[The Mathematics Enthusiast](https://scholarworks.umt.edu/tme)

Volume 16
Number 1 *Numbers 1, 2, & 3*

Article 2

2-2019

On the definition of periodic decimal representations: An alternative point of view

Giuseppina Anatriello

Giovanni Vincenzi

On the definition of periodic decimal representations: An alternative point of view

Giuseppina Anatriello
University of Naples "Federico II", Italy

Giovanni Vincenzi
University of Salerno, Italy

ABSTRACT: Until the seventeenth century, rational numbers were represented as fractions. It was starting from that century, thanks to Simon Stevin, that for all practical purposes the use of decimal notation became widespread. Although decimal numbers are widely used, their organic development is lacking, while a vast literature propagates misconceptions about them, among both students and teachers. In particular, the case of the period 9 is especially interesting. In this paper, changing the point of view, we propose substituting the usual definitions of the periodic decimal representation of a rational number – the one obtained through long division introduced in the secondary school and the one obtained through the series concept for undergraduate level – with one which, instead of infinite progressions, uses a known property of fractions deducible from Euler's Totient Theorem. The long division becomes a convenient algorithm for obtaining the decimal expansion of a rational number. The new definition overcomes the difficulties noted in the literature. An elementary proof of Euler's Totient Theorem will be given depending only on Euclidean Division Theorem.

Keywords: Periodic decimal representation, Totient's Euler Theorem, 0.999...

The Mathematics Enthusiast, **ISSN 1551-3440, vol. 16, nos. 1, 2 &3**, pp. 3–14

Introduction

The introduction of decimals in mathematics came about through a long and arduous process. *The first systematic algebra to use positive and negative numbers, zero and the decimal system was developed by Hindu mathematicians in India during the seventh century A.D.[...] This worldwide system is universally known as the "Hindu–Arabic numeral system" that was discovered by the Indian Hindu mathematicians, and then adopted and transmitted by the Arab mathematicians to Western Europe* ([Deb]). But at the end of the fifteenth century not even the writing of natural numbers in decimal system had yet been established.

The Hindu-Arabic arithmetic for natural numbers became definitive and widespread thanks to the publication of printed texts, while fractions still remained in use because they were easy to name, though their calculations required constant conversions. Tenths, hundredths and thousandths were used only casually.

At the end of the sixteenth century Francois Viéte (1540-1603) became an ardent defender of them, but it was Simon Stevin (1548-1620) who, with his work *De Thiende*, first published in Dutch in 1585 and translated into French as *La Disme* (translated into English as *Decimal Arithmetic*), was able to convince his contemporaries of the benefits of systematically splitting the units into tens, substituting the calculations of ordinary integers with conversions and reductions to the same denominator. In the same year Stevin published *Arithmtique*. In this work he presented a unified treatment for solving quadratic equations and a method for finding approximate solutions to algebraic equations of all degrees (bringing to the western world for the first time a general solution of the quadratic equation, originally documented nearly a millennium previously by Brahmagupta in India).

Moreover, with his works, *De Thiende* and *Arithmtique*, Stevin *initiated a systematic approach to decimal representation of measuring numbers, marking a transition from a discrete arithmetic as practiced by the Greeks, to the arithmetic of the continuum taken for granted today* ([KK]).

About this subject, in [Mal] the author writes: *A Euclidean number is always what we call a "natural" or positive integer number, and the numerical operations involving them (for instance, the determination of proportional means) were severely restricted to ensure that the answers were acceptable that is to say, that the answers were positive integer numbers. As far as we know, not only was the neat and consistent separation between the Euclidean notions of numbers and magnitudes preserved in Latin medieval translations (see below), but these notions were still regularly taught in the major schools of Western Europe in the second half of the 15th century. By the second half of the 17th century, however, the distinction between the classical notions of (natural) numbers and continuous geometrical magnitudes was largely gone, as were the notions themselves. We cannot find in the 16th century a theory of numbers and magnitudes that confronts the classical notions head on not until we get to Stevin's "Arithmétique" of 1585.*

We could say, according to [Fow], that Stevin was a thorough-going arithmetiser. Anyway, his general notion of a real number was accepted, tacitly or explicitly, by all later scientists ([vdW]). But acceptance of the symbol with endless digits was possible only after the introduction of the concept of limit and the assertion of Cantor's theory of infinite sets.

The developments of analysis in the nineteenth century, especially concerning the theory of limits that was the basis of it, made it increasingly urgent to arrive at a precise and unequivocal definition of real numbers and their properties. "Arithmetization of analysis" is the term used in the history of mathematics to refer to the period (around 1870) in which an exclusively arithmetic foundation was given to the theory of real numbers, inducing mathematicians such as Karl Weierstrass, Richard Dedekind and others to want to re-found the entire theory of numbers, freeing its introduction from any intuitive aspect and conceiving of real numbers as conceptual structures rather than as intuitive parameters inherited from Euclidean geometry ([Boy], [PS]). This objective was reached by different paths that led to, among other things, the definition of infinite decimals through the series concept.

