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Graphs of Two Variable Inequalities: Alternate Approaches to the Solution Test

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Graphs of Two Variable Inequalities: Alternate Approaches to the Solution Test

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Abstract: In this article I suggest some alternatives to the solution test for the understanding of graphs of inequalities in two variables. Based on the work done with two groups of preservice secondary mathematics teachers, and by using the idea of variation and the framework of action and process conceptions of inequalities, I explain how graphs of inequalities can be viewed as a collection of rays or curves. I also explain the potential benefits of these alternative explanations in the solving of optimization problems and in the instruction of functions as a medium of one variable and two variable functions.
Keywords: Inequality; Graph; Variation; APOS theory; Preservice teachers

1 Introduction

Mathematical inequalities are important in mathematics due to their connections to other mathematical topics, such as mathematical equations, and their applications to real-life situations. There has however been a general lack of attention from the mathematics education community on mathematical inequalities, despite the community's awareness of their importance (Boero & Bazzini, 2004; Halmaghi, 2011; Vaiyavutjamai & Clements, 2006). Further, most of the studies on inequalities center on the understandings and/or difficulties related to the problem solving of algebraic inequalities in one variable with no graphical representations involved (Almog & Ilany, 2012; Schriber & Tsamir, 2012; Tsamir & Bazzini, 2004). A few studies that involve graphs utilize the graphs of one variable functions to explain the solutions of inequalities in one variable—for example, graphs of $y=f(x)$ and $y=g(x)$ for the solutions of the inequality, $f(x)>g(x)$ (Dryfus & Eisenberg, 1985; Verikios & Farmaki, 2010). Research that concerns

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graphical representations themselves or the connections between algebraic and graphical representations of inequalities in two variables is almost non-existent to this date, despite the importance of graphs in mathematical understanding (Even, 1998; Leinhardt, Zaslavsky & Stein, 1990). In fact, there were only 20 articles found (since 1998) in the Eric search engine with the command "inequalities, graphs, mathematics," and of those, only one article by Switzer (2014) attended to the graphical representations of inequalities in two variables.

In many secondary and post-secondary algebra textbooks (see, for example, David et al., 2011; McKeague, 2008), graphs of algebraic inequalities in two variables are explained through either the one-point solution test or its variation that involves multiple points. For a graph of $y<f(x)$, for example, an individual performs a series of steps: drawing a graph of $y=f(x)$; selecting a point from one of two regions divided by the graph of $y=f(x)$; and substituting the x and y coordinates of the point into the inequality $y<f(x)$ to determine the truth value of the inequality at the coordinates. If the inequality is true at the coordinates, the entire region from which the point is selected is the graph of $y<f(x)$. If not, the other region is the graph of $y<f(x)$. Its variation using multiple points instead of one is similar. An individual selects multiple points, tests the truth values for multiple coordinates, and determines the graph as the region from which the coordinates of the selected points make the inequality $y<f(x)$ true.

Whether it uses one point or multiple points, the solution test promotes instrumental understanding rather than relational understanding (Skemp, 1976), giving instructions on "what to do" to represent graphs of inequalities (instrumental understanding), but falling short of providing the mathematical "reasoning of why" graphs should be so (relational understanding). In addition, this explanation method can potentially impede students' sound development of conceptions about mathematical proof. As shown in several studies, many students and teachers erroneously derive the truth-value of a mathematical sentence from the truth-value(s) of one or more of particular cases (see Harel & Sowder, 1998, for example). This solution test is not much different from such an error in that it determines the truth value of an inequality $y<f(x)$, with x and y representing variables, from one or more of the truth values of $b<f(a)$, with a, b representing constant coordinates.

The goal of this paper is to suggest different ways to explain the graphical representations of inequalities in two variables, $R(x,y)<0$ or $R(x,y)\leq 0$, the kinds of explanations that can promote relational understanding. The ideas presented are based on the results from my investigation with two groups of preservice secondary mathematics teachers, combined with the idea of variation and the framework of the action, process and objective perspectives (Breidenbach, Dubinsky, Hawks, & Nicholas, 1992). This paper also discusses the applications of alternative explanations and implications for future practice and research.

