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Clockface polygons and the collective joy of making mathematics together

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Abstract: The social and embodied nature at the heart of all knowing, doing, and learning contrasts with the images that pervade our cultural imagination of mathematical work as a solitary, cognitive activity. This article describes a playful experiment by the author group to *do collective mathematics*, in an extended effort to construct alternative images, instincts, and practices for ourselves. We present a pair of episodes of mathematical exploration that come from our work together and that we have seen as an early success, intimating features of a stabilized collective mathematics that we hope to continue pursuing. Coming from a single investigation of our group, these episodes offer narrative accounts of the parallel inquiries of subgroups, working to define and characterize a mathematical space we had collectively identified, and then to formulate and investigate conjectures about that space. The narratives are followed by a discussion of themes within and across them and reflections on their significance as a step toward self-organized collective mathematics.

Keywords: Collective mathematics; collaboration; problem-finding; problem-posing; problem-solving

Introduction

Mathematics is often treated as an “individual sport.” A strong image persists in our cultural imagination of the solitary mathematical genius, engaging with abstract, disembodied entities in Platonic realms. This image persists in spite of strong empirical evidence that professional mathematical scientists work in teams and invoke body-based resources when tackling difficult problems (Gonzales, Jacoby, & Ochs, 1994; Ochs, Gonzales, & Jacoby 1996); that the basic conceptual infrastructure of mathematical thinking, like all thinking, implicates the body and lived social experience (Frejd & Bergsten, 2016; Lakoff & Nuñez 2000), and that learners and teachers think through communicating socially (cf., Sfard, 2007) and through embodied acts that provoke, accompany, or even precede learning (Goldin-Meadow, 2003; Cook, Mitchell, & Goldin-Meadow, 2010).

Historians and philosophers of mathematics have also made the compelling case that disciplinary and representation-embedded thinking is social, at least in the sense that mathematical thinkers must generate a “dance” of disciplinary agency (Pickering, 1995) in their work, simulating the presence and response of the field and of the subject matter itself. In this way, mathematical

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actions and conjectures have features in common with fundamental aspects of other collective “forms of life” (Wittgenstein, 1953). In particular, mathematical actions are *human* actions and inherently social, sharing the “dialogic” nature of all “utterances” (cf, Bakhtin, 1982) in that they are posed in anticipation of a response - in this case, a disciplinary response. Analysis of the historical record substantiates this conception of mathematical activity as a speculative, dialogic exploration of the potential of models and representation systems (MacLeod & Nersessian, 2018; Gooding, 1990), and theories of learners’ activity increasingly recognize this pattern of “co-action” (Moreno-Armella & Hegedus, 2009; Moreno-Armella & Brady, 2018), particularly where technology can support and illuminate processes of learning in interaction.

Moreover, collective cognition and socially distributed inquiry has begun to affect the working practices of professional mathematicians in potentially transformative ways. Fields Medal winner Timothy Gowers posed the question of the feasibility of what he described as “Massively Collaborative Mathematics” in the journal, *Nature* (Gowers & Nielsen, 2009), and in his widely-read blog. The question was ambitious: could a collective mathematical entity prove new theorems, where individuals had failed in the past? Since then, thirteen “Polymath” projects have run, involving medium and large groups, sometimes including non-professional mathematicians and hobbyist participants, have sprung up in direct connection to Gowers’s blog alone (Cranshaw & Kittur, 2011). Each of these projects has followed a pattern where a number of topics are proposed, and discussed in an open forum on Gowers’s blog. When a critical mass of speculative comments gathers, a date is set, and a concentrated collective effort begins (Cranshaw & Kittur, 2011). This approach has yielded published papers and proved theorems. (e.g., Polymath, 2014). Polymath is thus not just a speculative experiment: it is an emerging new way of doing mathematics. Moreover, the connections between Polymath activities and the recent advances on the chromatic number of the plane by amateur mathematician Aubrey de Grey (de Grey, 2018) suggest that collective modalities may also open the way to broader participation in mathematical work.

Given the success of the Polymath Projects and the strong and increasing emphasis on the social dimensions of mathematics thinking and learning in education research, then, we are led to ask whether there are ways to bring these approaches into classroom-based learning. As participants in Polymath have learned, there are dispositions, skills and expectations that favor productive, creative collaboration; and students in our classrooms may benefit greatly from developing those skills and that experience. The potential impact of integrating collective mathematics learning into students’ experience on the image of mathematics and of mathematicians is a desirable and democratic one; and the potential cost of not doing so could be large. On the upside, collaborating groups have the potential to develop new forms of emergent, collective intelligence (cf., Lévy, 1999; McGonigal, 2008). On the downside, as futurist Howard Rheingold (2002) put it, the tools and practices of collectivity have the potential to introduce a new digital divide, separating “those who know how to use new media to band together from those who don’t” (xix). While Rheingold was describing “smart mobs” using early social media to coordinate movements and actions, the polymath projects can also be seen as indicating a new approach for using coordination and collectivity to enter into discursive fields in powerful new ways. If only some of our students have experiences that prepare them to participate effectively in such collective action, the set of “those who don’t” may come to be defined along lines that correlate highly with race, gender, or socioeconomic status.

In this paper we describe episodes from a social and mathematical experiment being conducted at Vanderbilt University, in which a small group of participants have been exploring the possibility of collective mathematical inquiry in our own experience. We begin by giving an overview of our group's history to date, along with an indication of some of its activity and the conceptual terrain it has explored. In the main body of the paper, we provide a narrative account of the group's ability to do the important early-stage mathematical work of identifying a rich conceptual domain, articulating questions that afford mathematical articulation and exploration, developing notation, and formulating and proving conjectures. This example episode also shows the potential of a group to shape and guide its own inquiry, with minimal facilitation. Finally, we close the paper with some reflections on the implications for mathematics education and the creative mathematical lives of mathematics enthusiasts who are not professional mathematicians.

History of the Group

Our group began in January of 2017 as a Math Club, responding to a desire by students in our graduate education programs to maintain or re-establish contact with their mathematical histories. From the inception of the Club, a key objective was to create bridges between our own creative mathematical experiences and our lives as education researchers. To that end, we began with by considering the text, *Roots to Research* (Sally & Sally, 2007), which provides highly structured investigations that (a) begin in fundamental mathematical ideas that are either in the K12 curriculum or are quite accessible to middle and high school student ("roots"), but that (b) develop a line of inquiry that reaches to areas of active current mathematical research ("to research"). This approach resonated well with a core idea and hope of the group, that substantial mathematics could be found in questions, problems, and ideas encountered in accessible situations.

The group began with a study of rational right triangles, meeting weekly for two-to-three hour sessions. The structure and expectations for these sessions were emergent, but from the beginning collaborative exploration was foregrounded. All preparatory work was done in pairs or triples (with the small size of these out-of-class groups being determined by scheduling constraints, rather than a desire for pair or triplet work). We quickly realized that our most exciting discussions in Club meetings came when we "left the garden path" of the text - to pursue our own side investigations, to critique the proofs of the book, or to raise and explore new questions.

After the Spring Semester of 2017, we declared the Club a success, although we found that as class pressures rose toward the end of the semester, students found it difficult to sustain their participation. Accordingly, we decided to formalize our work as a course in the 2017-18 school year. We called the course "Mathematical Thinking and Learning" and we established it as a year-long class carrying a single semester's worth of credit. The authors of this paper were the participants in that course, and the episode of mathematical inquiry we describe here came from near the end of the Fall 2017 Semester.

Context for the Paper and Our Research Question

While we began the semester with a second investigation from *Roots to Research*, we increasingly focused on investigating ways in which our own group could be a resource for mathematical inquiry. For instance, we spent two sessions reflecting on how our group might collectively *embody* mathematical concepts and work together to explore those concepts. (This

inquiry led to an exciting exploration of orientable and non-orientable surfaces, the creation of a group cylinder and a group mobius strip, and enactments of ideas about how to ‘animate’ paths on our surface, in order better to think about our construction as having no “privileged” or “special” point.) Here and throughout the semester, we discussed the possibilities for collective mathematics and for pursuing collective inquiry without structured support. In particular, the episode presented in this article involved exploring and refining our emerging idea that a collaborative group may be able not only to *develop* mathematical investigations together, but also to *identify and formulate* interesting problems.

We see this as an important *complementary* direction to the one that Gowers’s work with Polymath has explored. In Polymath, the power of the group is leveraged to gain new perspectives on existing problems. In contrast, in our explorations, our group has aimed to develop its ability (a) to identify promising situations to mathematize, and (b) to mathematize those situations in diverse ways.

We argue that if we can cultivate a group-level ability to seek out rich spaces for mathematical inquiry, this could be an important finding for mathematics education for several reasons. First, this ability is an intrinsically valuable skill for learners to develop. It implies a high level of agency in mathematical work and it indexes a strong sense of mathematical judgment, an understanding that one can construct systems of objects and relations in flexible ways to generate mathematical structures whose properties and behavior can then be explored.

Second, the ability to seek find and/or create one’s own mathematical systems addresses what we might call the “problem problem” in ambitious mathematics instruction - the belief that we are limited in pursuing more adventurous approaches to mathematical teaching and learning by a shortage of “good problems.” If we are able to cultivate a group-level ability to mathematize, we may rely much less heavily on carefully framed and tested problems and instead trust the group to find or create the conditions for its own fruitful inquiry. This could enable mathematical collectives to form to do recreational mathematics or engage in other non-traditional mathematical inquiry, without depending on a more knowledgeable facilitator or a written authority to structure their problem-solving work. Thus, we formulated our course-level research question:

Can we develop a group culture where finding and framing mathematical questions and areas for inquiry is a distributed ability of the group?

