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Petr Eisenmann

Martin Kuril

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Number series and computer

Petr Eisenmann[1] and Martin Kuřil
Jan Evangelista Purkyně University in Ústí nad Labem, Czech Republic

Abstract: This article describes the discovery and the subsequent proof of a hypothesis concerning harmonic series. The whole situation happened directly in the course of the process of a maths lesson. The formulation of the hypothesis was supported by the computer. The hypothesis concerns an interesting connection between harmonic series and the Euler's number e. Let H_n denote the n'th partial sum of the harmonic series. Let's notice the sums H_1, H_4, H_{11}, . . ., where the partial sum reaches 1, 2, 3, . . . for the first time. Let's mark the relevant indexes 1, 4, 11, . . . as p_1, p_2, p_3, ... So p_n is the index of such a partial sum for which the following is true: $H_{p_n-1} < n$, $H_{p_n} \geq n$. The hypothesis $\lim \frac{p_{n+1}}{p_n} = e$ has been proved, in an elementary way, in the article.

Key Words: harmonic series, partial sums, Euler's number, undergraduate mathematics course, computer use in mathematics education

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[1] Petr.Eisenmann@ujep.cz

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In this contribution we intend to describe a fascinating situation that occurred directly in an undergraduate course in the first year of university studies, in a situation when we together with the students discovered and afterwards also proved an interesting hypothesis. The discovery of the hypothesis needed a computer.

It all began when we were introducing the concept of infinite number series in the initial mathematics course. The concept of the sum of convergent series was introduced through the usual definition as a proper limit of the sequence of partial sums of the series. We also showed the typical examples of convergent and divergent series, especially geometric ones. We discussed the necessary condition of convergence of series $\sum_{n=1}^{\infty} a_n$, i.e. the condition that $\lim_{n\to\infty} a_n = 0$. This condition is met also by the notoriously well-known and very important harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots.$$

On the basis of numerical experimentations on the calculator the students tried to guess whether the series was convergent or divergent. They discovered that the sequence of partial sums of the harmonic series

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

increased very slowly. Using the computer we showed that e.g. $H_{1000} = 7.48$ ... or $H_{1000000} = 14{,}39$ ... . In literature we also came across the following fact: It takes more than $1.5 \cdot 10^{43}$ terms for the partial sum to reach 100 (e.g., Boas & Wrench, 1971, Table 1). We could not infer any conclusions about convergence of the harmonic series. That is why we presented the conclusion to the students and showed them one of the usual proofs of divergence of this series.

When studying the following table with the terms of the sequence H_n the students noticed the following interesting phenomenon.

$H_1 = 1.0000000$	$H_6 = 2.4500000$	$H_{11} = 3.0198773$
$H_2 = 1.5000000$	$H_7 = 2.5928517$	$H_{12} = 3.1032107$
$H_3 = 1.8333333$	$H_8 = 2.7178571$	$H_{13} = 3.1801338$
$H_4 = 2.0833333$	$H_9 = 2.8289683$	$H_{14} = 3.2515623$
$H_5 = 2.2833333$	$H_{10} = 2.9289683$	...

We were attracted by the sums H_1, H_4, H_{11}, . . ., where the partial sum reaches 1, 2, 3, . . . for the first time. We designated the relevant indices 1, 4, 11, . . . as p_1, p_2, p_3,... . So p_n is the index of such a partial sum of the harmonic series, for which the following is true:

$$H_{p_n-1} < n, H_{p_n} \geq n. \quad (1)$$

The calculator, however, is not a sufficient tool for obtaining further values of the members p_n of the sequence $(p_n)_{n=1}^{\infty}$, since this sequence grows very quickly. For that reason we used the computer and succeeded to determine the first 18 members of the sequence $(p_n)_{n=1}^{\infty}$:

1	227	33617	4989191
4	616	91380	13562027
11	1674	248397	36865412
31	4550	675214	...
83	12367	1835421	