2 Framework of action, process, and object perspectives

The theoretical framework I use in this paper is the APOS theory, which has been widely used for research and curriculum development in undergraduate mathematics (Asiala, Brown, Devries et al., 1996; Breidenbach, Dubinsky, Hawks, & Nicholas, 1992; Cottrill et al., 1996; Dubinsky & Harel, 1991). In particular, I primarily use the action and process conceptions for inequalities to explain graphs of inequalities, $R(x,y)<0$ or $R(x,y)\leq 0$.

An *action* is a mental or physical manipulation of objects that can transform one object to another, which is somewhat external (Breidenbach et al., 1992). It can be a single-step action, such as recalling memorized facts, or a multi-step action that involves a number of steps without conscious control of the transformation (Cottrill et al., 1996). When actions are repeated and interiorized, the actions collectively become a *process* (Breidenbach et al., 1992). That is, an internal representation of the same actions is constructed in an individual's mind, but not necessarily with extra stimuli.

In working with the graph of a one variable function $y=f(x)$, an individual with the action conception has the ability to plug in numbers for x to find the corresponding values for y by applying necessary actions—for example, multiplying 2 to the values of x and then adding 1 to the results in the case of the function, $f(x)=2x+1$. The individual may also have an ability to convert the results of the actions above to ordered pairs, represent the pairs as points in the Cartesian plane, and connect them to sketch a graph of the function $y=f(x)$. It is also possible that an individual may simply come up with a shape from memory associated with the function expression $y=f(x)$.

With the process conception, an individual understands that a graph of $y=f(x)$ is the result of the representational transfer from ordered pairs to points in the plane via *the Cartesian connection*—'a point is on the graph of a line if, and only if, its coordinates satisfy the line equation' (Moschkovich et al., 1993). Unlike an individual with only the action perspective, who may associate the graph of $y=f(x)$ to a shape envisioned from memorized facts or see the graph as a shape formed by connecting a few plotted points, an individual with the process perspective of function acknowledges the dynamic nature of variables—i.e. variation which denotes the idea of variables as varying objects—in the graph of the function $y=f(x)$, and hence understands the presence of "all" points in the graph of the function, which is the result of the "running through" a continuum of points (see Herscovics & Linchevski, 1994; Leinhardt, Zaslavsky, & Stein, 1990; Thompson, 1994, for the description of variables).

In regards to two variable functions, recent research unravels students' understanding of graphs of two variable functions with the lens of the APOS theory or covariational reasoning (Trigueros & Martinez-Planell, 2010; Weber & Thompson, 2014). Trigueros and Martinez-Planell (2010) showed that most students could not internalize actions into processes and had difficulty visualizing graphs of two variable functions in subsets of the domain. Students could not flexibly connect algebraic and graphical objects other than points in the 3-dimensional space and were unable to explain the plane $y=x+z+3$ as a set of points despite their knowing of the equation as a plane from their memorized facts. They also could not explain the graph of $z = x^2+y^2$ at $y=3$ as the intersection of the paraboloid $z = x^2+y^2$ and the plane $y=3$, even though they knew from their memorized facts that the former was a paraboloid and the latter was a plane.

The action and process conceptions related to graphs of two variable functions are also observed in the work of Weber and Thompson (2014), although the researchers used covariational reasoning—seeing a function as a covarying relationship among quantities involved in the function relation—as a framework. The researchers in particular investigated how students' understanding of graphs of one-variable functions influenced their understanding of two-variable functions.