Data and Methods

The episode that we describe in this article is indicative of a group-level phenomenon that we began to see with increasing regularity and reliability through the Fall. Our group was beginning to develop a collective ability to *mathematize*: to identify mathematical possibilities in conceptual situations; to refine shared articulations of those situations that enable mathematical work to be done; and to bring that work to a satisfying end that produced novel findings.

In the sections that follow, we provide a description of the work done by the two parallel subgroups that formed in the class, each pursuing a different, though related approach to mathematize and investigate a situation that the group itself identified. Based on the artifacts produced during the session, each group has reconstructed their inquiry process and presented it in a narrative form. In articulating it they reflected on their challenges and breakthroughs, from the point of view of their successful endpoint. Although the groups were interested in capturing

the in-the-moment feel of inquiry, this post-hoc narration also inevitably highlights their path toward a conclusion that was not obvious until they arrived there. In presenting these narratives, our aim is to give the reader a sense of the flow of exploration, while also analytically highlighting the emergent structure of inquiry and parallels between the two groups' work.

Leading Up to the Clockface Polygon Inquiry

The initial plan for the session that sparked the investigations detailed here was to explore topics in lattice geometry. This happened to occur on a day in the semester when three members of our class were absent. After introducing the integer lattice in the Cartesian plane and exploring some metric features of the space, our reduced group turned to a set of representational tools (in this case, a set of virtual geoboards, available at <https://apps.mathlearningcenter.org/geoboard/>), with a desire to identify structural features of those tools and to use our interactions with them to formulate a worthwhile question or problem to pursue.

A decisive moment occurred as one member of the group expressed a desire not to engage “any questions about perimeter and area” of figures on the integer lattice. This initially appeared to exclude an overwhelming number of questions of interest, and it certainly ruled out the directions the group had been heading. However, as has come to be a feature of the group, the idea was taken up seriously and in good faith. Either as a result of this re-direction or coincidentally, one of us became interested in the third geoboard environment on the website: a circular arrangement of twelve pegs arranged radially around a thirteenth at the center, like a clockface. (See Figure 1.)



Figure 1

Group members began tinkering independently with this environment, making shapes and exploring their properties. After placing several triangles on the board, the group became interested in how many different triangles could be created. Once a satisfactory answer had been achieved to this question, it was natural to move on to quadrilaterals. By this time, the our session was nearly ended, but there was sufficient time to recognize that the problem was challenging enough to be worthy of our attention. Moreover, there was already some sense of different ways to go about the inquiry. This led us to propose continuing the investigation in the next session, when more of our colleagues would be present to participate.

The next two sections of the paper give an analytic recounting of our two subgroups' investigations of “clockface polygons,” describing the process of inquiry and reflecting on decisive steps. We then close by analyzing themes across the two groups and discussing significance of these findings.

Group 1: Clockface Polygons Excluding the Center Point.

In this section, Group 1 narrates their inquiry in the first person. The members of the group refer to themselves by their initials: R, K, T, and L.

Opening the Inquiry, and What Polygons “Count”

We started by asking a deceptively simple question: What are all of the possible polygons we could make on the circular geoboard? R reiterated a tentative decision made in the previous week, that the polygons explored could not have edges that intersected away from pegs (no self-intersections are permitted) and could not use pegs more than once (exactly two sides must intersect at each vertex). Even so, there seemed to be different kinds of polygons that could be made on the geoboard following these rules, and very quickly we narrowed our scope of inquiry further. The first distinction came from an intuition that the outer 12 points were different from the center point: we decided to focus our inquiry on describing all of the polygons that could be made only using the outer 12 points as vertices. As we played with the consequences of this limitation, we realized that this meant we were only exploring “convex” polygons, because the center vertex was needed in order to construct a non-convex polygon on the clockface.

12-Sided Polygons and Setting Out to Count

With a problem space that seemed sufficiently narrow, we began to formulate an organized plan of action, starting with what we thought was the easiest entry point: finding all of the “distinct” 12 sided polygons (again, using only the radial points of the clockface geoboard as vertices). This choice departed from the inquiry of the week before, when the group had begun with triangles. However, our discussion around decisions to structure the space led us to attend to many-sided polygons.

Since the number of sides a polygon has is the same as its number of vertices, we found a trivial answer to this first sub-question: only one 12-sided polygon could be made because it needed to use all 12 vertices. At around this time L drew a chart to keep track of the number of vertices used, not used, and the number of distinct polygons made (Table 1). The way of labeling the columns shows how we were conceptualizing our inquiry, focusing on the vertices as a representation (or equivalence) for the sides of the polygons. The relationship between vertices used and not used (always summing to 12) becomes more apparent and interesting later on as the number of vertices not used increases.

Number of Vertices Used	Number of Vertices Not Used	Number of (“Distinct”) Polygons
12	0	1
11	1	1

Table 1

11-gons and What Polygons Count as “the Same”

We then sought to find all of the distinct 11-sided polygons. As with 12-gons, this was straightforward. If 11 points are needed, only one point is left out and the rest are connected to make an 11-gon. Since only one point is left out, the only choice is where it will go. Moreover, all choices lead to 11-gons that are “the same” except for a rotation. In exploring this claim, we determined that we would consider any congruent polygons to be “the same.” Although the case of the 11-gon did not raise the need for reflections, we also decided here that congruent polygons with different orientations would also not be counted as distinct polygons. Thus, even though we could choose any of the 12 clockface vertices to exclude in order to make an 11-gon, all twelve

of these 11-sided polygons were “the same” for us, since they are congruent (rotations of one another): the number of distinct 11-gons is just 1.

10-gons and 9-gons and Looking for Ways to Keep Track

Advancing to the next two steps in our inquiry, our work began to grow more complex and to seem worthy of group attention. We continued our earlier approach, imagining 10-gons by thinking about how to partition or distribute the vertices *not* used in the polygon. In creating a 10-gon, 2 vertices are not used; we reasoned that they could either be arranged next to each other or they could be separated by 1, 2, 3, 4, or 5 “used” vertices. This logic resulted in six distinct 10-gons. Arguing that these were the only 10-gons involved exercising our ideas of congruence through rotations and reflections. As an indicator of our developing thinking, early on in our work we treated “not separated” and “separated” as different cases, only later unifying them by counting the number of pegs separating unused vertices, in generating or moving over the polygon, clockwise. In this later way of thinking, “adjacent” unused vertices have zero “used” pegs between them (see Figure 2a), while “separated” ones have a non-zero number of “used” pegs between them (three, in the case of Figure 2b.)

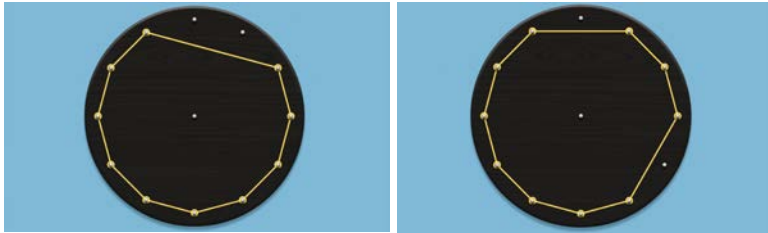


Figure 2

We extended our 10-gon logic to find that there were twelve distinct 9-gons and to create representatives of these polygons on our geoboards. This case was more complex, however, because the 3 vertices not being used could all be sequential, or they could be partitioned into 2 sequential and one separated, or the three vertices could all separated. So, instead of two options for vertex partition, there were now three. Initially as we struggled to conceptualize partitioning unused vertices, R, K, and T worked on virtual geoboards while L drew pictures of the partitions (Figure 3).

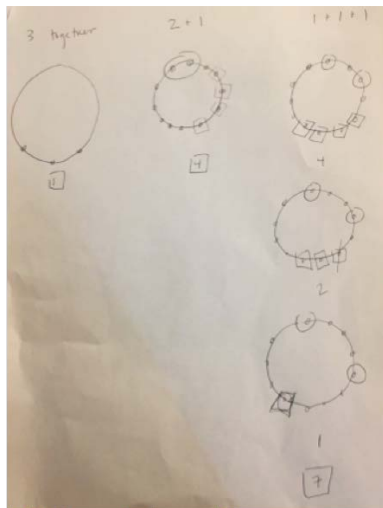


Figure 3

Figure and Ground: Getting Good at Attending to and Counting Unused Vertices

During the 10- and 9-gon work, we also began to get a feel for how to think about the partitioning of “skipped” vertices as an object in itself. This involved adopting figure-ground shift, in which we now attended to the parts of the geoboard that did *not* participate as vertices in the polygon. In looking back at our work and reconstructing this narrative, this was one of the most challenging aspects of our frame of mind to adopt again. And yet, it proved to be an extremely powerful mathematical stance for visualizing polygons. With it, we began to notice when, as we *moved a partition* clockwise around the vertices, it passed a point where it started to generate congruent polygons. For example, as shown in Figure 4, when creating a 10-sided polygon in which the two unused vertices are not sequential, you can separate the two unused points by 1, 2, 3, 4, or 5, vertices. Once you separate them by 6 pegs, the resulting polygon is congruent to one in which the unused points are separated by 4 pegs (As shown in Figure 4, the 4 vertices are just on the other side). If you separate the unused points by 7, 8, and 9, pegs, you continue to create polygons congruent to the other polygons made from the same initial separation, but in the opposite direction around the circle. For example, one could make the second polygon in Figure 4 by separating the two unused pegs by four pegs, but moving in a counter clockwise direction rather than clockwise as done in the image on the left. Together, through working in these two mediums (geoboard and personal drawings), the group began to develop a shared understanding, language, and combinatoric-based strategy for thinking about, and partitioning, the unused vertices in each polygon.

