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Ways of knowing school mathematics

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Abstract: Three ways of knowing; associative knowing, compositional knowing and contextual knowing are theoretically explored and illustrated. The main implication for mathematics education is that communication needs to focus on students' authorship. Classrooms in India and Sweden are compared and analyzed for these three ways of knowing.

Keywords: authorship, computational view, compositional view, flexibility, knowing, school mathematics, variation theory

Introduction

Two views on learning mathematics at school, the computational and the compositional, as well as two perspectives on teaching mathematics at school, the traditional instructional and the thematic project-oriented, are discussed as a background to the results of the empirical study presented in *Flexibility in Knowing School Mathematics in the Contexts of a Swedish and an Indian School Class* (Dash 2009). The main conclusions on flexibility in knowing school mathematics are discussed.

Computational view and compositional view on learning school mathematics

International research has shown that school mathematics often is organized to learn techniques. We mainly confront our students with structured text problems, so that there are possibilities to solve these with the help of already memorized routines. This over-emphasis on syntax leads to a kind of lexical learning, which Ernest (2004) has compared to language learning from a vocabulary, which can never be successful. *Numeracy* is often interpreted as a basic skill and as a basis for learning more advanced problem-solving skills and includes recall of number facts, mental arithmetic and written computational skills (English 2002).

Numeracy is also considered an important prerequisite for all further studies of mathematics and in that way it represents a computational perspective on mathematics. According to Gray and Tall (1991), the initial learning process is not concerned with learning definitions or concepts through visual perceptions, but a matter of dealing with mathematical symbolism as a process and the underlying mathematical concept. *Procepts* are 'mental objects', combinations

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of process and concept, produced by the process, which is a 'mental' process. Gray and Tall give a number of examples of procepts (p. 2, ibid):

> 1. The process of counting and the concept of number. Number 7 is for instance both a process of counting up to the number 7 and the number produced by that counting.
>
> 2. The process of 'counting on' and 'counting all' and the concept of addition. For instance: in 5 + 4, where the process of adding units and the result 9 is part of the same "procept".
>
> 3. The process of repeated addition and its product in multiplication.
>
> 4. The process of division of whole numbers and the concept of fraction.

Gray & Tall (1994) published a study of 72 students between the ages of 7 and 13 and concluded that high achievers engage in flexible thinking more than average achievers. In comparison between the 8 age group and the 10 age group there was no increase in how many additions the students had memorized, what Gray & Tall call *known facts*. The low achievers in this group were still counting. Either they were *counting all* or *counting on* (addition strategies), or *counting backwards* in subtraction, which are the first strategies to appear when children learn to count. *Counting all* means to count all the numbers, e g in number 4 there is 1,2,3,4 and then all numbers in number 7 1,2,3,4,5,6,7 and then begin to count all from the beginning. *Counting on* means to count 1,2,3,4 and continue with 5,6,7,8,9,10,11. *Counting backwards* is also difficult. It means that if we have 13 minus 7: 13,12,11,10,9,8. Low achievers learn a different mathematics and a more difficult one. The high achievers had forgotten about these counting strategies, they had developed processes of seeing patterns, understanding ideas e g about multiplication tables, or 'decomposing and recomposing' numbers flexibly. Low achievers need to memorize more since it becomes important to remember different methods.

A compositional view on mathematics is concerned with how learners understand mathematical relationships. If the key word for the computational view on mathematics is 'best fit', indicating how cognitive units fits into each other, then the key words for compositional view on mathematics is how one 'understands part-part-relationships'. Marton & Neuman (1990) discovered that, rather than doing additions or subtractions in one's mind one unit at a time, according to 'algorithmic rules', 'sudden restructuring' is crucial for how children understand number relations. The immediate grasp of numbers found in these studies can be called intuitive 'in so far it is certainly based on an implicit perception of the whole problem' (Marton & Neuman 1990 discussed in Marton, Fensham & Chaiklin 1994, p. 470). Their conclusions were based on studies of how children deal with tasks posed as questions in an everyday context, for example:

> If you have only 2 kronor in your purse and you want to buy a comic that costs 9 kronor, how many kronor do you need? ("2+ _=9") (p. 15).

Based on their results of interviewing eighty-two 7-year old children, all new school starters in four different first grade classes in Sweden, about how they solve different tasks, which were a variation of parts- and whole relations, Marton and Neuman (1990) provide evidence suggesting that the basis for arithmetic skills are non-computational. The outcome space consisted of 12 qualitatively different ways of experiencing numbers. Numbers were experienced in eleven different ways (p. 18) and there were only 5 cases of purely computational counting in 815 responses to 11 problems. One computational counting procedure was based on *counting fingers*, understanding the ordinal meaning only, counting 2+7, beginning with two fingers and 7 fingers thereafter, after which they could count all the fingers together (p. 52). This procedure was seen in 4 cases. A procedure based on *counting numbers* was seen in one case only of double counting. A more frequent strategy was *semi-computational*, where the children counted numbers both as unit-by unit, 'incrementing or decrementing', and with a grasp of the whole as a visual pattern. In two other strategies, named *estimated numbers* and *estimated fingers*, number words are uttered; there is an attempt to hear or see the cardinality of numbers, which is partly achieved in heard numbers, and totally in finger numbers (p. 53).