In the case of one student, Jesse, his understanding of the graph of one-variable function as a sweeping out of points in the plane—showing covariational reasoning and process conception—helped him generalize his understanding to the graph of two-variable functions—as a sweeping out of curves in 3-dimensional space. For example, in the case of a one-variable function, $y=f(x)=a(x^2-2x)$, with a acting as a parameter representing a distance of the graph from the x-y plane, his understanding of the graph of the one-variable function, $y=x^2-2x$, as a collection of points in a plane helped him see the graph of $f(x)=a(x^2-2x)$ as a vertical stretching of the points on the graph of $y=x^2-2x$ and at the same time as a moving object along the a-axis, with a working as a variable. Yet in the case of another student, Lana, her lack of process conception of a graph of a function—seeing a graph as a picture associated with an algebraic equation and the graph of $y = a(x^2 - 2x)$ as a graph of parabola that "molds differently" with the value of a changed one at a time (an action perspective)—confounded her visualization of two variable functions. Although she referred to a as a parameter, she could not visualize the graph of $f(x) = z(x^2 - 2x)$ as a sweeping out of the parabolas, $y = a(x^2 - 2x)$.

As shown above, memorization of graphs in the lower dimensions do not aid the understanding of graphs in the higher dimensions. When the understanding of variation and covariation is combined with the action perspective, the actions can potentially be interiorized into processes. In this respect, there is a need for alternative explanations for graphs of inequalities, which can replace the current method of teaching, the solution test. In fact, the solution test mainly promotes the action perspective by testing the truth value(s) for inequalities at one or multiple points, and the test has no specific emphasis on variation or process and objective perspectives.

3 Alternative ways to explain graphs of inequalities in two variables

The alternative ways I suggest here are based on the works of two groups of preservice secondary mathematics teachers in the U.S., incorporating the action and process conceptions of inequalities. One group consisted of 14 preservice teachers in a post-baccalaureate, teacher education program at a large research university in the West and the other group consisted of 15 preservice teachers on an undergraduate secondary teaching track at a small doctoral comprehensive university in the Southeast. For the former group, I analyzed their individual portfolios on the *Cookies* unit, which focused on the understanding and solving of real-life problems by using graphs of linear equations and inequalities, from a high school mathematics curriculum, *Interactive Mathematics Program*. Having been educated in a classroom where relational understanding was particularly emphasized, the preservice teachers were expected to explain critical ideas embedded in the problems and to solve the problems with detailed explanations on why they used certain procedures or ideas in their problem solving.

Despite having taken the class in which relational understanding was particularly emphasized, ten of the fourteen teachers did not explain graphs of inequalities at all. Two of the remaining four explained graphs only by using the solution test, while the other two explained graphs by using the solution test and the idea of variation. The two preservice teachers who explained the graphs by using variation and the solution test mentioned "near-by points" as part of their arguments. For example, one wrote, "a big idea that students realize is that 'near-by points in the feasible region also satisfy inequalities.' This big idea lets the students see the reason why it is necessary to test only one point in order to find the feasible reason."

I interviewed the latter group with three inequality problems (linear, parabolic, and circular) as part of a larger project that investigated big ideas in algebra. In the interviews, I asked them to represent the solutions of three algebraic inequalities— $x+2y-32<0$, $y<x^2+1$, and $x^2+y^2>1$—in the Cartesian plane and to explain why their graphs made sense. A complete analysis of their work is shown in a different paper (see BLINDED). In summary, only 2 of them represented all three graphs correctly—8 successfully providing the linear inequality graph, 5 the circular inequality graph, and 2 the parabolic inequality graph. For the explanations of why their graphs made sense, most of those who provided correct graphs used the solution test or an argument that "less than means lower part." There were some explanations that varied from the above, which were related in part to variation or the action and process perspectives. For the graph of $x^2+y^2>1$, three of the teachers mentioned that “any point” chosen outside of the circle $x^2+y^2=1$ would satisfy $x^2+y^2>1$ because the equation $x^2+y^2=c$, with c greater than 1, would represent a circle with the radius greater than 1, thus showing some understanding that the graph of $x^2+y^2>1$ is a collection of circles with the radii greater than 1 (as the result of a sweeping out process of circles). For $x+2y–32<0$ or $y<x^2+1$, three of them considered infinitely many values as the solutions of algebraic inequalities by taking an action of fixing x or y values as a constant, by saying “y-values have to be less than16 when x is 0” in the case of $x+2y–32<0$ or “if y is 1, then 1 is less than x square plus 1, which makes x greater than 0” (which had to be x^2 greater than 0 instead) in the case of $y<x^2+1$.