However, the "degeometrization" of the real numbers was not carried out without skepticism. In [Kli] Morris Kline quotes Hermann Hankel who wrote in 1867: *Every attempt to treat the irrational numbers formally and without the concept of [geometric] magnitude must lead to the most abstruse and troublesome artificialities, which, even if they can be carried through complete rigor, as we have every right to doubt, do not have a right scientific value.* Frege himself, the father of logicism, in the last years of his life went back on his position regarding geometry. In *Zahlen und Arithmetik* [Numbers and Arithmetic] (in [Fre], 1924-25) he recognized in geometry the true source of mathematical knowledge, arguing that the nature of numbers was geometrical because of the entirely geometrical nature of irrational numbers, numbers that mathematicians need; in the same work he stated that there is no bridge leading to irrational numbers by starting from natural ones (which were for Frege the "numbers of commerce", the first to be taught because a child must be prepared to keep accounts, buy and sell). Moreover, in his *Neuer Versuk der Grunderlegung der Arithmetik* [Retrying the Foundation of Arithmetic] (in [Fre], 1924-25) Frege began to formulate a geometric theory of proportions to define complex numbers in a purely geometric context. He died leaving this work unfinished (recently, in [ATV] and [AMV], the authors identify an axiomatic theory of a purely synthetic geometrical type in keeping with the reasoning initiated by Frege).

In school path it is passed from the definition of decimal representation of a rational number obtained by means of the long division, which can bring up all the periods except for the period 9, to that obtained by means of notion of the infinite series. Regarding the topic of the 'period 9' in the literature are assumed opposite positions: allowable [Rud], prohibited [Art].

Many studies establish that students at all levels, including preservice elementary and middle school teachers, have considerable difficulty understanding the relationship between a rational number (fraction of integer) and its decimal expansion(s). In particular, these studies examined the cognitive difficulties in accepting the relationship $0.\overline{9}$ $(= 0.999\ldots) = 1$, all attributable to the concepts of limit and infinity ([BLDKPS], [CD], [Eis], [Kal], [Ric], [Sie], [Sch], [YBL], [Yim], for example).

To solve the problems related to the notion of recurring decimals, and in particular to the period 9, in this work we propose an elementary definition of decimal periodic representation (see Definition 2.1 in Section 2) that includes the period 9 and that uses neither the concept of limits nor that of actual infinity. At the same time it will be proved that every rational admits a periodic decimal representation (see Theorem 2.3).

This change of viewpoint, as will be seen, makes it possible to actually overcome the problems which normally accompany the periodic decimal representation of rational numbers in the cited works. A key role will be played by the following Euler's Totient Theorem:

Theorem 0.1 (*Euler's Totient Theorem*). For every positive integer b, let $\varphi(b)$ be the number of integers in $\{1,\ldots,b\}$ that are coprime with b. If b be is coprime with a, then $a^{\varphi(b)} - 1$ is a multiple of b.

Euler's Totient Theorem is of major interest in number theory, as well as many other fields of mathematics, for example, in cryptography ([Chi]). His connection with periodic decimals has long been known (see for instance [Cat]), but at the scholastic level it is generally not referred to, perhaps because the usual proofs of the theorem are given by means of group theory, binomial theorem, or modular arithmetic (see [Sán], p. 188-189). In Section 3 we will provide a simple proof of Euler's Totient Theorem depending only on Euclidean Division Theorem.

In the note [Cat] E. Catalan converted periodic decimals into ordinary fractions without using infinite progressions ([Dic]) and wrote *La note suivante ne contient rien de neuf: si je me décide à la publier, c'est parce que la manière dont on présente ordinairement la théorie des fractions périodiques n'est, si je ne me trompe, ni trés-logique, ni très-rigoureuse. En outre, cette théorie s'appuie assez naturellement sur le théorème de Fermat, et sur d'autres propriétés intéressantes* (among which Euler's Totient Theorem) *qu'il serait peut être convenable de faire entrer dans les éléments (En 1835, M. Midy, alors professeur au college de Nantes, a public un peut mémoire que ma note reproduit en grande partie d'apies ce que j'ai dit ci dessus, cette conformité etait inevitable).*

For convenience of the reader, in the following section we will review the definitions of decimal representation via series and long divisions and known properties.