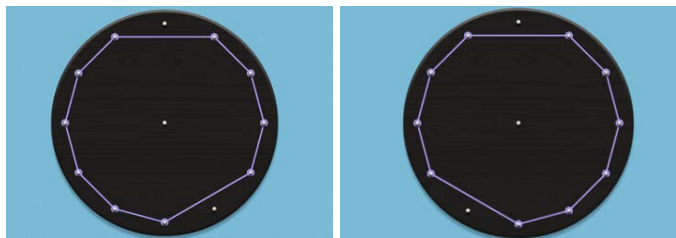


Figure 4

8-gons and the Need for a Shared, Descriptive Notation

Our inquiry continued to grow more complex as we pursued the challenge of counting octagons. L continued to draw her diagrams, starting with all 4 unused vertices together, followed by three together, two together, and then all four unused vertices separated (Figure 5). Keeping track of all of the different possibilities for partitions quickly became unmanageable especially when there were 3 or 4 different ways to partition unused vertices. Furthermore, it became increasingly difficult to communicate to others in the group and to coordinate actions across the different representations that each of us was inventing.

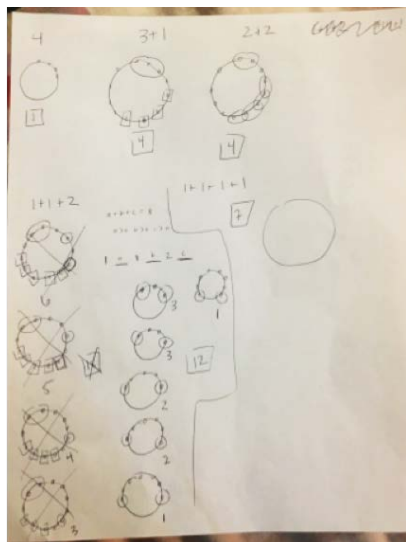


Figure 5

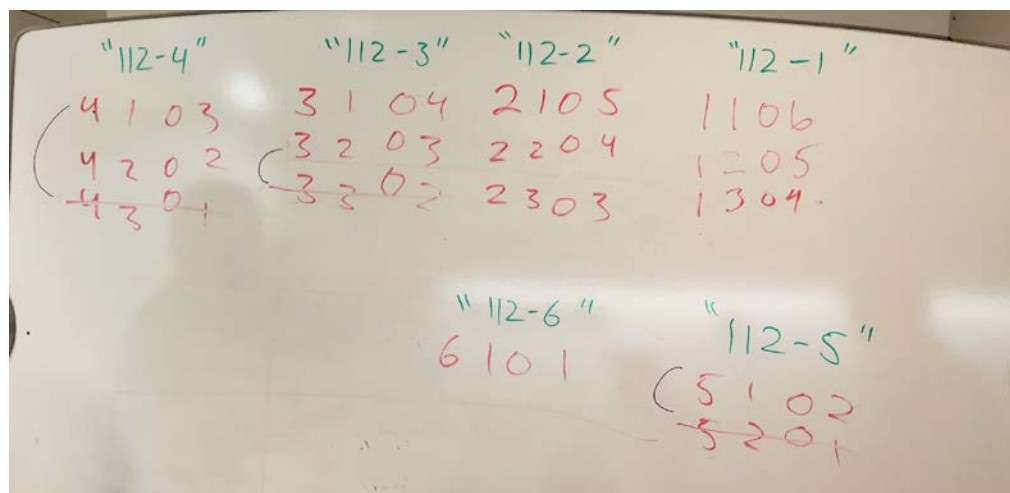


Figure 6

At about this time, the group began to work together to develop a shared notation for naming clockface polygons. Though our motivation was to support communication about our 8-gon work, we hoped that the notion would eventually extend to all of our polygon work. Our notation system began with an idea of *family names*, rooted in our approach of searching for polygons that realized different partitions of unused vertices. For example, “112” would be the “family name” for the 8-gons in which the unused vertices were partitioned into 1 vertex, 1 vertex, and 2 vertices together, each separated by at least one vertex in between as a partition. To find members of this family, we had the convention also to fix all but one partition and move the last one around to find all possibilities in the family. So for the “112 family,” the two single unused vertices would be stationary. Thus each family name had a second part to it, referred to by the group as the leaps (or the number of used vertices in between the fixed sub-partitions (here, the two 1s)). So in the 112 family, there are 6 different sub-families: 112-1, 112-2, 112-3, 112-4, 112-5, and 112-6. For example, Figure 7 represents all of the distinct 8-sided polygons in the

112-1 sub-family. Moving the 2-subpartition one more step than shown in Figure 7c would create a congruent polygon, a reflection of Figure 7c. We referred to such phenomena as “loops”—a name that showed the increasing influence of linear, string-of-numerals representations, since this reflection is formed by seeing the string “112” as a “loop” and reading it backwards, starting from the middle “1.” The transition to using the notation as a tool to generate or predict relations between polygons was slow in coming: we were not immediately able to recognize when congruent polygons were going to be created, and thus our working pages from this part of the investigation often had many proposed and then crossed-off entries, as seen in a whiteboard capture in Figure 6.

In finding all of the distinct polygons in a given family, the group also solidified a convention in orienting representatives of a class of polygons. The point at 11:00 on the clockface was always our first vertex used, so the 12:00 point was always the first unused vertex. On review, the fact that this convention seemed natural to us illustrates that we had deeply adopted a perspective in which the unused vertices constituted the *figure* for us and were the primary mathematical *objects* we were manipulating. This convention made it much easier for us to check work occurring across multiple computers and coordinate with handwritten representations. In this phase of inquiry, R and L both were making all of the polygons on their computers as a way to check each other’s work and the same orientation made it easy to compare visually. The set orientation also helped to promote our sense of when moving the unfixed partition crossed the midway point around the circle and the “loops” started creating polygons congruent to those made before by reflections.

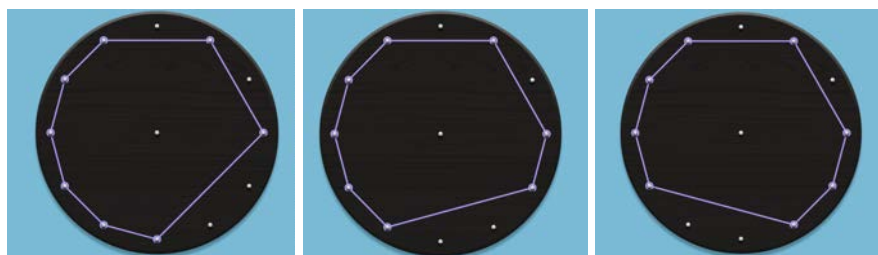


Figure 7

Our Notation Grows toward Completeness and Generativity

The set orientation for exploring each “family” also helped us to extend our notation system to include a name for each distinct polygon *within* the family. This helped us to understand if we had found all of the distinct polygons in a given family without relying as heavily on the geoboards, and it eventually gave us a parallel tool to the geoboards for predicting new family members and structures. Thus, the notation moved beyond being an accounting system and became a means of generating new polygon-cases and for reasoning about whether we had found all cases. In this sense, it became a full-fledged “peer” to the geoboard and the two began delivering the power and complementarity of “multiple representations” in our inquiry.

In extending our notation we continued our emphasis on attending to unused pegs, shifting from the attention to groupings that characterized “families” to an attention to sequencing and peg-skipping. Our notation described or generated a polygon by beginning in the standard start position and tracing or constructing the polygon according to how skipped pegs are distributed or separated by “used” pegs. Figure 8 provides an annotated example. Standard position dictates that 12:00 will always be unoccupied -- establishing the origin for “used” pegs that separate

skipped ones. The polygon in the figure is a member of the 112-1 family, and its representation in the new notation is “1106.” Starting at 12:00, you jump over 1 used peg (1:00) to reach the next unused one, then another 1 (3:00) to reach the next unused. Next, you jump 0 to get to the next unused peg, and finally you jump over 6 used pegs (6:00-11:00) to end up back at 12:00. Given the conventions for family naming, there is a large amount of redundancy or double-coding in the family-name and the new notation. (Recall that the “11” of “112-1” encodes that the first two pegs are singletons, and the “-1” captures that they are separated by one used peg. This corresponds to the “11” of “1106.” Then, the “2” of “112-1” communicates the fact that the next two pegs occur as a neighboring pair, which is captured in the “0” of “1106.” Finally, the “6” of “1106” implicitly specifies a unique member of the 112-1 polygon family; namely, the one where 6 “used” pegs remain between the 2-pair and the starting point.) To give the reader practice, the other two members of 112-1 shown in Figure 7 are captured in the new notation as 1205 and 1304, respectively.

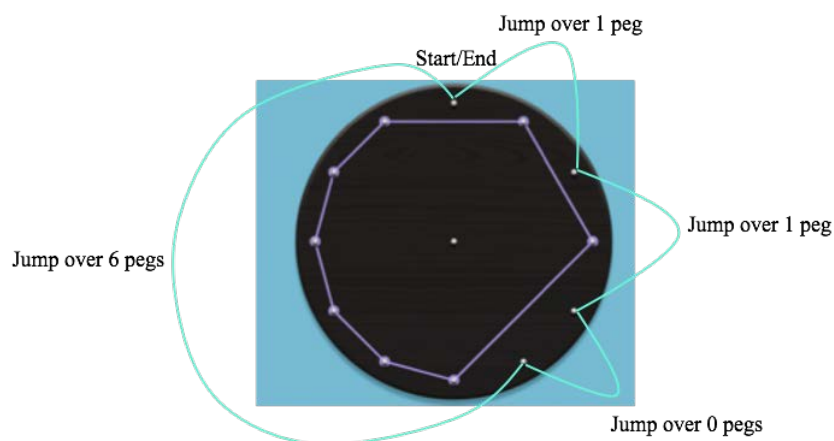


Figure 8: Illustrating the notation system for the 8-gon **1106**

Though they communicate much of the same descriptive information, the two notations supported complementary perspectives on the problem. While the “family” name (e.g., 112-1) focused attention on the partition of unused vertices (here, the 4 unused pegs are divided as 1, 1, 2), the new notation for polygons offered a dynamic perspective on how to “animate” the construction of the polygon (and its digits summed to the number of vertices in the target polygon, here 8). Thus, together the two notations provided a hierarchical structuring of the group enterprise.