Using the ideas embedded in the works of the preservice secondary teachers for graphs of inequalities—variation, the action of fixing the x or y constant, and the process of sweeping out—and by complementing the gap in their understanding, I hereby suggest alternative ways to explain graphs of inequalities in two variables. I use the inequality $y<x+1$ as an example, and the graphs of other inequalities in two variables, in the form of $R(x,y)>0$ or $R(x,y)\geq 0$, can be explained similarly.

Graphs of $y < x+1$

The graph of the inequality $y<x+1$ can be seen in the following ways (shown in Figure 1(*a*), (*b*), (*c*), respectively):

A. As a collection of vertical rays, $x=c$ and $y<c+1$, if the x variable is kept constant,
B. As a collection of horizontal rays, $y=c$ and $c–1<x$, if the y variable is kept constant,

C. As a collection of lines, $y–x=c$, with $c<1$, if $y–x$ is kept constant.

Figure 1(a). In order to see the graph of $y<x+1$ as in Figure 1(a), an individual needs to be able to do or understand the following:

- An individual assigns a value for the x variable and obtains infinitely many y values that satisfy $y<x+1$ (the action conception of inequalities in algebraic representation). This is an action similar to that involved in the graphical representation of $y=x+1$. However, instead of having one corresponding y value for each fixed x value in the case of $y=x+1$, the individual has infinitely many y values that satisfy $y<x+1$. The individual then represents the x and y values as ordered pairs in the Cartesian plane, which is represented as an open ray. This conversion still involves the action conception, but requires the connection between algebraic and geometric representations. As an example, when an individual assigns the value 0 for x, the individual has infinitely many y values that satisfy y<0+1. The individual then represents (0, y) with $y<0+1$ as an open vertical ray on the y-axis with the end point, (0, 0+1).
- The individual then repeats similar actions multiple times for other assigned values of x and realizes that for each assigned value of x, there are infinitely many y values satisfying the inequality $y<x+1$, which are represented as an open vertical ray with the end point (x, x+1) on the Cartesian plane. After repeating such actions, the individual then interiorizes these actions into a process and visualizes the graph of the inequality $y<x+1$ as a collection of all vertical open rays, which is an open half-plane with the boundary line of $y=x+1$. This idea is similar to that of Weber and Thompson (2014),

which interprets a three-dimensional surface as a sweeping out of two-dimensional curves in the fundamental curves ($x=c$ or $y=c$). For graphs of inequalities in two variables, the graph is the result of a sweeping out process of rays in the fundamental lines ($x=c$).

Figure 1(b). This case works similarly to Figure 1(a) and is left to the reader.

Figure 1(c). In order to see the graph of $y < x+1$ as in Figure 1(c), an individual needs to be able to do or understand the following:

- An individual sees $y<x+1$ as an equivalent statement to $y–x<1$. Instead of taking an action of assigning a value for x or y variables as in Figures 1(a) or (b), the individual in this case takes an action of fixing $y–x$ as a constant less than 1. For example, the individual assigns the value of 0 for $y–x$ and then visualizes the graph of $y–x=0$ as a line passing through the origin.
- The individual repeats the same actions for other values, such as $y–x= –1$ and $y–x = –2$ and visualize them as lines. The individual then can interiorize these actions into a process without externally representing them all, and sees the graph of $y–x<1$ as a collection of those lines, $y=x+b$ with $b<1$, which forms the lower half of the line graph of $y=x+1$.