1 Background

Today at the undergraduate level the decimal representation of a number real, and in particular of a rational number, is introduced by first defining a real number as an element of a complete ordered field and then by proving that every element of a complete ordered field is in absolute value the sum of a series of the following type:

$$\sum_{i=0}^{\infty} \frac{a_i}{10^i}, \tag{1.1}$$

where a_0 is a non-negative integer, and $a_1, a_2, \ldots$ are integers satisfying $0 \le a_i \le 9$, called the *digits of the decimal representation* (see for example, [Kal], [Rud]). As we have already said some authors forbid decimal representations with a trailing infinite sequence of "9"s, this restriction still allows a decimal representation for each real number, additionally, makes such a representation unique ([Kal]).

The number a defined by (1.1) is often written more briefly as

$$a_0.a_1a_2a_3\dots$$

Usually a_0 is called the *integer part* of a and $a_1, a_2, a_3, \dots$ are the digits forming the so-called *fractional part* of a.

In the case of rational numbers (and only in that case) it happens that from a certain index $i \geq 0$ onwards there is a cyclical repetition of $a_{i+1}\dots a_{i+t}$ ([Kal]). The decimal representation so obtained is called the *periodic decimal of period* $a_{i+1}\dots a_{i+t}$ and *anti-period* $a_1\dots a_i$. In particular, the series $\displaystyle\sum_{i=0}^{\infty}\frac{9}{10^n}$ converges to 1.

In secondary schools first the decimal representation is introduced only for rationals m/n ($m/n > 0$ and expressed in lowest terms) whose denominator n divides a power of 10.

In this case
$$\frac{m}{n} = \frac{v}{10^i} = a_0 + \frac{a_1}{10} + \dots + \frac{a_i}{10^i}$$

and to represent m/n in decimals the symbol $a_0.a_1\dots a_i$ is used, with natural a_0 and $a_1, \dots, a_i$ digits of the decimal system. The symbol $a_0.a_1\dots a_i$ is called the *finite decimal representation* of m/n, and m/n is called a *finite decimal.* To find $a_0, a_1, \dots, a_i$'s which make it possible to convert a rational m/n into a finite decimal by using the long division algorithm between m and n which, as is well known, consists in adding a point to the quotient and zeros to the dividend until a zero remainder gotten.

If n presents a prime divisor different from 2 and 5, then m/n cannot be represented by a fraction with a denominator power of 10^i and a remainder of zero is never obtained by using long division. It happens, as is known, that a remainder must necessarily be repeated and therefore all subsequent remainders are cyclically repeated.

In general, let be q_j and r_j the quotient and the remainder of the Euclidean division of $m \cdot 10^j$ with n, for every non-negative integer j, the following relations hold:

$$\frac{m}{n} = \frac{m \cdot 10^j}{n} \cdot \frac{1}{10^j} = \frac{nq_j + r_j}{n} \cdot \frac{1}{10^j} = \frac{q_j}{10^j} + \frac{r_j}{n \cdot 10^j}, \tag{1.2}$$

with $0 \leq r_j < n$ and then $0 \leq \dfrac{r_j}{n \cdot 10^j} < \dfrac{1}{10^j}$.

For the sake of simplicity, let us refer to the case of $0 < m/n < 1$ and m/n expressed in lowest terms.

The digits of q_j are at most j and, by possibly placing zeroes before the digits of q_j, we can consider q_j as constituted by j digits, i.e. we can place $q_j = c_1\dots c_j$ with digits $c_1, \dots c_j$, and the result is $m/n = 0.c_1\dots c_j + \dfrac{r_j}{n \cdot 10^j}$.

If n is coprime with 10, for each non-negative integer j, the remainder of the Euclidean division of $m \cdot 10^j$ with n is never zero, and the first remainder that is repeated by the long division between m and n is precisely $r_0 = m$. If the first remainder is repeated after t steps, then $r_t = r_0$, $q_t = b_1\dots b_t$ and after subsequent t steps we obviously get $r_0 = r_t = r_{2t}$ and $q_{2t} = b_1\dots b_t b_1\dots b_t$. After kt steps we get $r_0 = r_{kt}$ and $q_{kt} = \underbrace{b_1\dots b_t}_{1^{th}}\dots\underbrace{b_1\dots b_t}_{k^{th}}$. In this way we get endless representations of m/n (potential infinity) and we can write

$$m/n = 0.\overline{b_1\dots b_t} = 0.b_1\dots b_t b_1\dots b_t\dots b_1\dots b_t\dots \tag{1.3}$$

and (1.3) is called *periodic decimal representation of period* $b_1\dots b_t$.

If n is not coprime with 10, and s is the maximum between the exponents of the powers of 2 or 5 that divide n ($s > 0$).