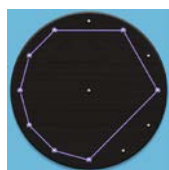
The enhanced notation system also supported our group in coordinating distributed work and in identifying and sharing insights. Each group member was operating with and on a different representation to animate their understanding of the inquiry. T was at a whiteboard writing down family names and numerical sequences to describe each of the distinct polygons in each family. He was supported by R and L on geoboards, and K writing character sequences to map the same terrain. As T began to write out our notation of polygons in an organized fashion, by sub-family, on the public display of the whiteboard (Figure 6) we began to get excited as some patterns began to emerge. For instance, as we started writing out the members of the 112-1 sub-family we saw that certain patterns in the order of the numbers might lead to predicting the next members in a different sub-family. We noticed that the sum of the numbers in each polygon was fixed and summed to the number of vertices used so for these 8-sided figures, the numbers would all sum to 8. We also noticed that in the 112-1 sub-family, the new-notation representations of polygons

all shared an initial 1, due to the “-1” of the sub-family name, 112-1. The second number (dash #) increased by 1 each time with each sub-family because we were moving the two unused vertices over by one peg each time, thus increasing its distance from the second unused peg by 1 each time. Additionally since the number of pegs used remained constant, the final number decreased by 1 each time.

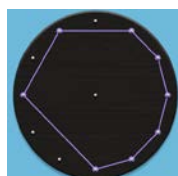
These patterns in the notation we were noticing both indexed and suggested a systematic process for listing polygons. Increasingly, we began to see the polygons as related and sequenced, and the notation as dynamic, generating them. By the time we got to the sub-family 112-3 we started to use the patterns we observed to *predict* the next polygon in the sequence. At this point, we recognized that the notation-driven polygon-generating process produced the symmetry-based repetitions we had learned to anticipate in our more primitive notation. We decided to write the first repeat and cross it out, instead of erasing it, to keep a record of when we started repeating and what the first repeated polygon was. Our goal was to reproduce the intuitions we had developed in prior work. But keeping these repeats visible also helped us notice that polygons whose notation could be read in a “loop” were equivalent since orientation (rotations) and reflections create congruent polygons. Loops generated congruent figures that were not counted as distinct polygons within a single sub-family, and hence we crossed them off. For example, the first picture in Figure 7 is read as 1106 within the sub-family of 112-1. A rotation creating the order 1601 would not be a distinct polygon and would be counted as identical to 1106. Similarly, 1160 (and its reverse loop notation 1061), 0116 (and its reverse loop notation, 0611), and 6110 (and its reverse loop notation 6011) are all congruent to 1106 in the sub-family of 112-1. The polygon 1106 can thus be created 8 different ways, by rotating and reflecting it, using our orientation starting with the first unused peg at 12:00 (Figure 9). All 8 of these figures represent the same congruent polygon and would only be counted once.

Counterclockwise Rotations of 1106

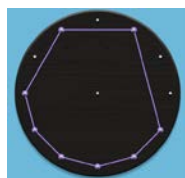
Reflections of left image



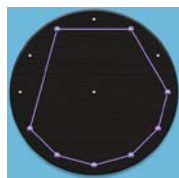
1106



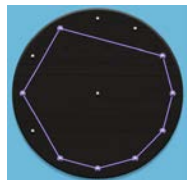
6011



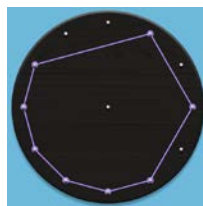
1061



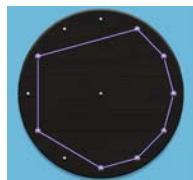
1601



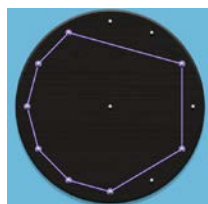
0611



1160



6110



0116

Figure 9

We noticed and verified these discoveries and patterns through distributed work. While R and L made polygons on their computers, K read off the numerical representation, and T transcribed them on the whiteboard. Developing and familiarizing ourselves with the new notation as a means to coordinate our efforts and celebrate discoveries, we solidified the notation system and energized our inquiry. We catalogued 28 octagons, and our notational advances gave us confidence that we had accounted for them all, without double-counting.

7-gons and Leaning on the Notation.

Exploring heptagons provoked the need to understand and describe all possible partitions of unused pegs (5 in this case) to establish families. L wrote them out and confirmed her work with R and K. L's system, illustrated in Figure 10, was structured by the largest subpartition. In the 7-gon case this meant starting with a single partition of 5 vertices; then the (one) partition containing a subpartition of size 4; then the two partitions including a subpartition of size 3; next, the two partitions that have a 2-subpartition as their largest subpartition; and, finally, the partition into singletons.

Largest Sub-Partition	Partitions
5	5
4	1-4
3	2-3 1-1-3
2	1-2-2 1-1-1-2
1	1-1-1-1-1

Figure 10

The complexity of partitioning carried forward to provoke the need to become more sophisticated in our family-naming strategy. For instance, as we began working with the 1112 partitioning, we discovered that this family had additional sub-family structure: with four sub-partitions, separating the unused pegs with a different number of pegs created different polygons. As we had done before, we found that if we fixed the separation of all partitions except the last one, we could find all of the possibilities. In recognizing our strategy as a generalization of our prior system, we recognized it as a counting method. For example, in the family 1112, we created sub-families that fixed the separation between the first three partitions of 1,1, and 1. The first sub-family separated the first two partitions by 1 and then included all of the possibilities of separating the second and third partitions: 1112-1+1, 1112-1+2, 1112-1+3, and 1112-1+4. Thus, in this family, the two “modifier” numbers after the dash specify the two separations between the 1, 1, 1 singleton subpartitions. Fixing these separations then allowed us to vary the separation between the final singleton and the pair (i.e., between the third “1” and the “2” in “1112”).

In working with this notation in the 1112 sub-families, we realized that some sub-family categories were duplicative. For example we found that 1112-2+1 was the same as 1112-1+2. This led us to look at “looping” issues in the family notation, as in this case 1112-2+1 is the same as 1112-1+2 if we read backward from the third “1” and loop. Thus, after 1112-1+4, the list skipped to 1112-2+2.

Figures 11 and 12 show work the group produced in exploring the 1112 sub-families of 7-gons, in the numeral-string and geoboard representations, respectively.