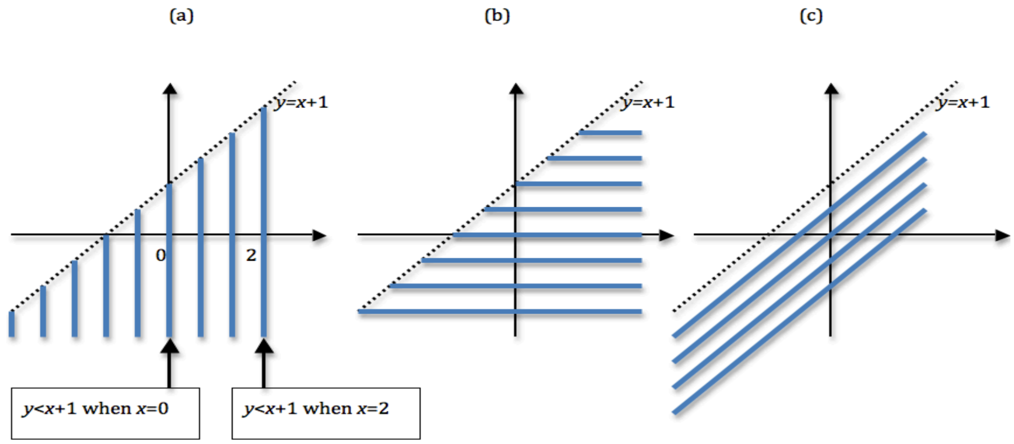

Figure 1 Graph of Inequality, $y < x+1$

4 Applications of the alternatives in mathematical problem solving

The line of thinking embedded in the explanations can offer different kinds of understanding for prospective teachers and calculus students. As an example, consider a problem that asks the maximum profit of sale, represented as a two-variable function, $f(x,y)=x+2y$, with constraints $x+y\leq 6$, $2x+y\leq 10$, $0\leq x$, and $0\leq y$, a similar problem to those in the *Cookies* unit in the reform-based high school curriculum, *Interactive Mathematics Program*.

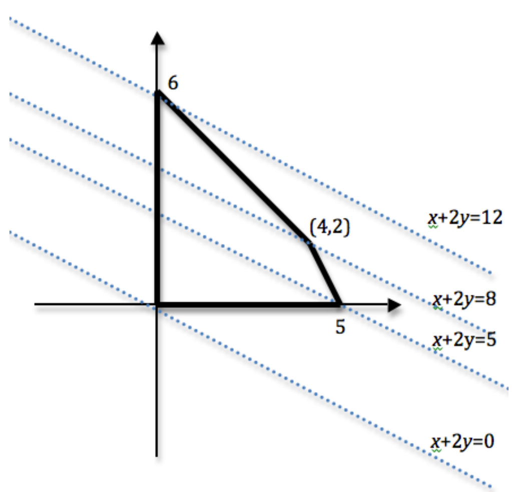

Figure 2 Maximization of $f(x,y)=x+2y$ with Constraints, $x+y\leq 6$, $2x+y\leq 10$, $0\leq x$, and $0\leq y$, Using Level Curves

One way to solve the problem, which is also shown in the *Cookies: Teacher's Guide* (Fendel et al., 1998) is to find the region that meets all constraints by using the solution test for each of the four inequalities and then by running a family of lines $x+2y=k$ over the constrained region. The core idea behind this strategy is the Cartesian Connection (Moschkovitch et al., 1993), which stipulates that for every point on the line graph of $x+2y=k$, its x and y values satisfy the equation $x+2y=k$ and thus yield the same profit of k. That is, the profit function $f(x,y)=x+2y$ has the value 0 when (x,y) is on the line graph of $x+2y=0$, and has the value 5, 8, and 12 when (x,y) is on the line graph of $x+2y=5$, 8, and 12, respectively (see Figure 2). As a result, the maximum profit with the given constraints is 12.

An alternative way to solve this problem is to use the line of thinking involved in the action and process conceptions for graphs of inequalities, which sees the constrained region as a collection of vertical or horizontal segments by fixing x or y as a constant. In this case, an individual can consider the values of $f(x,y)=x+2y$ on each of the vertical (or horizontal) line segments by keeping x (or y) constant, for example $x=1$, and realizes that the maximum of $f(1,y)=1+2y$ on that segment occurs when y is the greatest, which happens at the upper boundary point of the segment (see Figure 3).

