these objects "open a crack between reality and imagination" (p. 9, *our translation from the French*), inviting us to live simultaneously in the imperfect world of the material object *and* the perfect one of mathematical abstraction. In this view, mathematics is not about mining and mastering preset truths, but about developing and imposing strategies for interpreting and organizing aspects of the world.

As we have already flagged, mathematical engagements that are exemplified by use of a 3D pen might be seen as situated at a nexus of several threads of discussion within contemporary mathematics education research—including, most obviously, embodied cognition, spatial reasoning, gesture, and manipulative materials. Amid these discussions, we see the unique contribution of the 3D pen to be the immediate material trace of one's motion in the world—that is, the generation of a physical model of one's thoughts/actions that is then available for elaboration, analysis, and other sorts of interrogation. Phrased differently, in addition to being adaptable to a range of topics in school mathematics (as illustrated in the preceding section), 3D drawing engenders a sort of mathematical craftwork as it presents opportunities to "build"—literally and figuratively—object–concepts. What sorts of possibilities for understanding might arise when one is able to step into a self-amplifying loop of interpreting a concept and materially representing aspects of that concept?

In this frame, computation is displaced as the focus of mathematics learning and repositioned as an element of mathematical inquiry. For example, in addition to its many uses in studying topics in geometry, a "simple" 3D-drawn cube presents multiple opportunities to enumerate, measure, and calculate—that is, to impose mathematical notions onto emerging artefacts, where the point isn't just to apply the appropriate procedures to characterize an object (i.e., computation), but also to explore how percepts and concepts co-develop.

To perhaps put a finer point on this discussion, consider the topics of exponentiation. A "math is computation" mindset almost inevitably pulls that topic toward a definition-bound and rule-driven study that is anchored to the limiting definition that "exponentiation is repeated multiplication." But within a space of drawing a square into existence through combining lines, and drawing a cube into existence by combining squares, it might become more apparent that the exponentiation that is happening is something that is much more than repeatedly *timesing* a number. In this instance, it is a journey across dimensions, an emergence of new units. Moreover, the act of computing in this instance is clearly and profoundly an act of modeling. The mathematics to be learned is not an external body of fixed knowledge to be acquired, but an ever-developing system to interpret, organize and manipulate one's world.

Will 3D pens be taken up?

We do not mean to overstate the case. The use of 3D pens presents many issues, including matters of cost, usability, accuracy, and development of necessary proficiency. We are thus not particularly hopeful that the pens will be embraced on any grand scale. That is, our point is not that 3D pens should be incorporated across topics in school mathematics, but that mathematics educators should be attending to the possibilities being presented in emerging technologies that enable tracking and recording of gestures and other actions that might be associated with mathematical cognition.

An important element in these considerations, as Francis and Whiteley (2014) have developed, is that the relationship between 2D images and 3D objects is neither natural nor intuitive. As they noted,

> Recognition of the 3D object from its 2D representation cannot be assumed, and spatial reasoning about 3D objects requires an unambiguous reconstruction of the object (at least mentally). Moving back and forth between 2D representations and 3D objects requires acculturation to conventions used, and developed practice of when and how to move between 3D and 2D. Fluency with moving between 2D and 3D space is essential for reasoning and connecting representations of scientific concepts. (p. 134)

How might such issues and problematics be transformed if the movement between the millennia-old emphasis on 2D scribing in mathematics is enabled through 3D drawing?

On those matters, we believe the same principles and possibilities apply to other emerging technologies. An obvious parallel can be seen in the recent elaboration of virtual reality headsets, which have evolved from devices used for looking/watching into tools for creating as they make it possible to retain digital traces of the wearer's motions. How similar are such virtual experiences to the more obviously material experience of drawing with a 3D pen? How, for example, might the similar enactive/gestural set associated with using a 3D pen to draw tangent and move it along a curve compare with doing it virtually? That is, how much does the physical object matter in these learning encounters? When is it better? When does it get in the way?

The more overarching queries here are clearly linked to questions and concerns raised by thinkers interesting in the co-evolutions of ideas and tools associated with those ideas (e.g., symbols, algorithms, digital devices). Rotman (2000), for example, has offered a compelling account of how mathematics since its inception been engaged in a two-way co-evolutionary traffic with machines—from which he predicts that mathematics will move increasingly toward the non-alphanumeric.

Of parallel interest on the individual level, and perhaps closer to the actualities of mathematics educators, are the ways that tools and experiences contribute not just to conceptual possibilities but to actual brain health and development. As the educational establishment becomes more aware of and accumulates evidence on the tight linkages of modes of experience and preferences for sense-making, it apparently will soon have to grapple with the intersections of insights developed in virtual worlds and the actual encounters with physical realities.

Of course, educators have long been dealing with versions of these questions, albeit that the issues have tended to be framed more in terms of the ideal/real than the virtual/physical. Somehow, however, the lines between these paired realms seem to be more troublesome when the ideal and virtual is permitted to escape the plane and drawn into the space we perceive ourselves to inhabit.

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