Marton and Neuman conclude that the origin of arithmetic skills can be seen in the use of finger numbers and the understanding of the number relations; the oneness, twoness, threeness or possibly the fourness. They write:

> When using finger numbers, all numbers up to ten are made immediately perceptible and so are the relations between the numbers; they are not only seen *in* each other but they are also felt in a tactile and body anchored way. This capability is uniquely human and it is in our understanding the essential element in the development of arithmetic skills. Although counting is a necessary prerequisite to the acquisition of arithmetic skills, the skills themselves are fundamentally non-computational, and difficulties with mathematics seem to evolve when children are led to believe the reverse is true. (p. 53)

Some of the children gave an explanation to that the answer is 7 to the question 2+ _ = 9: "Because I know that 9 - 2 = 7". The interpretation of this and other statements about similar problems, was that 'the children understand numbers in terms of other numbers, of which they are composite parts, or which are composite parts of them' (p. 35). Number 9 is composed of two parts of which one is given. The solution procedure is guided by the actual numbers, and *not by the type of arithmetic operation* (Marton & Neuman 1990, p. 47, my italics).

The conclusions made from the mentioned studies based on the two views, computational and compositional, are helpful to the discussion on flexibility in knowing (Dash 2009). Traditional teaching is often based on a computational view, but other mathematics educators present mathematics with open-ended questions, which demand bringing in external information, deep analysis or experimental investigations and a compositional perspective. Underlying these different pedagogical approaches are different philosophies of mathematics according to Ernest (1991), viz., the absolutist and the fallibilist views. The main difference between these is that the absolutist view contributes to an objective and neutral perspective whereas the fallibilist view acknowledges human dialogue and development of mathematics through history. More recent views are also found that are subsumed within a humanistic philosophy of mathematics (Sriraman, 2017).

Boaler's often cited longitudinal studies in two different schools (2002) showed these different pedagogic approaches' impacts on student learning. In one of them, the Phoenix Park School, the class 9 students were taught mathematics according to a project-based approach and the other class 9 at Amber Hill School, mathematics was taught according to a so called traditional approach. The project based approach meant that students were assigned projects which comprised of open-ended tasks that needed mathematical methods. The traditional approach meant that students were shown methods and then practiced them. The exam results showed that students from Phoenix Park succeeded better than those from Amber Hill. A reason for this was that the students had gained flexible tools to solve problems. As an example of how the students described their math approach Lindsay said (p.58):

> Well if you find a rule or a method, you try and adapt it to other things. When we found this rule that worked with the circles we started to work out the percentages and then adapted it, so we just took it further and took different steps and tried to adapt it to new situations.

Boaler concluded that the traditional approach was the less successful. As Lindsay gave voice to, research has shown that flexible ways to use aspects of mathematical models lead to more successful ways of solving problems.

In a study conducted by Dash (2009) a classroom in India was observed where mathematics was represented in a, what can be called, traditional teaching environment. The teacher, mentioned as Teacher A in the text, used a *unison dialogue*. The dialogue followed a certain structure, simplistically described as teacher talk, teacher questions and students' answers in unison. The underlying goal, conceived by the students, was to make them understand the *direction* of how to solve the problem. The emphasis on the direction is the essence of this method. The direction was important in two ways. When the students were given a mathematics problem, they perceived that they should evaluate their solution according to how it appeared from beginning to the end. They would also focus on if there were any parts missing in the totality of the solution. There is a big difference between this focus and a focus on stepwise solutions. If the steps are focused, then there is a fragmented view on the solution and the bigger picture is not seen. The students could in many cases re-capture the entire solution, but at the same time they had an intuitive feeling that they hadn't understood it fully.

Both the problem solving method used by the teacher and described in the textbook as well as the learning approaches, were characterized by the intention to use repetition. The repetition method used entailed writing and revising the entire solution, with different focus on different aspects each time. This method led most of the students to conclusions about how different aspects, of for instance a mathematical relationship, could be formulated as different mathematical problems. These concluding variations on tasks were by the students called opposite questions (in English, not in the mother language Odiya), but the qualities of these are more accurately caught in the expression *reverse problems*.

The students had probably derived this explicit expression from the textbook Geometry and Application (p. 69) where a few lines indicates that there are opposite variations on Pythagoras' theorem:

For an example in a triangle ABC the angle B is the right angle, then $AC^2= AB^2+ BC^2$
Any *opposite* of this example is also true.

I asked N about opposite questions. In the following interview extract N has just mentioned that she is in the habit of making, what she and many students at the school call, 'opposite questions'.