In this case, as is known, the first remainder that is repeated in the long division between m and n is r_s. Let us suppose $r_s = r_{s+t}$ (with $r_s \neq r_{s+i}, 0 < i < t$), if $q_s = a_1\dots a_s$ and $q_{s+t} = a_1\dots a_s b_1\dots b_t$, for (1.2), we get

$$m/n = 0.a_1\dots a_s + \frac{b_1\dots b_t}{10^{s+t}} + \frac{r_s}{n \cdot 10^{s+t}}$$

$$= 0.a_1 \dots a_s + \frac{b_1 \dots b_t}{10^{s+t}} + \frac{b_1 \dots b_t}{10^{s+2t}} + \frac{r_s}{n \cdot 10^{s+2t}} = \dots$$

and for m/n we can use the representation

$$0.a_1 \dots a_s\overline{b_1 \dots b_t} = 0.a_1 \dots a_s b_1 \dots b_t b_1 \dots b_t \dots b_1 \dots b_t \dots, \tag{1.4}$$

(1.4) is called the *periodic decimal representation of period* $b_1 \dots b_t$ *and anti-period* $a_1 \dots a_s$ of m/n.

If $m/n > 1$ then $m/n = q + r/n$, where q and r are the quotient and remainder of the Euclidean division of m by n, and it suffices to apply to r/n the above-said and replace "0." with "q." in the periodic representations (1.3) and (1.4). As is known, the decimal representation of the rational numbers introduced through long division does not yield the period 9.

2 An alternative definition

We now propose an elementary definition as an alternative to the periodic decimal representation of a rational number - that we have just recalled in the above section - which also covers the case of the period 9 and which offers the advantage of requiring neither the concept of limit nor that of ongoing infinity, notions wherein lie, according to the literature refered to in the introduction, the reasons for the problems connected with the misconceptions about $0.\overline{9} = 1$. Long division, as will be noted in Remark 2.2, takes on the role of a convenient algorithm (as occurs for finite decimals).

In the sequel we will refer only to positive numbers.

Definition 2.1. Having assigned arbitrarily the digits $m_1, \dots, m_n$, $a_1, \dots, a_u$, $b_1, \dots, b_v$, $b_1, \dots, b_v$, where some b_j different from 0 we establish

$$m_1 \dots m_n \,.\, a_1 \dots a_u\overline{b_1 \dots b_v} = m_1 \dots m_n + \frac{a_1 \dots a_u}{10^u} + \frac{b_1 \dots b_v}{10^u(10^v - 1)}. \tag{2.1}$$

We call the first member symbol of (2.1) the *periodic decimal representation of the integer part* $m_1 \dots m_n$, *anti-period* $a_1 \dots a_u$ and *period* $b_1 \dots b_v$ of the rational number to the second member.

The expression

$$\frac{a_1 \dots a_u}{10^u} + \frac{b_1 \dots b_v}{10^u(10^v - 1)},$$

that is less than 1, is called *decimal part* of $m_1 \dots m_n \,.\, a_1 \dots a_u\overline{b_1 \dots b_v}$.

It may help to think of it as similar to mixed numbers. Here an example could be useful:

Example 2.1. $62.0417\overline{34} = 62 + \dfrac{417}{10000} + \dfrac{34}{(10000)(99)}$.

In Introduction we have presented the conceptual problems related to the classical interpretation of the period 9. In the next remark we highlight that with the new definition of periodic decimal representation we don't need to force that $0.\overline{9} = 1$, as the equality is just a consequence of the definition itself.

Remark 2.1. If a periodic decimal representation of a rational number m/n has period 9, then m/n coincides with the finite decimal obtained by removing the period 9 and increasing by 1 the digit that precedes the period 9. Indeed:

$$m_1 \dots m_n . a_1 \dots a_u\overline{9} = m_1 \dots m_n + \frac{a_1 \dots a_u}{10^u} + \frac{9}{10^u \cdot 9} \tag{2.2}$$

$$= m_1 \dots m_n + \frac{a_1 \dots a_u}{10^u} + \frac{1}{10^u}.$$

Conversely every finite decimal has a periodic decimal representation with 9 as period. To see this, let m/n be a rational number with $nk = 10^s$, for some integer positive s. Put $h = mk$, then for each non negative integer t, it yields

$$m/n = \frac{h(10^t - 1)}{10^s(10^t - 1)} = \frac{(h-1)(10^t - 1) + (10^t - 1)}{10^s(10^t - 1)} = \frac{h-1}{10^s} + \frac{10^t - 1}{10^s(10^t - 1)}.$$

Therefore m/n allows a periodic decimal representation of type (2.1) of anti-period $h-1$ (even in this case we can assume that the digits of $h-1$ are exactly s, $h-1 = c_1 \dots c_s$) and of period $d_1 \dots d_t$, with $d_i = 9$, for $0 < i \le t$.