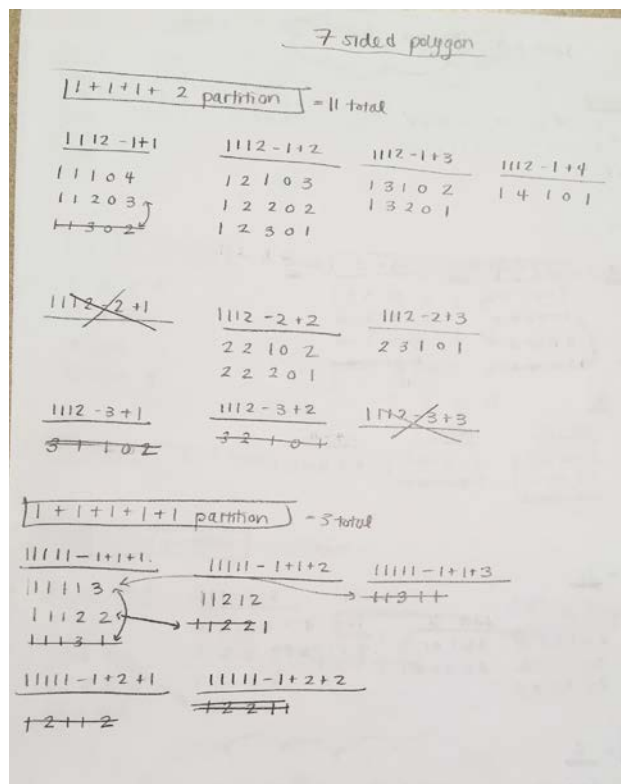


Figure 11

1112-1+1 1112-1+2 1112-1+3 1112-1+4 1112-2+2
 1112-2+3

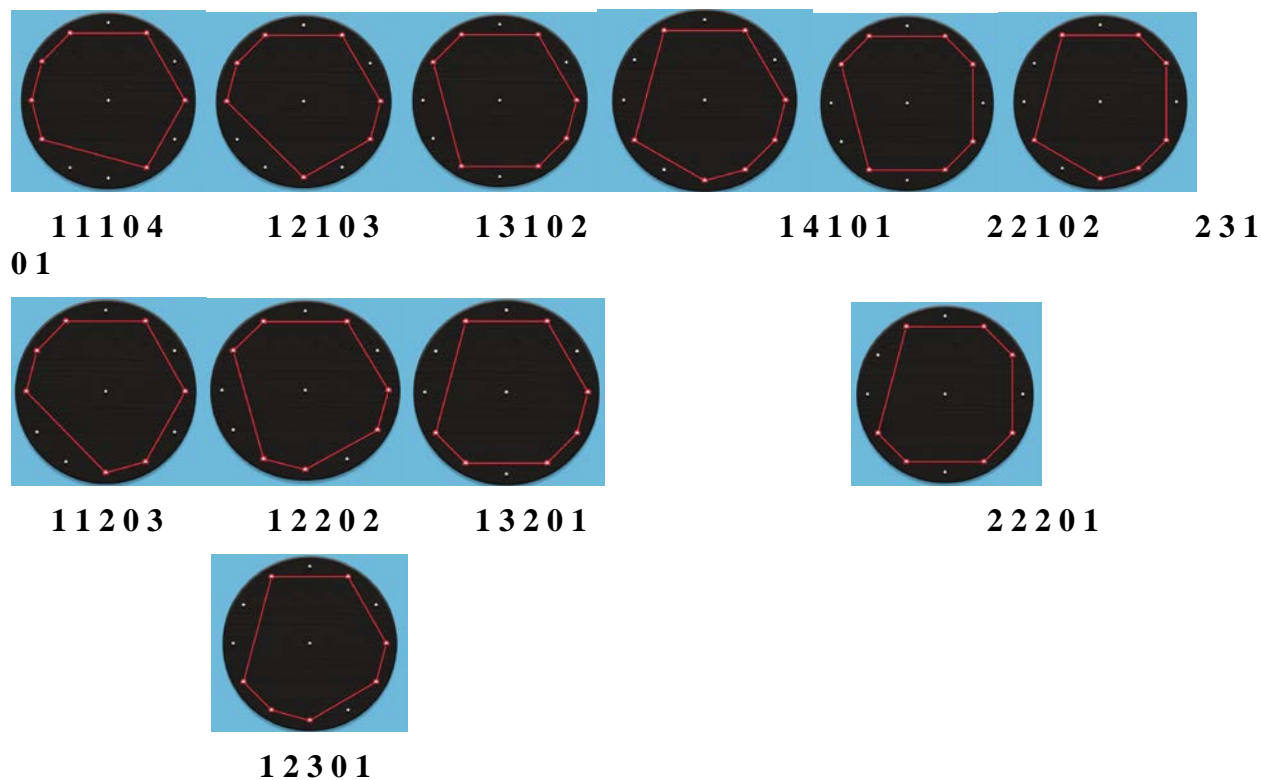


Figure 12

Our additional focus on the sub-family notation and its dynamic character also helped us to reach a greater level of shared understanding of the numeral-string representation for individual polygons. For instance, L had sometimes neglected to use a 0 in the polygon notations, but at this point recognized it played an important role in the notation system for distinct polygons and allowed them to be read forwards and backwards and in loops, holding “place value” in describing the trace of the polygon.

6-gons, Duality, and the Push to a Complete Count

At this point, we had hit our representational stride and we had a consistent process for finding clockface polygons. The class period was coming to a close, and the group was exhausted. However, L argued that completing the hexagon case would bring us to a rewarding “halfway” point.

Number of Vertices Used	Number of Vertices Not Used	Number of ("Distinct") Polygons
12	0	1
11	1	1
10	2	6
9	3	12
8	4	28
7	5	38
6	6	

Table 2

On the other hand, we predicted this exploration could be overwhelmingly complex; we expected that 6-gons would have the most possible constructions so far, beginning from the additional sub-families produced by partitions of 6 unused pegs. However, we pushed on and eventually used our process to find all possible 6-sided polygons on the geoboard. As before we started by writing out all of the possible 6-partitions. K was our designated scribe and became so adept in using the notation system that eventually she was able to see patterns and predict polygons before they could be constructed on the geoboard. R and L created the polygons on their computer geoboards and also became more fluent in seeing visual patterns. The place value of 0 had an increasingly important role in the notation used for constructing the various hexagons and recognizing distinct and repetitive polygons. This helped us understand how the 0's in the notation held the place for unused vertices. Although we had constructed our notation to use 0 to mean "jumping over 0 pegs to get to the next unused vertex," it wasn't until working on hexagons that we really experienced fluency with this feature of the notation, becoming expert in reading numeral-strings and geoboard configurations forwards, backwards, and in loops.

After a long effort, but more quickly than we expected, we found that there were 43 distinct clockface hexagons. Figures 13 and 14 show the breakdown of these 43 by partition type and by subfamilies.

6	1
1+5	3
2+4	3
3+3	3
1+1+4	6
1+2+3	10
2+2+2	2
1+4+1+3	6
1+1+2+2	5
1+1+1+1+1	3
1+1+1+1+1	1

Figure 13

Hexagon = 43

All 6 = 1 total

1+5 ✓ 2+4 ✓ 3+3 ✓ 1+1+4 ✓
 1+2+3 ✓ 2+2+2 ✓ 1+1+2+2 ✓
 1+1+1+3 ✓
 1+1+1+1+2 ✓
 1+1+1+1+1+1

$1+5 = 3$ $2+4 = 3$ $3+3 = 3$

100005 100050 100500
 200004 200040 200400
 300003 300030 300300
~~400002~~ ~~400020~~ ~~400200~~

$1+1+4 = 6$

$114-1$ $114-2$ $114-3$ $114-4$
 110004 210003 310002 410001
 120003 220002 ~~320001~~

$1+2+3 = 10$

$123-1$ $123-2$ $123-3$ $123-4$
 101004 201003 301002 401001
 102003 202002 302001
 103002 203001
 104001

$2+2+2 = 2$

$222-1$ $222-2$
 101040 ~~201030~~
 102030 ~~102020~~

Figure 14

After completing the hexagon exploration, we made what was initially a wild conjecture—that the *table* we had been building would itself be symmetric. The core technique of our investigation (switching “figure” and “ground”) resonated with this idea. Specifically, we reasoned that the number of polygons with a given number of vertices, should be the same as the number of polygons that *omit* that given number of vertices. If this were true, then we could produce Table 3.

To accept Table 3, we needed to absorb this idea of duality and its consequences, and we needed to overcome some puzzling interpretations and contradictory evidence. The final three rows of the table were confusing, as we needed to think about polygons with 2 vertices, 1 vertex, and 0 vertices. But in each case, we were able to come up with a geoboard representation of the indicated objects, and in the case of 2-gons (line segments), the geoboard representation was interesting. For the 3-gon case, our table agreed with the work the group had done the week before, in finding 12 unique clockface triangles, when the center point was ignored. In the 4-gon case, the group had found only 27 quadrilaterals the week before, but we confirmed our belief in 28. With duality as a visualizable overlay to the geoboard representation of many-sided polygons, we had increasing confidence in the symmetry of Table 3, not only as a numerical quirk but as reflecting a mechanism by which the corresponding rows were connected.

Number of Vertices Used	Number of Vertices Not Used	Number of ("Distinct") Polygons
12	0	1
11	1	1
10	2	6
9	3	12
8	4	28
7	5	38
6	6	43
5	7	38
4	8	28
3	9	12
2	10	6
1	11	1
0	12	1

Table 3

Group 2: Clockface Polygons Including the Center Point.

Group 2 consisted of two students, Fai and Isaac (F and I), who started off in a different direction immediately from Group One, admitting into their consideration clockface polygons that included the center point. On the other hand, like Group One they did (provisionally) adopt a definition of polygon that prohibited self-intersections and that required vertices to be the intersection points of exactly two sides of the figure. The account that follows is narrated by Isaac.

Starting Off

F and I decided to start working with many-sided figures on the clockface, naming and counting them. Starting with a 13-gon, we labeled it a ‘pacman’ figure because of its resemblance to the arcade game icon. Notating this:

13-gon

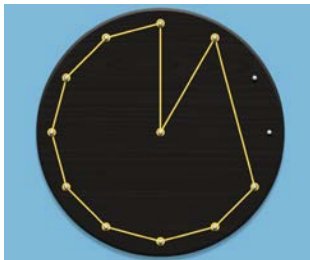
1 - pacman

Next we looked at the 12-gon. We knew that if the figure used all the vertices around the clock face (without the center vertex) this would create a regular 12-gon. Because of the constant curvature of this figure we labeled it a “pseudo-circle.” We also knew that if we made the mouth of our 13-gon pacman bigger, by leaving one peg in the ‘mouth’ of the pacman, this would also

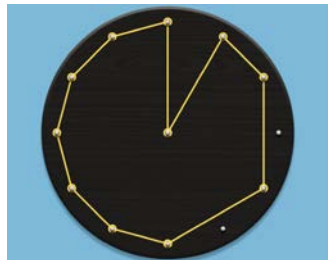
create a 12-gon. We decided to label this figure “pacman₁” where the 1 signified the number of vertices in the pacman’s mouth, and accordingly we changed the name of our 13-gon to “pacman₀.” Lastly, we found that we could modify our pacman₀ to exclude one of the vertices around the perimeter of its ‘head’. This posed a new problem of symmetry that we had not yet faced. Not including the first vertex resulted in the “same” figure as not including the last vertex, simply reflected by a mirror. Four more distinct figures were simple to count on the geoboard using our eye to see the line of symmetry. We decided to call the flat part on the pacman head a ‘distortion’ and to label the group of these five figures as “pacman₀ dis₁.” The 1 signifies the “size: of the distortion: the number of vertices excluded around the pacman’s head. We notated our 12-gons as follows:

13-gon	12-gon
1 - pacman ₀	1 - pseudo-circle
	1 - pacman ₁
	5 - pacman ₀ dis ₁
	(with distortions proceeding around the head to the symmetry point)

Moving along, we felt we were tackling the 11-gons with a fairly robust set of descriptive tools. We used our new distortion descriptor on the new pseudo-circle, and we did not run into much trouble counting the different pacmen. We decided to make a new distinction between “pacman₀ dis₂” and “pacman₀ dis₁ dis₁” where the former signifies a single distortion (a single flat part) in which two adjacent vertices are skipped and the latter signifies two distortions, each with one skipped vertex.