ID: How do you make these 'opposite-questions' when you practice for exams?
N: There is a question about *this* side (N points to one side of the right-angled triangle in her textbook) and *that* side is given, (N points to another side of the right-angled triangle), then I can take the formula $a^2 = b^2 + c^2$ and then it comes out! But it can be that *this* side and *this* side that is given, then *this* and *this* side are *related to* this side. Then it comes out. These are opposite questions.

In this extract, N pointed out that two opposite questions can be posed, given Pythagoras' theorem. She used the word for 'related', *samanthara* and that indicates that it is understood as a mathematical relationship between the sides. Later during the interview, N told me that there are three possible variants of questions that can be asked from this theorem. When N talks about the 'opposite questions' she uses the expression *saman*, which in Oriya means 'equal to', to describe the relationship between a, b and c in the theorem. N also uses another expression, *bhariba*, for what comes out as a result of a calculation. These are two different ways to express what lies behind an equal sign. The language use and the understanding that three questions can emerge from this composition, can be seen as an indication that N understands Pythagoras' theorem as a relationship between the sides a, b and c. She sees the theorem as three different relationships, depending on if side a, b or c is focused, hence, the term *reverse problems* grasps the sense of mirroring aspects in a mathematical relationship so that it creates a variation of the problems that can be posed. That this is crucial for any kind of learning is by Booth, Wistedt, Halldén, Martinsson and Marton (1999) concluded in that:

> Encountering variation of one sort or another can bring a person to see new dimensions of potential in a phenomenon that were previously taken for granted, and this spying new aspects of a phenomenon is fundamental to learning (p. 73).

And that's exactly what the majority of the students did. Through repetition, they saw different aspects against the background of the whole and changed the view with every repetition, just like looking at individual jigsaw puzzle pieces to see how every part is related to the whole picture. Although the teaching method at a first glance seemed to have characteristics of a traditional approach, the aim of the presentations were interpreted in ways that encouraged the students to learn by understanding parts-whole relationships instead of memorizing bits of information- a compositional view on mathematics.

Learning through variation

The learning approaches in the classroom context were based on repetition as a method. The flexibility in knowing lies in how well the parts discerned by the learner were experienced to be a 'match', what in-coherences can be observed, and how the variation in a material is experienced and used. The students interviewed made reflected guesses on which formula could be used. All of the students talked about the importance of repetition, and explained what they did when they did repetition. Sa explains (ID stands for interviewer and Sa is the student):

> ID: How do you learn mathematics?
> Sa: It is given in the books. One has to practice.
> ID: How do you memorise all?
> Sa: In the theorem you mean?
> ID: Yes
> Sa: First we learn words, then sentences, in class one we learn ah, kha, gha (letters of the Oriya alphabet) After many times' repetition, one can learn the theorem. Mathematics is to understand. One doesn't need to memorise much. One should not just 'massage it in' (learn by rote). If one tries to remember, for instance, that ABC is equal to ADC, or that AB is equal to DC, if one writes that on an exam, and that BD is the common side of the two triangles, and understands exactly, then one doesn't need to remember anything.
> The premise, the mathematics and the proof is needed.
> ID: How many times do you have to write?
> Sa: This one doesn't need to learn by heart. (she reads the task aloud again)
> One has to know what is demanded in the task, what one should do. One has to understand the proof and then one understands.

All students stated that understanding comes first, and then memorization. A few students also felt that in order to be able to really 'make a picture' about the problem so that they could 'see' whether it all came out right, they have to practice and do the problem several times, at least two, three times 'Doing tasks' and repetition was crucial to understanding the mathematical

logic. The understanding these students aimed at was concerned with *how to solve* the present mathematical problem. Most of the students compared how the formula had been used in one problem, and tried to figure out how to use the formula in another way in a new problem. In a similar fashion, some students visualised and 'saw' if the task had been done correctly or not.
In this study the traditional teaching method with elements of unison dialogue and text book presentation of variations on mathematical relationships as a starting point, was experienced by the students as ways to figure out how mathematics is used to solve problems.

Empirical results were also found in observations of what we can call project-based teaching in a Swedish school class context (2009 ibid). Teacher M often gave the students questions in order to promote understanding. She calls these questions 'steps in-between', and argued that such steps make the students take a bigger leap towards understanding.

> Two students were working with the task 1/5 + 1/15.
>
> M asked: Which one is biggest?
>
> S: 1/5.
>
> M placed the two fraction-sticks, with the proportional lengths 1/5 and 1/15, on the table in front of the students. The students placed the fraction-stick 1/5 over the fraction-stick 1/15. They counted 1, 2, 3, and proceeded to write and talk about the solution.