By (2.1) it yields

$$m/n = 0.c_1 \dots c_s \underbrace{\overline{9 \dots 9}}_{t},\ \forall t \in \mathbb{N}. \tag{2.3}$$

In order to show that every rational number admit at least one 'periodic' decimal representation of type (2.1) we need the following results, that can be detected by the Euler Totient Theorem:

Lemma 2.2. For any positive integer n, there exist positive integers j and k and a nonnegative integer i such that $kn = 10^i(10^j - 1)$, where $i = 0$ if neither 2 nor 5 divides n, and $j = 1$ if no prime other than 2 and 5 divides n.

Theorem 2.3. Every rational number m/n allows a periodic decimal representation of type (2.1).

Proof. By Remark 2.1 we just have to show that if a rational number m/n is not finite decimal - that is n has a divisor coprime with 10 - then it admits a representation of type (2.1).

If m_1 and m_2 are quotient and remainder of the Euclidean division of m by n, then $m/n = m_1 + m_2/n$, $m_2/n < 1$, therefore we can limit ourselves to analyzing the case of $0 < m/n < 1$, with m and n coprime.

By Lemma 2.2 there exist a non negative integer s and positive integers t and k such that $nk = 10^s(10^t - 1)$, thus:

$$m/n = \frac{km}{10^s(10^t - 1)}. \tag{2.4}$$

Let q and r be respectively the quotient and the remainder of the Euclidean division of km with $10^t - 1$:

$$km = (10^t - 1)q + r. \tag{2.5}$$

If $r = 0$, then by (2.4) and (2.5), we have $\dfrac{m}{n} = \dfrac{q}{10^s}$, so that each divisor of n divides 10^s. This contradiction shows that $r \neq 0$. The number of digits of q are at most s, since $m/n < 1$, those of r are at most t, since the remainder is less than the divisor; therefore, possibly by placing zeros before the digits of q and r, we can consider q made up of s digits, i.e. $q = c_1 \dots c_s$, and r made up of t digits, i.e. $r = d_1 \dots d_t$. It yields

$$m/n = \frac{km}{10^s(10^t - 1)} = \frac{q(10^t - 1) + r}{10^s(10^t - 1)} = \frac{c_1 \dots c_s}{10^s} + \frac{d_1 \dots d_t}{10^s(10^t - 1)} \tag{2.6}$$

and, by (2.1), it yields

$$m/n = 0.c_1 \dots c_s \overline{d_1 \dots d_t}.$$

□

Remark 2.2. The long division between numerator and denominator becomes a convenient algorithm for determining the periodic decimal representation of a rational number m/n and therefore the classical definition and that of type (2.1) of periodic decimal representation coincide. For the sake of brevity, we limit ourselves to considering the case of $0 < m/n < 1$ and n coprime with 10.

We observe that if $h < 10^t - 1$ then

$$h \cdot 10^t = h(10^t - 1) + h,$$

therefore h is both the quotient and the remainder of the Euclidean division of $h \cdot 10^t$ with $10^t - 1$. In addition, if $m/n = h/(10^t - 1)$, $h = km$ and $m \cdot 10^t = nq_t + r_t$, with $0 < r_t < n$, then $h \cdot 10^t = (10^t - 1)q_t + r_t k$, con $0 < r_t k < 10^t - 1$, which implies $h = q_t$.

Remark 2.3. In [Bar] Baruk stresses the difficulty of understanding the reasons for certain rules that are used for decimals, that in some sense appear to be "magical". Here we will see that those rules referred to current periodic decimal rapresentation are satisfied by the definition of periodic decimal representation of type (2.1).

i) An elementary computation shows that the current decimal representation and the one of type (2.1) have the same *generative fraction*:

$$m_1 \dots m_n \,.\, a_1 \dots a_u \overline{b_1 \dots b_v} = \frac{m_1 \dots m_n a_1 \dots a_u b_1 \dots b_v - m_1 \dots m_n a_1 \dots a_u}{10^u(10^v - 1)}.$$

(ii) $m_1 \dots m_n \,.\, a_1 \dots a_u \overline{b_1 \dots b_v} = m_1 \dots m_n \,.\, a_1 \dots a_u \; b_1 \dots b_v \; \overline{b_1 \dots b_v} = \dots.$ This follows from the identity:

$$\frac{b_1 \dots b_v}{10^v - 1} = \frac{b_1 \dots b_v}{10^v} + \frac{b_1 \dots b_v}{10^v(10^v - 1)}$$

(iii) For any i integer, $1 \le i < v$, we have

$$m_1 \dots m_n \,.\, a_1 \dots a_u \overline{b_1 \dots b_v} = m_1 \dots m_n \,.\, a_1 \dots a_u b_1 \dots b_i \overline{b_{i+1} \dots b_v b_1 \dots b_i}.$$