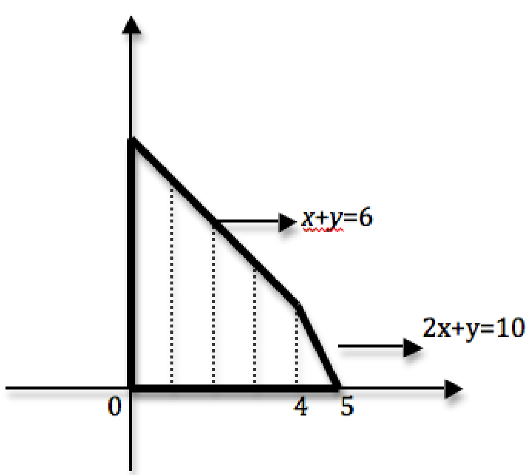

Figure 3 Maximization of $f(x,y)=x+2y$ with Constraints, $x+y\leq 6$, $2x+y\leq 10$, $0\leq x$, and $0\leq y$, Using the Action, Process, and Object perspectives

The individual then repeats the same actions for other values of x and can interiorize the actions into a process and realizes that for any vertical segment, the maximum occurs at the upper boundary point, which is in fact on the line $x+y=6$ when $0\leq x\leq 4$, and on the line $2x+y=10$ when $4\leq x\leq 5$. That is, the original problem becomes the maximization problem of $f(x,y)=x+2y$ with the constraint $x+y=6$ when $0\leq x\leq 4$, and with the constraint $2x+y=10$ when $4\leq x\leq 5$. When $0\leq x\leq 4$, $f(x,y)=f(x,6-x)=x+2(6-x)=12-x$ takes values between 8 and 12, inclusive, as $x+y=6$ is equivalent to $y=6-x$. When $4\leq x\leq 5$, $f(x,y)=f(x,10-2x)=x+2(10-2x)=20-3x$ takes values between 5 and 8, inclusive, as $2x+y=10$ is equivalent to $y=10-2x$. As such, the maximum profit of $f(x,y)=x+2y$ is 12, which occurs at $x=0$ and $y=6$. This problem can also be done by fixing y as a constant and by seeing the constrained region as a collection of horizontal line segments. As the steps involved in this process are similar to those above, the details are left to the reader.

Another example is from a multivariable calculus book by Larson, Hostetler, and Edwards (2006), which finds the extreme values of $f(x,y)=x^2+2y^2-2x+3$ with a constraint, $x^2+y^2\leq 10$. The solution in the book utilizes the Lagrange multipliers to find the maximum and minimum values of the function f on the circle boundary, uses the first and second partial derivatives to find relative maximum and minimum for the inside of the circle, and combines the two results to determine the maximum of 24 and the minimum of 2 (see p. 972 for a detailed explanation).

This problem can also be solved when the constrained region is viewed as a collection of vertical line segments (or horizontal line segments), with mathematical understanding at the secondary level. When an individual considers $f(x,y)=x^2+2y^2-2x+3$ at a constant value of x, for example $x=1$, the constraint $x^2+y^2\leq 10$ at $x=1$ becomes$1^2+y^2\leq 10$, or equivalently $-3\leq y\leq 3$, which is geometrically representing the closed line segment connecting (1,-3) and (1,3) (see Figure 4). As such, $f(1,y)=1^2+2y^2-2+3=2y^2+2$ has the maximum value of 20 at y=3 or -3 (i.e., at the end points of the line segment) and the minimum value of 2 at y=0 (at the point on the horizontal diameter) on that segment. When the individual repeats the same actions for other vertical segments and can interiorize the actions into a process, she can find that the function $f(x,y)=x^2+2y^2-2x+3$ must have the maximum on some points on the circle and the minimum on some points on the horizontal diameter.