Because the practical materials were available, in this case the fraction sticks, some questions were posed by moving the material around. Teacher M encouraged discussions in connection with the production of and work with concrete material, helped students visualize mathematical concepts, and worked with documentation of thematic project work. She said that it is important that students are free to express themselves, and see that mathematics can be negotiated and the meaning agreed upon: 'Precisely this thing about daring to make hypotheses, and daring to test (…)so they formulate hypotheses and then they have created their own problems, then they have carried them out and reported their results, and then at the end we have made a joint summary, always so that the children get everything we have written, so we should agree then, so that I feel it is a good point of departure, and that it is what we also do in maths, with joint discussions, and that does develop language and feeling free and open-minded' (Dash 2009, p. 135)

Teacher M used concrete materials when the students were beginning to work with a mathematical idea, and then proceeded to encourage the students to try to make a picture of what they knew:

> Teacher M: Those of you who don't have 12th-parts, can maybe figure
>
> out how to do?
>
> T: I know what to do!

Teacher M: What do you think he has done? He has folded 6th-parts.

Can one think without doing it?

Teacher M stressed that it is important that the students try to express their thoughts verbally. Sometimes she helps them with words, and asks them if she has understood what they meant. It appears from her teaching methods that Teacher M thinks she can get information about the students' thoughts and a good assessment of their knowledge in classroom talk. The theme documentation is another assessment method. A third method for assessment is diagnostic tests on basic arithmetics.

Here follows an extract of a lesson on fractions, which nicely illustrates how Teacher M gives the students a picture, so that they can understand the meaning of fractions. The extract (Dash 2009, p.136-137) also shows how Teacher M works with narration and negotiation (P stands for different students, if not otherwise mentioned).

Teacher M writes on the whiteboard: 2/6; 4/9; 3/5; 2/4

Teacher M: What can you tell about these fractions?

How can one re-write them so that it is easy to compare them?

P: One can divide into twos.

Teacher M: How?

P: Shadow 2 squares out of four.

M draws a quadrangle with two shadowed parts out of four parts and next to it a similar quadrangle which is divided into two parts, whereof one part is shadowed:

Teacher M: Are these two different?

P: No, I think they are the same.

P: I think it is equally right.

P: It is the same as half.

Teacher M: Then we can put an equal sign between them, can't we?

If I compare these two figures, I think they are the same. The shadowed

space is same. Can you compare them?

Teacher M makes a pause here and all students sit in silence and look at the board, as if they were trying to make a picture in their minds.

Teacher M: Now, we'll do like this. I want to explain. If we do like this, then what do you think, if you only look at this picture?

Teacher M draws a figure of a pizza with 3 shadowed pizza slices out of a total of 8 slices.

Teacher M: Now I pose my question, because I want to 'trick' you a

little. What is 8 minus 3/8?

P: 5/8 of 8 are left.

Teacher M: Oh, you haven't eaten them all, have you?

P (same student again): It is 5/8 that is not shadowed.

Teacher M: As it says in the book, '1-3/8'. What does that mean?

P: What?

Teacher M: Yes, it says like that in the book. What does 1-3/8 mean?

P: One should take 1/8 away so that 2/8 are left.

My reflection: This student is perhaps thinking that one can take a first step towards the solution, seeing that 2/8 are equal to 1/4. 2/4 was illustrated and dealt with at the beginning of the extract. But Teacher M returns to the original meaning of the task and rephrases it

Teacher M: One whole and 3/8, yes, I can see that…

Teacher M: Now, we have got to have a picture in our mind otherwise

we only have numbers.

What stories do you have?

P: One has a cake and then one eats 3/8

P: If I go to the bakery and want to buy cake slices and there are only

5/8 left of a whole cake.

Teacher M: Write and draw on the back (of the paper). Do it quickly. It

has to be about reality, and this is important for you, so that we can play

the (fraction) game afterwards.

Teacher M walks around and looks at the students' work. She asks them: 'What stories do you have?'

During the interviews the students in general talked about *what they did*, about calculations, and about steps. Despite this focus on procedure in their explanations, the resistance to the absolutist view on mathematics that they experienced with their previous teacher is strong. All the students talked about how meaningful their documentations on their thematic project works were and that Teacher M:s way of teaching is based on that they have to think and take responsibility for their solutions, build up argumentations for their problem-solving and validate their conclusions in collaboration with other students. The students said that they 'have to think more' since M became their teacher. Most of the students mentioned that it is important to think in 'intermediary steps' (mellanled) until 'one knows it'. They also felt that discussions in class, talking about the steps between, stating the problem and reaching the solution, are ways to practice so that one can gain 'real knowledge'.

Teacher M: Does anybody know what T does when he doesn't have 12th parts?
B: He cuts 6th parts into halves.
B: How does one make 85th parts, then?
Teacher M: How do you reason about that, B?
B sits for a while in silence. Then he brings a calculator, a ruler and a paper and tries to solve it.
By the end of the lesson teacher M says:
Teacher M: 'B' was thinking about how to make an 85th part. How do you think he did it?
P: He measured with a ruler and divided into 85 parts.
P: they must have become small.
P: He perhaps took this side of the paper (points to the longer side of a blank paper).
Teacher M: How did you do it, B?
B: I did like that, but I took a paper with squares (cm^2-squares). I took 22 squares and divided each square into 4. Then I didn't get exactly 85.
Teacher M: How can we get exactly 85?
Silence.
Teacher M: would you like to hear my suggestion?
All students: Yes.
Teacher M: I took 5 rows of eleven squares and 3 rows of ten squares.
Here, teacher M is part of the collaborative authoring. The next extract illustrates how the students' authorship is both collaborative and independent.