A pacman₀ dis₂



A pacman₀ dis₁ dis₁

Pacman₀ dis₂ figures could be counted in a similar manner to the 12-gon, but pacman₀ dis₁ dis₁ posed a new difficulty with placing the two distortions. We counted these using the geoboard, setting a distortion in the first place (clockwise) and moving the second distortion around the pacman head (also clockwise). Next we would move the starting distortion one vertex clockwise, and count the second distortion around again. This systematic counting method became very easy, but the difficulty was in knowing when to stop moving the first distortion so we did not double count because of symmetries. We used our eye to be able to tell there were 8+7+6+5 of these figures, excluding symmetric duplicates. Our board thus became:

13-gon	12-gon	11-gon
1 - pacman ₀	1 - pseudo-circle	1 - pseudo-circle dis _i
	1 - pacman ₁	1 - pacman ₂
	5 - pacman ₀ dis ₁	5 - pacman ₁ dis ₁
		5 - pacman ₀ dis ₂
		26 - pacman ₀ dis ₁ dis ₁

At this point, F made a connection between the two vertices of our distortions and the known counting problem (and notation), ${}_{10}C_2$ -- we have 10 options of vertices to exclude and we need to choose 2 of those vertices. Computing this gave us 45 figures. We knew this method double counted some congruent figures, but because the result was an odd number we knew it could not have been double counting them all. Moreover, we recognized that this “choose” counting calculation would consider $\text{pacman}_0 \text{ dis}_2$ and $\text{pacman}_0 \text{ dis}_1 \text{ dis}_1$ as the same type of figure, so the comparable count from our other method was $26 + 5 = 31$. Through playing with this idea, we realized we were not double counting figures who were internally symmetric (rather than paired with another figure through symmetry), but we only found 5 such symmetric polygons, which would lead us to the answer of $(45 - 5) / 2 = 20 + 5 = 25$, not 31. Thus, we had to decide which count and method was correct. We soon realized that in our former method we had actually double counted some of our $\text{pacman}_0 \text{ dis}_1 \text{ dis}_1$ figures. These were mirror images of the ones we had counted before (in the “6” of our “8 + 7 + 6 + 5” count, above). This led us to settle on the 25 as the correct answer for the combination of $\text{pacman}_0 \text{ dis}_2$ and $\text{pacman}_0 \text{ dis}_1 \text{ dis}_1$ 11-gons.

This process took a fair amount of discussion, and by the end of it, we were seriously questioning our ability to see the symmetries in the figures we were generating. We began to feel an acute need for a notational system that would allow us to make counting the different types of pacmen and their distortions a systematic process. To begin in this direction, we decided to use ordered number tuples to represent the number of one-unit edges, i.e. edges that connect directly adjacent vertices. These represent the ‘non-distortion’ edges around the pacman’s head and are broken up by the distortions. Thus, each noted class of pacmen would specify a specific set of parameters on the ordered number tuples. We started with $\text{pacman}_0 \text{ dis}_2$ which would be described by ordered pairs with a sum of 8: the first element indicating the number of uninterrupted edges before the distortion and the second the number remaining after the distortion. We listed these out, going back and forth between our notation and the geoboard:

(8,0) (7,1) (6,2) (5,3) (4,4)

We then moved to $\text{pacman}_0 \text{ dis}_1 \text{ dis}_1$, where we used ordered 3-tuples in the same way. These summed to 7 and we fiddled with the logic of this sum, settling on the formula: For 11-gons, start with 11, subtract the pacman subscript, then for each distortion, subtract 1+ the distortion subscript. Furthermore, we reasoned that the number of distortions +1 gave the dimension of the n-tuple for that class.

Using this notation, we started to generate notation-figure pairs to better understand where the symmetries lay, so we could systematically generate all unique figures. With some trial and error, we settled on mirror equivalence, i.e. (A,B,C) is the same figure as (C,B,A). Thus by setting a notational rule that $A \leq C$, we knew we would not double count figures. Using this approach, we generated the following list of $\text{pacman}_0 \text{ dis}_1 \text{ dis}_1$ 11-gons:

(0,0,7)	(1,0,6)	(2,0,5)	(3,0,4)
(0,1,6)	(1,1,5)	(2,1,4)	(3,1,3)
(0,2,5)	(1,2,4)	(2,2,3)	
(0,3,4)	(1,3,3)	(2,3,2)	
(0,4,3)	(1,4,2)		
(0,5,2)	(1,5,1)		
(0,6,1)			

(0,7,0)

This led us to realize we had also been double counting the last figure step when we were moving the distortion around, i.e., when the second distortion landed in the position closest to the pacman mouth this was the same as the first distortion being in the first position (also closest to the pacman mouth but on the other side). Thus the correct number of figures aligned with F's connection with the linear form, double counting all but symmetric figures. To check to see if this held true for 10-gons, we moved forward with our new notation.

For 10-gons we continued as we had, first noting the all the different types of pacmen and pseudo-circles and then beginning with counting the 'easy' figures:

10-gon

1 - pseudo-circle dis₂
 5 - pseudo-circle dis₁ dis₁
 1 - pacman₃
 4 - pacman₂ dis₁
 4 - pacman₁ dis₂
 - pacman₁ dis₁ dis₁
 4 - pacman₀ dis₃
 - pacman₀ dis₂ dis₁
 - pacman₀ dis₁ dis₁ dis₁

We checked both our counting and our notation on the 'easy' figures, which were figures resulting in ordered pairs. The more difficult figures were those represented by 3-tuples and 4-tuples. We notated these as we had done earlier, starting with pacman₁ dis₁ dis₁, which was similar to the 11-gon's pacman₀ dis₁ dis₁ except summing to 6 instead of 7. We also began to note the symmetric figures with an asterisk (*) to help us check duplicates with the choosing method:

(0,0,6)	(1,0,5)	(2,0,4)	(3,0,3)*
(0,1,5)	(1,1,4)	(2,1,3)	
(0,2,4)	(1,2,3)	(2,2,2)*	
(0,3,3)	(1,3,2)		
(0,4,2)	(1,4,1)*		
(0,5,1)			
(0,6,0)*			

We next moved to pacman₀ dis₂ dis₁. This resulted in 3-tuples also summing to 6, but now our notation was ambiguous, since the order mattered in a new way. Before, we had distortions that were the same and thus symmetric, but here we had two different types of distortions (one of size 1 and the other of size 2). Thus, our equivalence of (A,B,C)=(C,B,A) was no longer correct, because the distortion occurring at the first “,” was excluding 2 vertices and the distortion occurring at the second “,” was excluding 1 vertex. So, we built on top of the pacman₁ dis₁ dis₁ adding the extra figures in a different color.

(0,0,6)	(1,0,5)	(2,0,4)	(3,0,3)*
(0,1,5)	(1,1,4)	(2,1,3)	(3,1,2)
(0,2,4)	(1,2,3)	(2,2,2)*	(3,2,1)

(0,3,3)	(1,3,2)	(2,3,1)	(3,3,0)
(0,4,2)	(1,4,1)*	(2,4,0)	
(0,5,1)	(1,5,0)		
(0,6,0)*			

Note that we did not think to extend our numeration to include (4,0,2) etc, even though those are valid for $\text{pacman}_0 \text{ dis}_2 \text{ dis}_1$. We also did not notice at this point that the symmetric figures annotated this way within the category of $\text{pacman}_1 \text{ dis}_1 \text{ dis}_1$ were *not* symmetric figures for $\text{pacman}_0 \text{ dis}_2 \text{ dis}_1$.

The Notation Isn't Helping

As we were generating these lists, we were discussing our notational system and F kept expressing a distaste for it. She seemed to want the notation itself to better guide the enumeration process of the figures. I did not know what this meant, nor could I think of a frame of reference to act as an example of this type of work. I thought that the current system with the correct rules could have developed into such a system, but we were not there.

Class was coming to a close. F and I started to enumerate $\text{pacman}_0 \text{ dis}_1 \text{ dis}_1 \text{ dis}_1$ using the equivalence $(A,B,C,D)=(D,C,B,A)$ to form enumeration rules. We established $A \leq D$ and, after some struggles, a second rule that if $A=D$, then $B \leq C$. We began to generate cases but we were never satisfied with our list.

Finally, at some point we went back and checked our counting of $\text{pacman}_1 \text{ dis}_2$ and $\text{pacman}_1 \text{ dis}_1 \text{ dis}_1$ with the linear choose formula ${}_9C_2$ with 4 symmetries from $\text{pacman}_1 \text{ dis}_1 \text{ dis}_1$ and 0 symmetries from $\text{pacman}_1 \text{ dis}_2$. This confirmed that our conjectured formula worked. We tried to confirm this with the pacman_0 s as well but calculated that we should be getting 60 figures, and we never found all of them in our other enumeration method.

Reflecting On Features of a Desirable Notation

About 5 days later, C and I talked about the notational work that both groups in the class were doing. We reflected together on the advantages and disadvantages of F's and my notational system. It was a static, descriptive system, but it had a flavor of humor in it as well. We talked about F's drive towards a different type of system, one that was less static and purely descriptive, and which instead could support us in animating or generating polygons. We talked about the path-perspective notational system that the other group had developed, and we discussed some of the advantages of that system.