To see this it is enough to observe that

$$\frac{b_1 \dots b_v}{10^v - 1} = \frac{b_1 \dots b_i(10^v - 1) + b_{i+1} \dots b_v b_1 \dots b_i}{10^i(10^v - 1)}$$

$$= \frac{b_1 \dots b_i}{10^i} + \frac{b_{i+1} \dots b_v b_1 \dots b_i}{10^i(10^v - 1)}.$$

(iv) $m_1 \dots m_n \,.\, a_1 \dots a_u \overline{b_1 \dots b_v} = m_1 \dots m_n \,.\, a_1 \dots a_u \overline{b_1 \dots b_v b_1 \dots b_v} = \dots.$
This can be checked in a similar way like in (ii).

(v) *The decimal point shift rule: If one multiplies a decimal number by* 10^i *the decimal point shifts* i *places from left to right, conversely dividing a decimal number by* 10^i *the point decimal shifts* i *places from right to left.*

Here some examples should be exhaustive to see how we can use the new definition to check it:

1) $2.4\overline{328} \cdot 10^3 = \left(2 + \frac{4}{10} + \frac{328}{10^{3+1}} + \frac{328}{10^7(10^3 - 1)}\right) 10^3 = 2000 + 400 + 32 + \frac{8}{10} + \frac{328}{10(10^3 - 1)} =$ $2432.8\overline{328}$

2) $$2.4\overline{328} \cdot 10^{-3} = \left(2 + \frac{4}{10} + \frac{328}{10(10^3 - 1)}\right) 10^{-3} = 0.0024 + \frac{328}{10^4(10^3 - 1)}$$
$$= 0.0024\overline{328}$$

In order to compare two decimal numbers with this new point of view, it will be useful the following definition.

Definition 2.4. Let $\alpha := m_1 \dots m_n.\; a_1 \dots a_u \; \overline{b_1, \dots, b_v}$, be a periodic decimal representation of type (2.1). Then we will call $a_1, \dots, a_u$ respectively, the *first, second, ..., u-th decimal digit of* α. Moreover, for every integer $r \ge 0$, we will call:
b_1 the *(u+1+r v)-th, decimal digit of* α,
b_2 the *(u+2+r v)-th, decimal digit of* α,
$\vdots$
b_v the *(u+ v-1+r v)-th, decimal digit of* α.

The following remark allows to establish when two decimal of type (2.1) represent the same rational, by their decimal digits.

Remark 2.4. Let α and β two periodic decimal representations. Then α and β represent the same rational if and only if for every positive integer i, the i-th digit of α coincides with the i-th digit of β. To see this, write

$$\alpha := m_1 \ldots, m_n.a_1 \ldots a_u\overline{b_1 \ldots b_v} \text{ and } \beta := m'_1 \ldots m'_{n'}.a'_1 \ldots a'_{u'}\overline{b'_1 \ldots b'_{v'}}.$$

The decimal parts of α and β are less then 1, therefore if α and β represent the same rational then they must have the same integer part: $m_i = m'_i, \quad i = 1, \ldots, n = n'$. Now if c_i and c'_i are the i-th decimal digit respectively of α and β, then by the point shift rule (see Remark 2.3, (v)) they are respectively the unit digits of the integer part of $\alpha 10^i$ and $\beta 10^i$, therefore they coincides. In a similar way we can use the decimal digits of α and β to establish which of the two is greater than the other.

We conclude this section noting that for periodic decimal representations of a rational number m/n the natural extension of calculation's rules - that are valid for finite decimals - are lost. For example:

- $0.4 + 0.6 = 4/10 + 6/10 = 1$, instead $0.\overline{4} + 0.\overline{6} = 4/9 + 6/9 = 10/9$.
- $0.4 \cdot 0.6 = 4/10 \cdot 6/10 = 24/100 = 0.24$, instead $0.\overline{4} \cdot 0.\overline{6} = 4/9 \cdot 6/9 = 24/81 = 0.\overline{2962}$.

Finally we note that the periodic decimal representation provides all the lower approximations of m/n in finite decimals.

3 An elementary proof of Euler's Totient Theorem

In Introduction and along the paper we have highlighted the strong connection between periodic decimals and the Euler's Totient Theorem (Theorem 0.1, Introduction); moreover, we have recalled the difficulties that (at the scholastic level) students meet about its proof, so that teachers usually avoid to explain this crucial result. This causes the lay out of the lessons on decimals not very clear, or at least incomplete. Here we propose an elementary and educationally feasible proof of such important theorem, that can be given in secondary schools. For the sake of simplicity we will prove the theorem for $a = 10$.