M has given a question on a paper, which says: 'How much is 1/5 minus 2/15?'
Teacher M asks A and T: 'Can you make 5th-parts into 15th-parts?'
Students A counts aloud 1, 2, 3 and 4, 5, 6 and 7, 8, 9 and 10, 11, 12 and 13,
14, 15.
T: Then we have got 3, 3, 3, 3 and 3.
A writes 3/12, erases and writes: 3/15 – 2/15 = 1/15
A: Yes, now we are finished. T agrees.
My reflection: 1/5 corresponds to 3/15. They seem to work with the idea that 1/15
should be divided into groups of threes to arrive at 1/5. They are actually working
with the lesser known denominator, without formally writing it down.

What variation did the students see and use?

It was noticed that variation was used in the overall learning approaches. In the Indian class context the variation was important to how the students studied mathematics, by using the variation to understand mathematical relationships and how to use mathematics to solve problems through repetition. In the Swedish class context the variation that was focused was the varied ways of seeing problems and how to reach a solution through intermediary questions. There were common features shared in these ways of knowing, although the classroom context and the understanding of the teachers' pedagogic approaches were different.

In dealing with a mathematical problem, the variation represented within the mathematical task is crucial to what possibility the learner has to experience variation. The varied aspects focused by the learner and the constancy of the unfocused aspects are simultaneously experienced. Marton, Runesson & Tsui (2004) discusses Voigt's example of how two students, Jack and Jamie, went about solving a group of arithmetic tasks (Voigt 1995, pp. 173-174, quoted in Marton, Runesson & Tsui 2004 (p. 36):

1. 50-9=41

2. 60-9=51

3. 60-19_41

4. 41+19=60

5. 31+29=60

6. 31+19=50

7. 32+18=_

Two students were working with seven tasks. They saw a pattern of how the numbers in the given tasks were related to each other through, what Voigt referred to as *negotiation of mathematical meaning*. Voigt concluded that a new strategy was developed through interaction between the students. Marton, Runesson and Tsui agree that interaction undoubtedly influences the strategy in the case described by Voigt, but argue that other elements are also at hand. Marton, Runesson and Tsui suggest that within the given tasks, it is possible to experience a certain structural variation. Voigt's study has been explored in more detail from the point of view of variation theory by Runesson (1999a), who points out that in the series of task something is kept invariant and something is varied. The three first tasks are all concerned with subtraction. The second number in the task ends with 9, but the number of tens are varied. Between task number 1 and 2, only the ten is changed (50 and 60) and between 2 and 3, the second term is changed (from 9 to 19). Runesson also observed that in the following three tasks, the arithmetical operation is addition, and the numbers end with 1 and 9. Between tasks 4 and 5 the number of tens in the first and second figure changes (from 41 to 31, and 19 to 29, respectively). The sum of these two additions, however, is the same: 60. Task number 6 contains and combines numbers from tasks 4 and 5. The first number is the same as the first number in task 5, while the second number is the same as the second number in task 4. In task number 7, the number 31 increases with 1 and makes 32, while 19 decreases with 1 and makes 18. Runesson concludes that there is a variation in the arithmetical operations presented in the tasks (subtraction and addition), as well as in the numbers involved in the operations. She looks more carefully at how Jack and Jamie solved the last task (32 + 18), and notices a difference in how the students solved task 5, where they added units and tens separately and how they understood 32 and 18 in relation to other numbers. Jamie saw that the sum in task 6 and task 7 are the same: 50. He can understand the relation between the parts and wholes within and between the tasks, and notices that one number increases with 1 and the other number decreases with 1, which means that the difference is zero (pp. 74-76). Runesson draws two important conclusions (p. 76). First, the change in strategy between tasks 5 and 7 can be described as a result of a change in the students' ways of experiencing the variation. Secondly, the change in the ways of experiencing was related to the character of the group of tasks the students were working with.

Four different patterns of variation in how we experience something in relation to something else are described by Marton, Runesson & Tsui (2004):

- Contrast
- Generalisation
- Separation
- Fusion

The aspects we take for granted and the aspects which we focus upon make up the relation to the whole experienced phenomenon. *Contrast* is the simplest experience of variation. As an example, when contrast is experienced, the individual can understand what 'three' means in relation to 'two' and 'four'. *Generalisation* is when we understand that 'three' has different appearances, for instance three apples or three honey bees. There are other characterizations of generalization in the context of seeing mathematical structures (e.g., Sriraman, 2004). *Separation* means separating variable aspects from invariable, as in the example above, when Jamie saw that there were some aspects which were invariable, like the arithmetical operation, while some aspects were variable; a number increased. Fusion means that many different aspects are simultaneously focused. The fusion allows different cases to be separated within the same understanding. When Jamie understood, in a simultaneous way, that between tasks 6 and 7 one number was increased by 1 and the other decreased by 1, he understood that this implied that the sums would be 50 in both tasks. The forms of variation mentioned by Marton et al. can also be seen as examples of different ways to organise experiencing of certain phenomena in mathematics education.