The students were studying the numbers in the table for quite a long time when student Jan exclaimed: ”It’s always three times more, the value of the next term.” His classmate Marie replied: ”No, that’s not true, it’s four times more in the beginning, and also later it is one time more and one time less than a treble!” This was followed by a short discussion which other students entered. Then student Petr suggested the following: ”Let’s calculate it, let the computer enumerate the quotients.” This gave rise to the following table in whose columns there were the quotients $\frac{p_{n+1}}{p_n}$:

4	2.7136 ...	2.7182675 ...
2.75	2.7175 ...	2.7182862 ...
2.8181 ...	2.718040 ...	...
2.6774 ...	2.718021 ...	
2.7349 ...	2.7182825 ...	

It was in utmost surprise in which we were studying the terms of the sequence. After some time student Martin burst out: ”But this is the Euler’s number, this 2.7 and so on, this is the base of natural logarithm!” That is why we formulated the following hypothesis:

$$\lim_{n\to+\infty} \frac{p_{n+1}}{p_n} = e. \tag{2}$$

The question was if the hypothesis was a true hypothesis and if we would be able to prove it with our students.

Let us now make a digression from our narration of what happened in the seminar and let us look into mathematics literature.

In the well-known book (Graham, Knuth, & Patashnik, 1994) we can find Exercise 49 (p. 493):

Prove that if n is a positive integer such that $H_{n-1} \leq \alpha \leq H_n$, then

$$\lfloor e^{\alpha-\gamma} \rfloor \leq n \leq \lceil e^{\alpha-\gamma} \rceil .$$

The solution of the exercise is given on page 600.

The Greek letter γ denotes here the Euler’s constant,

$$\gamma = \lim_{n\to+\infty} (H_n - \ln n) = 0.5772156649 \dots .$$

We have $H_{p_k-1} < k \leq H_{p_k}$. We put $n = p_k$, $\alpha = k$ and obtain

$$\lfloor e^{k-\gamma} \rfloor \leq p_k \leq \lceil e^{k-\gamma} \rceil . \tag{3}$$

With (3) the value of p_k is determined almost exactly. There is only one question left: whether p_k is equal to $\lfloor e^{k-\gamma} \rfloor$ or to $\lceil e^{k-\gamma} \rceil$. This question is studied in (Boas & Wrench, 1971, p. 865), and also in (Baxley, 1992, p. 311).

It follows from (3) that

$$e^{k-\gamma} - 1 \quad < \quad p_k \quad < e^{k-\gamma} + 1$$

$$\begin{array}{ccccc} e^{k+1-\gamma}-1 & < & p_{k+1} & < & e^{k+1-\gamma}+1 \\ \dfrac{1}{e^{k-\gamma}+1} & < & \dfrac{1}{p_k} & < & \dfrac{1}{e^{k-\gamma}-1} \\ \dfrac{e^{k+1-\gamma}-1}{e^{k-\gamma}+1} & < & \dfrac{p_{k+1}}{p_k} & < & \dfrac{e^{k+1-\gamma}+1}{e^{k-\gamma}-1}. \end{array}$$

Now,

$$\lim_{k\to\infty}\frac{e^{k+1-\gamma}-1}{e^{k-\gamma}+1}=\lim_{k\to\infty}\frac{e^{1-\gamma}-\frac{1}{e^k}}{e^{-\gamma}+\frac{1}{e^k}}=\frac{e^{1-\gamma}}{e^{-\gamma}}=e,$$

$$\lim_{k\to\infty}\frac{e^{k+1-\gamma}+1}{e^{k-\gamma}-1}=\lim_{k\to\infty}\frac{e^{1-\gamma}+\frac{1}{e^k}}{e^{-\gamma}-\frac{1}{e^k}}=\frac{e^{1-\gamma}}{e^{-\gamma}}=e.$$

Using the Sandwich limit theorem we obtain

$$\lim_{k\to\infty}\frac{p_{k+1}}{p_k}=e.$$

But this computation is based on the relation (3) derived in (Graham, Knuth, & Patashnik, 1994) with the use of Euler's summation formula which are nontrivial means.