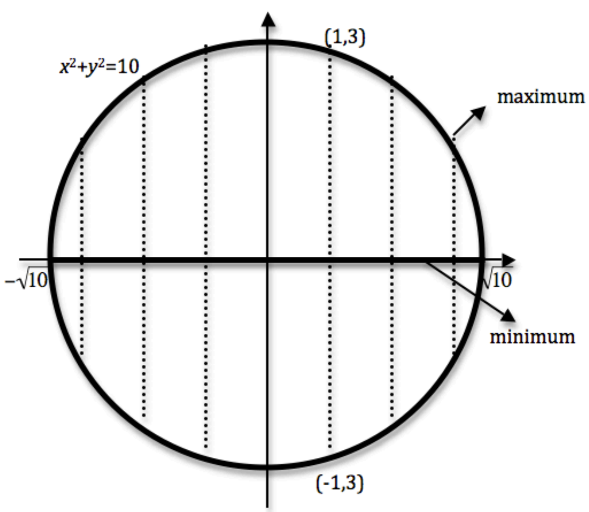

Figure 4 Maximization of $f(x,y)=x^2+2y^2-2x+3$ with a Constraint, $x^2+y^2\leq 10$, Using the Action, Process, and Object Perspectives

In order to determine on what points of the circle $f(x,y)=x^2+2y^2-2x+3$ has the maximum, the individual rewrites $f(x,y)= x^2+2y^2-2x+3=x^2+2(10-x^2)-2x+3=-x^2-2x+23=-(x+1)^2+24$, using the constraint $x^2+y^2=10$ (or equivalently, $y^2=10-x^2$) and by completing the square, and finding the maximum of 24 at x=-1. For the minimum, the individual considers $f(x,y)=x^2+2y^2-2x+3$ at points on the horizontal diameter, represented as y=0 with $-\sqrt{10} \leq x \leq \sqrt{10}$. She then writes

$f(x,0)=x^2+2y^2-2x+3= x^2-2x+3=(x-1)^2+2$ by completing the square and finds the minimum of 2 at $x=1$.

5 Discussion and conclusion

In this article, I suggested alternative explanations to the solution test for graphs of algebraic inequalities in two variables by using the framework of action and process conceptions by Dubinsky and others (Breidenbach et al., 1992). Unlike the solution test, which basically provides multi-step instructions on how to graph inequalities in two variables, the suggested alternatives offer a relational understanding (Skemp, 1976) for graphs of inequalities in two variables by incorporating the critical concept of the variable (see for example, Leinhardt et al., 1990, for the concept of variable). Further, the line of thinking embedded in the alternatives could bring a different kind of understanding and solving for optimization problems with constraints (with constraints represented as inequalities), by enabling an individual to see graphs of inequalities locally and globally, similar to the ideas discussed in the function concept by various researchers (see for example, Even, 1998; Weber & Thompson, 2014).

One potential benefit of the alternatives is that they may be used as the medium in instruction, transitioning from graphs of one-variable functions to graphs of two variable functions. To elaborate, in order to graph such functions and inequalities, a learner performs an action of fixing a variable as a constant value, $x=c$ for example, and then finds the value of the other variable in the case of $y=f(x)$ or the relationship between the other variables for the case of $z=f(x,y)$. Such an action then yields a geometric object—a point $(c,f(c)$ in a plane in the case of

$y=f(x)$; a ray, which is the graphical representation of $\{(c, y)|y<f(x)\}$, in a plane in the case of $y<f(x)$; or a curve $z=f(c,y)$ in a 3-dimensinal space in the case of $z=f(x,y)$. The graph is then a collection of all geometric objects, with the x variable running through a continuum of values in the domain. In this regard, the alternatives provide a consistency in mathematical thinking through the concept of the variable and the process perspective of functions and inequalities. They thus have great potential to help students strengthen their understanding of the role of variables in graphs of functions, which researchers have shown many students to lack (Breidenbach et al., 1992; Demana, Schoen, & Waits, 1993; Even 1998; Weber & Thompson, 2014).

The ideas proposed here are my suggestions based on preservice teachers' understanding of graphs of inequalities and some research on the graphs of one variable and two variable functions, but the effects of implementations using the alternatives are yet to be determined. Future research should examine students' understanding of and difficulties in understanding graphs of inequalities using the alternatives, as well as the effects of alternatives on students' understanding of one variable and two variable functions.

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