Ways of knowing

Three different ways of knowing school mathematics were the outcome of the qualitative analysis of the field material obtained from two school class contexts described in Dash (2009).These ways of knowing were related to how the students approaches their mathematical learning (authorship), their learner identities and the mathematical content, because of that it was important to try to capture the situation and the context of learning so that it would make the ways of knowing more comprehensible. The ways of knowing were not looked upon in isolation from the context, and at the same time they were not viewed as part of inherent qualities of particular students. This perspective is relational. The general meaning associated to flexibility in knowing (2009) was to "see, deal with, assess and to understand which aspects are critical in the process of delimiting parts and wholes, and understanding their relations. It also means to be on the look-out for new aspects within changing wholes" (p.105). In other words, that which belongs to a compositional view on school mathematics.

The three ways of knowing were:

Associative flexible experiencing, also called here ***Associative knowing***

Compositional flexible experiencing, also called here ***Compositional knowing***

Contextual flexible experiencing, also called here ***Contextual knowing***

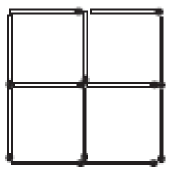
In the following I describe what qualities were attributed to the different ways of knowing.

Associative flexible experiencing or *Associative knowing*

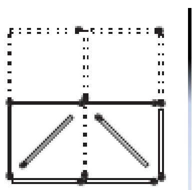
Associative knowing means that a learner approaches her/his learning situation so that there is an experienced *difference between arbitrarily* discerned aspects in a mathematical problem. This difference makes the aspects interesting and the focus. The problem-solving strategy is only constituted by what the learner perceives to be relevant from this particular criteria; difference. So the aspects are arbitrarily chosen and the learner is convinced that the difference has to be a key to the solution of the problem. The different aspects are discerned by this association to difference. But there is no consistency in the reasoning and therefore the flexible quality of how the learner experiences the learning situation is characterized by that 'any' other aspect that are contrasting can be focused. The strategy is based on seeing *contrast* (Marton, Runesson & Tsui 2004). *Contrast* can be experienced between two numbers; e.g. 2 and 3. There is a narrowing of focus to cues that can be found in the immediate task, or from the teacher's presentation/discussion. The way of only seeing contrast makes understanding of content undifferentiated. Since no invariant aspects are distinguished from variant aspects, all variation is perceived as equally arbitrary. Contrast is the simplest experience of variation. As an example, when contrast is experienced, the individual can understand what 'three' means in relation to 'two' and 'four'.

One exemplar of *associative flexible experiencing* is illustrated in the following extract, from an interview with D, from the Indian school class.

We look at D:s arbitrary shifts in how she combines numbers in arbitrary ways to produce self-invented formulas. Here I have posed the question to calculate the area of one small square, given that the length of one matchstick is 2 cm (the length of the side in a small square is 2 cm):



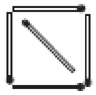
Then the following figure was placed with matchsticks:



Here, student D is asked about the area of an isosceles triangle inscribed as two diagonals in a rectangle with the length 4 cm. (The length of one match-stick is 2 cm. Note that two of the matchsticks that were placed to make two sides of the triangle were a piece of a broken matchstick). There was a possibility for the students to see that the inscribed triangle's area is equal to the area of two small squares in the first example, or half the area of the rectangle.

ID: How large is the area of the inscribed isosceles triangle?

She writes and speaks aloud while writing:
D: The area of the quadrangle is 3 into 2 is equal to 6 centimetres ('into' here means 'multiplied by').
My reflection: Does she mean that the length is 3 cm and the width 2 cm? The length is actually 4 cm. Two match-sticks long.
D: The area of the triangle is 4 cube, is equal to 4 into 4 into 4 is equal to 64.
D writes:
4^3=4x4x4=64 (Ans)
My reflection: What does she mean by 4 to the power of 3, or as D says, 4 cube? Does she experience the relation between the 4 and the cube as a relationship between side and area? In this case, the length, which is 4 (cm) is cubed. Or does she merely manipulate with the numbers to get 4 and 3? In that case 4 and 3 can mean anything or maybe nothing at all.
I place the following figure in front of us and ask how about the total area of the two triangles (the length of one match-stick is 2 cm).