I played around with a lot of different notational systems after meeting with C, attempting to capture a path-oriented system that would also lend itself to ways of listing and counting types of figures. I made a fair amount of progress on the problem in the process, arriving at a count for the number of 10-, 9-, 8-, and 7-gons, but I never truly got away from our classes of pacman and distortions.

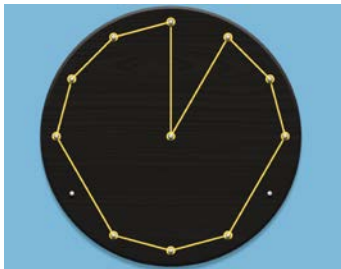
Several days later, I showed F my latest path-perspective notational system. She liked it but did not feel like it was really capturing what she had been driving at from the beginning. She told the story of her high school experience when a student solved the problem: If x people are going to sit around a round table, how many unique ways can they be seated? She described the "a-ha!" feeling when her classmate changed the problem from one with lots of different symmetries to a simple linear problem by fixing the location of one of the people. She wanted to do something similar with our clockface polygon system. Furthermore, we talked about the counting method

we had developed and wondered if there was a notational system that would connect to and describe this better than what we had been doing.

Inventing a Readable, Path-Based Notation

We tried this by actually notating the vertices in a line (rather than a circle) and deciding to include only the vertices of the pacman's head, i.e., not the two vertices connecting to the center (pacman's "lips") or any vertices between these (i.e., inside pacman's mouth). We experimented with different types groupings of these dots and bars between dots. Finally, we realized that putting a bar instead of a dot could notate that the corresponding vertex was being excluded—i.e., was part of a distortion. This connected directly to our counting method where we were counting the choices of vertices to exclude. After talking with F, I developed our idea a little further using ordered sequences of dots and bars, i.e. tuples. For instance, the following dot-bar inscription describes the clockface 11-gon on the geoboard below it.

• • | • • • | • • •

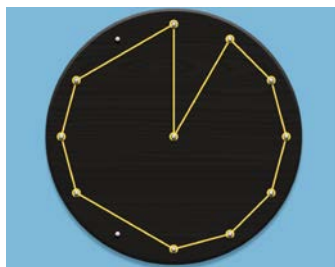


The notation of pacman_i was still meaningful for context because it determined the number of dots in the tuple, equal to $12-2-i$. I expanded this to create classes similar to the pacman distortion classes from the first classification, but where there was not a distinction between pacman_i s where the total distortion was the same (e.g., between $\text{pacman}_0 \text{ dis}_2$ and $\text{pacman}_0 \text{ dis}_1 \text{ dis}_1$). I used ordered pairs with a semicolon between to signify these. For instance, $(10;2)$ represented the class of $\text{pacman}_0 \text{ dis}_2$. One nice thing about this notation is that it is simpler to go from it to the number of sides in the figure. For any $(A;B)$, the number of sides in the figure is $A-B+3$. It also lines up well with the counting notation of ${}_{10}C_2$, but we still need to find a way to count the number of symmetric figures (which we call "symmetries"), since these are not double counted. At this point the formula for the size of the $(A;B)$ family was:

$$|(A;B)| = ({}_A C_B + \text{symmetries}) / 2.$$

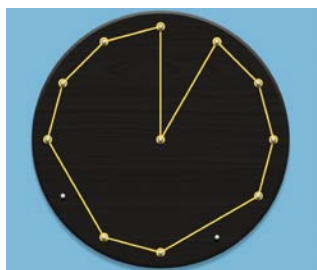
I played around with trying to figure out how to count the symmetries and found that this revealed an unexpected affordance of the linear ordered dot-bar tuples. I first looked at $(10;2)$ polygons. For example, this figure, which is not a symmetry:

• • • • • | • • • • |

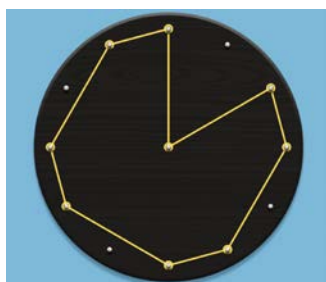


Now, the dot-bar notation for a symmetric figure in (10;2) will *itself* have a symmetry between the 5th and 6th positions, such as:

• • • | • • | • • •



Thus, the dot-bar notation (which was economical to write out) could reveal symmetries without requiring us to visualize or literally produce clockface polygons on the geoboard. Further, I began to find patterns. For instance, looking at (10;3), I could not find *any* symmetries, but (9;2) and (9;3) both contained symmetries. I realized that this had to do with whether the first number was even or odd. For instance, I noticed that in (10;2) the symmetries were completely determined by the first 5 places, i.e. they were of the form (ABCDEEDCBA) and thus there were ${}_5C_1$ options for this case. But for the (10;3) case, there was *no way* to create a symmetry because I could not evenly divide up the 3 skipped vertices, and there was no “middle position” for the odd vertex to be placed. For either the (9;3) case or the (9;2) case, there *was* a middle position that could either be a dot or a bar as needed so the number of distortions to be placed before and after the middle could be made even. Thus, both of these cases have the same number of symmetries, ${}_4C_1$.



• | • • | • • | •

A symmetric (9;3) will have a bar in the middle position.

Producing Results and General Formulas

Now we could write down a formula for the number of symmetries in (A;B), for the different cases of the parity of A and B:

Case 1: $A=2p$ and $B=2q$, then symmetries $= {}_pC_q$

Case 2: $A=2p$ and $B=2q+1$, then symmetries $= 0$

Case 3: $A=2p+1$ and $B=2q$ or $B=2q+1$, then symmetries $= {}_pC_q$

Thus, putting these together, along with our base formula, $|(A;B)| = ({}_AC_B + \text{symmetries}) / 2$, we are able to calculate the number of 10-gons:

$$\begin{aligned} |(10;3)| &= ({}_{10}C_3+0)/2 = 60 \\ |(9;2)| &= ({}_9C_2+{}_4C_1)/2 = 20 \\ |(8;1)| &= ({}_8C_1+0)/2 = 4 \\ |(7;0)| &= ({}_7C_0+{}_3C_0)/2 = 1 \\ &= 85 \end{aligned}$$

This technique works for all figures except when either when the pacman's mouth is exactly 5, (i.e., $(5;a)$ for any a), or when the number of excluded vertices exceeds 4. In the first case, the 'mouth' of the pacman creates a line through the center point, which I would argue is not actually a pacman, but a case of the pseudo-circle. The second criterion creates self-intersecting figures. It would be possible to revise our inquiry to allow these, but then we would have dramatically undercounted in much of our work, because we assumed that polygon was determined by the set of included/excluded vertices. If self-intersections are allowed, then more than one shape can be determined by a given set of excluded/included vertices.

However, there is an advantage to considering the special case of self-intersecting where the intersections involve sides connected to the center point. In discussing the initial findings of the Group One with C, he mentioned the duality they discovered and asked whether this case of self-intersection could be hiding a duality in our inquiry. This would be like considering the center vertex to be in the third dimension, slightly above the rest of the clock face, so that edges on the clock face would not be able to intersect with the edges connecting to the center vertex. (This is not a perfect metaphor because the vertices "in the pacman's mouth" are automatically excluded.) This line of thinking led to the following theorem:

Theorem: For any \mathbf{A} and \mathbf{B} in the natural numbers such that $0 < B < A$, $|(A;B)| = |(A;A-B)|$.

Proof: Because of the nature of the symmetries formula, it is helpful to consider this by cases:

Case 1: $A=2p$ and $B=2q$ for some p and q in the natural numbers.

Now by the formula and the symmetries for this case, discussed above, we know $|(A;B)| = ({}_{2p}C_{2q+p}C_q)/2$. By rules of counting, we also know ${}_{2p}C_{2q} = {}_{2p}C_{2p-2q}$ and ${}_pC_q = {}_pC_{p-q}$. Furthermore since A and B are even, it follows $A-B$ is also even, so $|(A;A-B)| = ({}_{2p}C_{2p-2q+p}C_{p-q})/2 = ({}_{2p}C_{2q+p}C_q)/2 = |(A;B)|$.

Case 2: $A=2p$ and $B=2q+1$ for some p and q in the natural numbers.

It follows $|(A;B)| = ({}_{2p}C_{2q+0})/2$, and since A is even and B is odd, it follows that $A-B$ is odd. Hence, $|(A;A-B)| = ({}_{2p}C_{2p-2q+0})/2 = ({}_{2p}C_{2q+0})/2 = |(A;B)|$.

Case 3: $A=2p+1$ and $B=2q$ or $B=2q+1$ for some p and q in the natural numbers.

If B is even, then $A-B = 2(p-q)+1$ is odd, and if B is odd, then $A-B=2(p-q)$ is even. Either way the symmetries for $(A;B)$ and $(A;A-B)$ would number by ${}_pC_q$ and ${}_pC_{p-q}$, respectively and by the rules of counting these are equivalent. It follows $|(A;B)| = ({}_AC_{B+p}C_q)/2 = ({}_AC_{A-B+p}C_{p-q})/2 = |(A;A-B)|$.

This duality is somewhat reflected in the figures on the clock. For example, there are the same number of 10-gons of the form (10;3) as there are 6-gons of the form (10;7). But it is not reflected in the total number of 10-gons and 6-gons because the 10-gons stop at (7;0) and the 6-gons at (3;0).

Discussion

These episodes in the work of members of our group are remarkable in several ways. They illustrate how a collaborative community can engage in intensely productive mathematical inquiry with very little need for structuring by an external facilitator; and they provide some insights into the kinds of challenges and discoveries that such groups encounter in undertaking the pioneering work to mathematize a new conceptual terrain.