Proof of Euler's Totient Theorem.
Let

1. $m_1 < \ldots < m_{\varphi(b)}$ be the integers in $\{1, \ldots, b\}$ that have no factors in common with b.
2. $r_1, \ldots r_{\varphi(b)}$ be the remainders of the Euclidean division of $10\, m_1 < \ldots < 10\, m_{\varphi(b)}$ by b, i.e. let the following relationships be valid

$$\begin{array}{c} 10\, m_1 = q_1\, b + r_1, \quad 0 \le r_1 < b \\ \ldots \qquad \ldots \qquad \ldots \\ 10\, m_{\varphi(b)} = q_{\varphi(b)}\, b + r_{\varphi(b)}, \quad 0 \le r_{\varphi(b)} < b \end{array} \tag{3.1}$$

We note that the remainders r_i, $1 \le i \le \varphi(b)$, are:

I) all different from zero and having no factors in common with b. In fact, for every $i \in \{1, \ldots, \varphi(b)\}$, a common factor between b and r_i would be a factor of $10m_i$, but $10m_i$ has no factors in common with b.

II) distinct. In fact, if it were $r_i = r_j$ for some $i > j$, $i, j \in \{1, \ldots, \varphi(b)\}$, by subtracting member to member in relations (3.1), we would get

$$10(m_i - m_j) = b(q_i - q_j).$$

Since b is prime with 10, b should divide $m_i - m_j$, and this is a contradiction as $0 < m_i - m_j < b$.

Hence

$$\{r_1, \ldots, r_{\varphi(b)}\} = \{m_1, \ldots, m_{\varphi(b)}\}$$

and therefore, by multiplying member to member relations (3.1), we get:

$$10^{\varphi(b)} m_1 \cdot \ldots \cdot m_{\varphi(b)} = (q_1\, b + r_1) \cdot \ldots \cdot (q_{\varphi(b)}\, b + r_{\varphi(b)}) = qb + r_1 \cdot \ldots \cdot r_{\varphi(b)} = qb + m_1 \cdot \ldots \cdot m_{\varphi(b)},$$

which yields:

$$(10^{\varphi(b)} - 1) m_1 \cdot \ldots \cdot m_{\varphi(b)} = qb$$

But b has no factors in common with $m_1 \cdot \ldots \cdot m_{\varphi(b)}$, and therefore b is necessarily a factor of $10^{\varphi(b)} - 1$.

4 Conclusions

As we have seen in the brief historical analysis presented in the Introduction the two usual definitions of decimal came about through a long path of formalization of the concept of number. These definitions involve the use of two different types of infinity: the potential infinity is used in the definition utilizing the long division (introduced at secondary school level and above) and the actual infinity is used in the definition that uses the notion of series (introduced at undergraduate level). The reader that has less familiarity with these concepts can see [MW], and reference therein. Studies conducted on the problems caused by the $0.\overline{9} = 1$ relation show the difficulty that students have in accepting as actual "infinite ongoing processes" (see [MW], for example).

We think that the problems related to the understanding of periodic decimals may reside in how they are defined, which is satisfactory for mathematicians professionals but not for students. In [Vin] the author writes: *The Definition represents, perhaps, more than anything else the conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concepts of acquisition [...] The teaching should take into account the common psychological process for acquisition of concepts and logical reasonings*

This is why we believe that it can be reasonable to introduce a periodic decimal representations of rational numbers that does not use infinite processes. Our approach is an attempt in this direction as it eliminates both potential and actual infinity.

Acknowledgment. *This research has been supported by Indam (Ministry of Education, Universities and Research, Italy).*

References

[ATV] Anatriello G., Tortoriello F.S., Vincenzi G. (2016). On an assumption of geometric foundation of numbers. *International Journal of Mathematical Education in Science and Technology,* 47(3), 395–407.

[AMV] Anatriello G., Martini H., Vincenzi G. (2018). A Synthetic Way to Geometrize the Method of Coordinates. *Journal for Geometry and Graphics*, (22)(1), 1–13.

[Art] Artin, M. (1991). *Algebra*. New Jersey: Prentice Hall.

[Bar] Baruk S. (1992). *Dictionnaire des Mathmatiques lmentaires.* Paris: Seuil.

[Boy] Boyer C.B. (1968). *A History of Mathematics.* NewYork: John Wiley Sons.

[BLDKPS] Buchholtz N., Leung F.K., Ding L., Kaiser G., Park K., Schwarz B. (2013). Future mathematics teachers? Professional knowledge of elementary mathematics from an advanced standpoint. *ZDM*, 45(1), 107–120.