D writes and at the same time as she says:
D: The triangle's area is: 64/2 =32
D: The triangle's area is 32

Here, D sees that the area calculated before must be double the size if the two triangles' area. So, she perhaps was calculating the rectangle's area at first, although it was cubed rather than squared. D speaks with certitude. She must believe that she is doing mathematics, but these 'self-invented' strategies do not have a consistent inner logic. She did not see first that the isosceles triangle's area and the area of the quadrangle were related to each other and could have been calculated with ease, if D had understood the concept 'area'. This lack of understanding she compensated with manipulation of numbers and formalization into algebra. The consequence of her first attempt to formalize the calculation of the area of the triangle is that she starts with a numerically wrong area, although the strategy of dividing the area of the right-angled triangle into 2 to get the area of the two triangles in the small square is correct. The problem is that she most arbitrarily chooses the answer she got from her calculation of the area of the isosceles triangle and divides it into two.
We cannot know how she reasoned when she did these calculations, or what her conception of area is. The flexibility is here characterized by a high degree of discontinuity in focus. We can see that although D writes and speaks with a trust in her ability, she uses tools which are not useful in order to solve the area-problems. Her concern is mostly with manipulation with numbers and denotations. These tools seem to be experienced as external to her authoring. Her knowing is dependent of others' authoring.
The associative way of knowing shows us that the teacher needs to promote an active engagement in mathematics, with the main focus on mathematical authoring. There is a need to constantly ask questions like: Why did you choose to do like that? What was your plan? What is important to think about when solving this problem? What is the problem about? Have you solved the problem?

Compositional flexible experiencing (*compositional knowing*) and contextual flexible experiencing (*contextual knowing*)

When a learner experiences simultaneously structural and referential aspects of variation (Marton & Booth 1997), it leads to an increasingly differentiated understanding. This is what characterises both *compositional* and *contextual flexible experiencing*. When the learner uses this experienced variation, it can be used on the basis of an experienced relevance within the mathematical logical content, as in compositional flexible experiencing, or of an experienced contextual relevance, as in contextual flexible experiencing.

The *compositions*, the combining of different parts to make a whole, are sometimes 'view-turned'. Here, I use Ahlberg's (2004) definition of *view-turns* (*synvändor*): 'what people experience constitutes changes in ways of experiencing something'. In compositional flexible experiencing, the change in how the students experience a whole is based on which *compositions* they discern within the mathematical problem. 'Compositions' are here taken to be any relatively stable set of relationships between aspects that the students discern, and which are used as an argument in their reasoning. By 'stable' I here mean that there is a certain measure of consistency in the student's explanations. The following examples represent compositional knowing in that it shows that the mathematical content is treated in both original ways and in interconnecting ways. Here is an extract from an interview with S in the Swedish study.

ID: Sometimes M says 'Now you should discover something'. What does she mean by 'discover'?

S: Yes, well, 'discover' one should perhaps do to get it easier and to see the system, the pattern.

ID: Can you remember a pattern right now?

S: Eh, well, no.

ID: Is there something else one can 'discover', something you remember, where you learnt something? When you felt that you learnt something?

S: Yes in grade four, when we had multiplication table. Nine times six and nine times seven, the difference is nine between them.

ID: Yes, that is a pattern. Do you remember when you worked with factorisation?

S: Yes, there was an example of 3 times 3, times 3. 3 times 3 is 9. And 9 times 3 is 27.

ID: Do you remember when you have worked with this, when M didn't talk about factorisation?

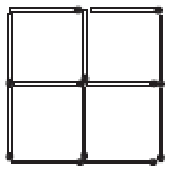
S: It was when we were working with cubes. Length times width times height. If the length was 5, width 2 and height is 2, then it becomes 20.

When student S says that she recognises the pattern used in a multiplication in the formula to calculate length · width · height, she has observed the mathematical composition. A contextual knowing means that she would have additionally understood differences in volumes when the three sides in a three-dimensional figure vary (are multiplied). We cannot from the interview extract say if she has understood how the mathematical figure looks if the lengths of the three sides vary.

The next exemplar is taken from an extract from the same interview with D, from the Indian study, mentioned earlier. We can see that the focus is on formulas and what is given in the example.

I have placed four squares with match-sticks. These squares can be seen as inscribed in a bigger square. (The small square's side is one match-stick long, i.e. 2 cm).

ID: How large is the area of the big square?



D writes: The side= a^2= 2·2= 4 4x2=8 8x2=16

D: We know that one side is 2 cm. And in the sutra (formula) it says that a side is squared. a^2, a^2, a^2, a^2 becomes $4a^2$.

My reflection: D wrote that the side is equal to a^2. It should be the area that can be expressed as a^2. The length of the side is a. But listening to D, she seems to have an idea of that area can be expressed as a squared side. She also points to each of the squares and counts a^2, a^2, a^2, a^2. She gets 4 a^2, four squares. A nice transition between the concrete squares and the algebraic expression, as well as an understanding of what the area of squares represents.

ID: If the side is 3 cm instead, how large is the area then?

D: Then I take 3x4. D writes: 3x4=12

My reflection: Now she might have confused the side and the area? 3 should have been squared to be consistent with previous reasoning.

ID: If the square with the side 2 cm was a side in a cube, how large would the volume be?