Productive, Independent Mathematical Inquiry with Little to No Facilitation

The two groups developed their inquiries essentially “from scratch,” having decided to depart from the conceptual terrain that the instructor had prepared for them (in Lattice Geometry). Thus, they moved from what might have been an *open* inquiry exploring a well-mapped domain, to a more *pioneering* one in which neither the students nor the instructor were aware of any prior work that might serve as a guide or as an indicator that there was “worthy” mathematics to be found or made in clockface polygons. In fact, it initially seemed (at least to the instructor) that the mathematical potential of the space they chose might be exhausted relatively quickly. Fortunately, the group had learned two important things in their prior work together: first, they had developed confidence in their instincts about what might be interesting; and second, they had come to trust that when any one member of the group felt strongly that an idea or direction could be interesting, they could benefit from encouraging and supporting that group member’s interest.

Further, the two separate sub-groups found very *different* mathematical structures by making their own decisions about how to structure their space of inquiry and pursuing these decisions tenaciously to bring them to a satisfactory conclusion (though not necessarily to a final close, as Group Two’s inquiry shows). For Group One, the “arc” of the inquiry took on its broad outlines within the three-hour period of the class session (though the group did stay late). In contrast, for Group Two, the class period surfaced key conceptual challenges of their exploration, which they continued to struggle with alone and together over the course of the following two weeks. The groups were very different in many ways, including their size (4 versus 2 members) and the mathematical decisions made to frame their exploration (excluding or including the center point radically changed the conceptual terrain). Moreover, their “styles” of inquiry seemed quite distinct, as were their distribution of work and their means of coordinating individual labor and shared discussion. The narratives above provide some insight into the “flavor” of the two inquiries, but the moment-to-moment affect – the bewilderment, frustration, excitement, boredom, and triumph – is hard to capture without experiencing it as a participant.

Key Emergent Aspects of the Two Groups’ Inquiry

The detailed narratives of the groups’ clockface polygons explorations also offer insights into the texture of learning and inquiry in such “pioneering” work by mathematical collectives. In particular, both groups’ narratives illustrate the challenges and power associated with inventing and refining notation systems, and they document the communal process of becoming fluent with and discovering the affordances of those notation systems.

Progressive Elaboration of Notation Systems. For both groups, notation and notation systems were key tools, both for illuminating the structure of the mathematical space they were attempting to map and for coordinating their independent and collective efforts. Yet these notations were not ready to hand or easy to create: they evolved in fits and starts, and they were in no way “objectively natural.” In fact, the instructor struggled to understand the systems that each group was developing as they emerged, and the groups themselves struggled to re-construct and re-animate them as they assembled reflective narratives of their inquiry. It seemed instead that the notation emerged interactively with (a) the group’s efforts at coordination and communication, and (b) their insights into the mathematical structures of the domain they were investigating. The “embroiled” relation between the notation and the group’s functioning highlighted a particular set of “meta-representational” (diSessa & Sherin, 2000) requirements and standards that the groups had for judging their notations. First, both groups created and favored hierarchical systems, where one level of notation specified a “family” or a class of polygonal figures, while another supported the enumeration of figures within a family or class, in an ordering that afforded awareness of sequence and of key “stopping points” such as the point where symmetry considerations led to double-counting. Creating or discovering these affordances of notations occurred in a social space where multiple people sought to communicate and coordinate their work. Second, both groups experienced an unsatisfied feeling about their notation systems until they reached a level of completeness and sophistication that enabled them to be tools for predicting and guiding polygon generation, rather than just accounting for found polygons. For Group One, this became salient as the digit-string representation came into its own alongside the digital geoboard, and operators of the two representations could mutually check each other’s work, dictate the search sequence, and catch errors. For Group Two, this became salient as the dot-bar representation and its surrounding system began to mesh sufficiently well with the standard “choose” notation, to benefit from the power of that notation, while maintaining “close” to the local and idiosyncratic texture of the problem with its “pacmans” and “distortions.”

Notational and Representational Fluency. Both groups experienced qualitative shifts in their sense of the problem concurrent with achieving a level of representational fluency. Although the representations were invented, it nevertheless came to appear that they exhibited the structure of the problem, so that facility with the notations emerged as intuitions about families of polygons. This affective experience raises important questions about the nature and role of “representational fluency” (e.g., Cramer, 2003). In this case at least, the representation flexed and was refined while the group members negotiated proper ways of operating it and reasoning with it. Thus, “fluency” might here be applied both to the group members (i.e., to their interface with the representation) and to the representation itself (i.e., to its interface with the emerging field of the group’s mathematical work). Over time, *for the groups*, their notation became more dynamic, and they developed the ability to see mathematical structures *through* these representations. Moreover, collectively, they distributed roles-with-representations across the social structure of the group, putting notations and representations in conversation with one another to move mathematical inquiry forward. This scene of work provides insight into a refreshed view on the role of “multiple linked representations” (Kaput, 1991; 1998). In emergent group procedures (predicting, generating, confirming, and notating generated polygons), different members of the group took on roles in which they participated in joint work by animating different aspects of representations and notation systems of “the same” mathematical

objects, conjuring up and fixing these objects in a “triangulated” conversation between representations.

These narratives suggest a means for gaining new insights into mathematical cognition as it occurs as a distributed social process. When groups need to construct the tools to support their own inquiry, important relations between inquiry and tools come into sharp relief. The interplay may not be adequately captured with exclusive attention to a scene of action where learners are being enculturated to tools that are established and that encode entrenched disciplinary practices. Of course, these processes are important for the study of learning, but they may not give us the whole story, and they may provide a particularly incomplete guide in a setting where learners construct, refine, or adapt tools to structure new forms of inquiry.

Clockface Polygons in the Context of Our Collective Mathematics Experiment

The clockface polygons inquiry holds an important place in the shared experiment in collective mathematical activity that has engaged the authors of this article for the past year. These particular episodes of collective mathematics occurred after our group had been working on open mathematical problems together for about 3-4 months during the Fall semester of 2017, and they represent a pivotal moment in pursuing our shared, course-level research question:

Can we develop a group culture where finding and framing mathematical questions and areas for inquiry is a distributed ability of the group?

While we have seen the clockface polygon experience as a remarkable piece of evidence toward answering this question in the affirmative, we recognize that it emerged out of an extended period of work together, which laid a strong foundation for collaboration. After beginning the semester with an inquiry into the Four Numbers problem (cf., Sally & Sally, 2000), which evolved to become increasingly independent from the book’s guiding ‘script.’ Leaving that study, we searched for other more experimental ways to use our group as a vehicle for mathematical exploration. For instance, responding to the research interest of one member of our group, we dedicated a session to explore what we might investigate mathematically using the group of our bodies. When someone suggested a Mobius strip, we spent the next two hours using physical enactment to investigate the idea of orientability. Through shared physical experiments and joint enactment we animated our ideas and developed shared understandings of the local and global properties of the Mobius strip.

Thus, our group-level ability to identify, formulate, and successfully pursue the two clockface polygon inquiries described in this article developed through a sequence of shared experiences that encouraged adventurous thinking, promoted risk taking, and built our individual and collective confidence. Along the way, we spent quite a bit of time talking freely about our experiences of being “stuck” and about how to avoid interpreting such moments as indicators that our mathematical abilities were inadequate. Knowing that fears and feelings of inadequacy were common affective experiences helped to motivate us, but even more important, we also began to wonder whether these feelings themselves might indicate something about the problem-refining and problem-solving process. That is, once we were able to question whether these feelings were telling us something about *ourselves*, we could ask whether or when they might be telling us something about the interface we had established with our shared mathematical work. Indeed, monitoring and “aggregating” the affective experiences and possibly-off-topic thoughts that occurred to us as we worked, has seemed to us an important area for future study, balancing the urge toward forward progress in shared inquiry with opportunities to make non-linear

connections or explore related ideas. As it becomes more practicable to think of a group as a collective mathematizing entity, we have emerging design opportunities to support that distributed “organism” in doing its work.

Conclusion

Through the 16th-19th centuries, European society, and particularly French society, repeatedly produced intimate social spaces for collective inquiry, known as “salons.” These intellectual collectives have been associated with fostering intense creativity and productivity in philosophy and the arts. Moreover, Jürgen Habermas (1989) credits the Salon with the formation of what he calls “the public sphere,” a critical factor in the democratization of society and in shifting the center of cultural gravity away from the royal court. Salons offered necessary ingredients to develop collectivity and provided a forum for exercising collective thought—where people came together to envision, formulate, and achieve things they might not have been able to do on their own.

In our present society, the ability to connect has been amplified enormously, and yet the experience of deep collaboration and shared thinking may be, if anything, diminished (cf., Turkle, 2011). On the other hand, the success of the Polymath projects and the trend toward larger-and-larger scale collaborations in Physics and other sciences suggests that collectives may be an increasingly important unit of scientific and intellectual action. It is thus promising to identify approaches that can enable groups of non-professional mathematicians to come together without expert facilitation to do novel mathematical work. Our experience offers an existence proof that groups, and in particular classroom groups, who are “activated” as learning collectives have the potential to work in this way.

If a coordinated group of amateurs can do more powerful mathematics than its members can, individually, then knowing when and how to form a collective to address a mathematical question or challenge may be an important form of knowledge for learners to develop. That is, learners may need to learn how to “wield” collective organization as a means to engage in inquiry, as this form of activity becomes more prevalent and as the challenges facing communities require sophisticated mathematization and quantification to tackle. For this reason, we propose that understanding the necessary conditions for community organization and action is an exciting frontier for collective, participatory mathematical work. Moreover, there is a real joy to doing collective mathematics, and having access to that experience of joy is also a public good.

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