[Cat] Catalan E. (1842). Sur les fractions dcimales priodiques. Nouvelles annales de mathmatiques. *journal des candidats aux coles polytechnique et normale*, 1, 457–470.

[Chi] Childs L.A. (1979). *Concrete Introduction to Higher Algebra.* Graduate Texts in Mathematics. New York: Springer-Verlag.

[CD] Choi Y., Do J. (2005). Equality involved in 0.999...and$(-8)^{1/3}$. *For the Learning of Mathematics*, 25(3), 13–15

[Deb] Debnath L. (2011). A brief historical development of classical mathematics before the Renaissance. *International Journal of Mathematical Education in Science and Technology,* 42(5), 625–647.

[Dic] Dickson L.E. (1918). *History of the Theory of Numbers, Volume I: Divisibility and Primality* Dover Publications. (ser. Dover books on mathematics. 2012 New York).

[DWMcB] Dubinsky E., Weller K., Mcdonald M.A., Brown A. (2005). Some historical issues and paradoxes regarding the concept of infinity: An Apos-Based analysis: Part 1. *Educational Studies in Mathematics,* 58(3), 335–359.

[Eis] Eisenmann P. (2008). Why is it not true that $0.999\cdots < 1$. *Teaching of Mathematics,* 11(1), 35–40.

[FMRR] Faltin F., Metropolis N., Ross B., Rota G.C. (1975). The real numbers as a wreath product. *Advances in Mathematics,* 16, 278–304.

[Fow] Fowler D. (1992). Dedekind's Theorem: $\sqrt{2}\cdot\sqrt{3}=\sqrt{6}$. *The American Mathematical Monthly*, 99 (8), 725–733.

[Fre] Frege G. (1969). *Nachgelassene Schriften und Wissenschaftlicher Briefwechsel* [Posthumous writing and scientific correspondence]. Hamburg: Felix Meiner Verlag.

[Hut] Hutchinson M.R. (1972). Investigating the nature of periodic decimals. *The Mathematics Teacher*, 65(4), 325–327.

[Kal] Kalapodi A. (2010). The decimal representation of real numbers. *International Journal of Mathematical Education in Science and Technology,* 41(7), 889–900.

[KK] Katz K.U., Katz M.G. (2012). Stevin numbers and reality. *Foundations of Science*, 17(2), 109–123.

[Kli] Kline M. (1972). *Mathematical Thought From Ancient to Modern Times.* Oxford: University Press.

[Mal] Malet A. (2006). Renaissance notions of number and magnitude. *Historia Mathematica*, 33(1): 63–81.

[MW] Moren L.E.A., Waldegg G., (1991). The conceptual evolution of actual mathematical infinity, *Educational Studies in Mathematics*, 2, pp. 211–231.

[PS] Petri B., Schappacher N. (2007). On arithmetization. In *The Shaping of Arithmetic after C.F. Gauss's Disquisitiones Arithmeticae*, pp. 343-374. Springer: Berlin Heidelber.

[Ric] Richman F. Is 0.999 ... = 1? (1999). *Mathematics Magazine*, 72(5), 396–400.

[Rud] Rudin W. (1976). *Principles of Mathematical Analysis.* New York: McGraw-Hill.

[Sán] Sándor J., Crstici B. (2004). The many facets of Euler's totient. In *Handbook of Number Theory II* (pp. 179-327). Netherland: Springer.

[Sie] Sierpinska A. (1994). *Understanding in Mathematics.* London: Falmer.

[Sch] Tall D., Schwarzenberger R.L.E. (1978). Conflicts in the learning of real numbers and limits. *Mathematics Teaching*, 82, 44–49.

[Vin] Vinner S. (2002). The role of definitions in the teaching and learning of mathematics. In *Advanced Mathematical Thinking.* Netherland: Springer

[YBL] Yopp. D.A., Burroughs E.A, Lindaman B.J. (2011). Why it is important for in-service elementary mathematics teachers to understand the equality $.999\ldots = 1$. *The Journal of Mathematical Behavior*, 30(4): 304–318.

[vdW] van der Waerden B.L. (1985). *A History of Algebra. From al-Khwarizmi to Emmy Noether.* Berlin: Springer-Verlag.

[Yim] Woo J., Yim J. (2008). Revisiting 0.999... and $(-8)^{1/3}$ in School Mathematics from the Perspective of the Algebraic Permanence Principle. *For the Learning of Mathematics*, 2: 11–16.

G. Anatriello, Department of Architecture, University of Napoli "Federico II", Via Monteoliveto, 3, I-80134 Napoli, Italy

Email address: anatriello@unina.it

G. Vincenzi, Department Mathematics, University of Salerno, via Giovanni Paolo II, 132, I-84084 Fisciano (SA), Italy
Email address: vincenzi@unisa.it