D writes: a^3= 2^3 =2x2x2= 8

D understands the algebraic formalisation of area and volume. The flexibility lies in that she can see the algebraic relation between the length of one side and the area or volume. She could not, however, deal with a change in the length of the side in the square and the transition to area. When D works with 3 cm as the given length of the side in the square, she all of a sudden makes a break in her consistent understanding. The length of the side is perhaps confused with the area. This is a good example of what happens when the compositions are focused and separated from context. D authors her knowledge with the help of algebraic expressions. By contrast, in contextual knowing, mathematics is connected to *meanings in different contexts*, and there is an understanding that the meaning varies with the context in which mathematics is used.

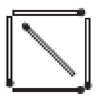
Contextual flexible experiencing is based on comparing and contrasting with other problems, an understanding of what the particular problem-situation requires, and what the mathematical principle or relation means, so that different aspects can be *fused* (Marton, Runesson & Tsui 2004) in relation to the specific context of the problem.

Al in the Swedish class had written a math story to the numbers 1 - 1/5.

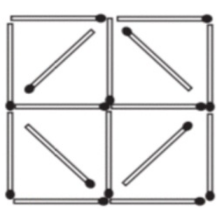
ID: How did you solve this problem with a story?

Al: I got 40. Because I had 50 candies from the beginning and then they ate 1/5. 40 were left. Here Al put the notion of ‘one whole’ into context. She could see the number ‘1’ corresponding to a visualised whole, represented by 50 candies. Al understands a whole in terms of 1 as a unit of 50 candies. She shifts between the number one, and unit 1 represented by the 50 candies. After eating 1/5 of the whole, 40 candies were left. In contextual knowing, aspects are contextually discerned and fused. Al derives from the number 1 the understanding that 1 is one whole represented by 50 candies. With this understanding it would perhaps not be difficult to solve 5- 1/5. The understanding that a ‘whole’ can correspond to many sub-items - in this case, candies - and be represented by the number, is an example of fusion.

R in the Indian class calculated the area of one right- angled triangle.



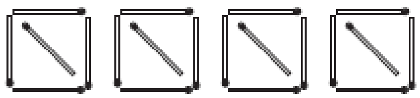
She stated that ½ multiplied by base multiplied by height is the area of a triangle. The area of one triangle was 2 cm^2 and that 8 multiplied by 2 (there were 8 triangles in total marked in a bigger square, see below) and got 16 cm^2.



For the next question about the total area of all the squares (above) she started to calculate the area of one triangle and multiplied that with 2 to arrive at the area of one square; 4 cm^2. Thereafter she multiplied with 8, which was the number of triangles. When she corrected her solution by the end of the interview, she multiplied with 4, the number of squares, and got the right answer, 16 cm^2.

In the beginning of her calculation she has not distinguished the area of one square from the area of one triangle. Her answer becomes double the area, since she counts with 8 triangles, instead of 4 squares. She knows that the area of one square is double the area of one triangle and makes corrections after being asked by ID why there is a difference in the total area - 16 cm^2 and 32 cm^2. R speaks with conviction and goes back to the tasks and explains in detail how she calculated. She finds her own mistakes.

When four squares were placed in a row, with each side one matchstick long (2 cm), R explains that 32 cm² is not really wrong.



ID: What is the total area of the squares?

R: 32 square centimetres.

R: This is right, because it can be like that.

ID: Can you make me understand?

R: If it is tables in a row. If there are tables in a room, then they 'take up more space' than if the tables are put in a square.

Discussion

The exemplar of *associative knowing* only demonstrated work with mathematical language at the *level of words*. Mathematical denotations were not translated into what they correspond to, but used in an arbitrary way.
Compositional knowing and contextual knowing means that the learner has distinguished parts and wholes and how these are related. It follows that these parts can be focused in view of the whole. When different parts (aspects) are focused in the light of a whole there is flexibility that can make up an understanding about the compositions or the context.
The exemplars from *compositional knowing* illustrated understanding of the *meaning structure*. Critical aspects were found for solving the mathematical task. In compositional knowing, mathematical principles or relations are seen as compositions. The strategy of view-turning can give good results, but in order to know in what way the compositions are understood mathematically there is a need to express about these.

Finally, the exemplars described in the category *contextual knowing* moved beyond the text/words and the structure. The *relevance structure*, based on experiences of what the learning situation demanded (Marton & Booth 1997) was focused. There is an understanding of a possible contextualisation to the mathematical task that the students are working with. R:s explanation of how she concluded that the area of the four squares in a row are 32 cm² (double the area of the sum of the squares) can only be discovered in dialogue.

Most importantly, these examples of ways of knowing show us that working with part-whole relations means discerning critical aspects of compositions and viewing them from a vantage point of a whole, a mathematical relation or a context.

It also can be seen that if these ways of seeing (knowing) are encouraged to be expressed in the mathematics classroom it paves way for the students to shape their mathematical authorship. Then the 'what and the 'who' of knowing can be developed continually in relationship to the learner's discursive positioning. In the process of authoring the students can confirm to themselves the validity of mathematics, rather than relying on the teachers to tell them so.

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