Similarly, Euler's formula is used in (Boas & Wrench, 1971). In (Baxley, 1992) Taylor's formula is used.

Let us now come back to the classroom and let us show a short and elementary proof of our hypothesis (2).

Looking at it more closely, this result is not so surprising, as we realize the connection between the partial sums of the harmonic series and the integral

$$\int_1^A \frac{dx}{x}=\ln A.$$

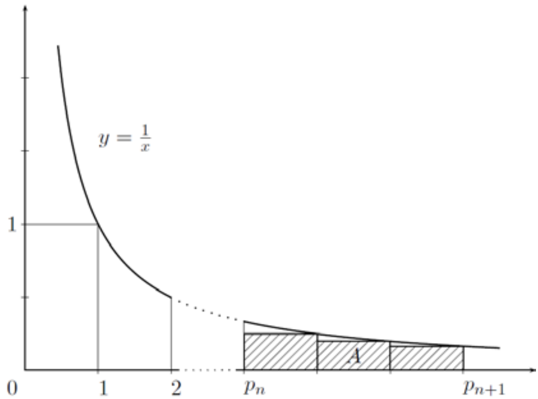

Figure 1. Rectangles inscribed in the graph of the function $\frac{1}{x}$

Let's denote the sum of the areas of the rectangles inscribed in the graph of the function $y = \frac{1}{x}$ with A (see Figure 1).

It holds obviously that

$$A = \frac{1}{p_n + 1} + \cdots + \frac{1}{p_{n+1}} < \int_{p_n}^{p_{n+1}} \frac{dx}{x} = \ln \frac{p_{n+1}}{p_n}.$$

If we set

$$B = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p_n} + \cdots + \frac{1}{p_{n+1}}$$
$$C = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p_n}$$

it holds according to (1) that

$$B \geq n + 1$$
$$C < n + \frac{1}{p_n}.$$

Thus

$$A = B - C > (n+1) - \left(n + \frac{1}{p_n}\right) = 1 - \frac{1}{p_n}. \qquad (4)$$

Let's denote in a similar way the sum of the rectangles circumscribed to the graph of the function $y = \frac{1}{x}$ with D (see Figure 2).

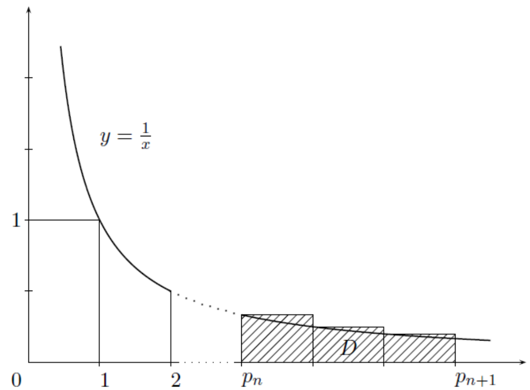

Figure 2. Rectangles circumscribed to the graph of the function $\frac{1}{x}$

It holds obviously that

$$\int_{p_n}^{p_{n+1}} \frac{dx}{x} = \ln \frac{p_{n+1}}{p_n} < D = \frac{1}{p_n} + \frac{1}{p_n + 1} + \cdots + \frac{1}{p_{n+1} - 1}.$$

If we set

$$E = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p_{n+1} - 1}$$
$$F = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p_n - 1},$$

it holds according to (1) that

$$E < n + 1$$
$$F \geq n - \frac{1}{p_n}.$$

Thus

$$D = E - F < (n+1) - \left(n - \frac{1}{p_n}\right) = 1 + \frac{1}{p_n}. \qquad (5)$$

As for every natural number n it holds that

$$A < \ln \frac{p_{n+1}}{p_n} < D,$$

we get according to (4) and (5)

$$1 - \frac{1}{p_n} < \ln \frac{p_{n+1}}{p_n} < 1 + \frac{1}{p_n}. \qquad (6)$$

Because the sequence $\frac{1}{p_n}$ converges to zero, out of the Sandwich limit theorem it follows that

$$\lim_{n\to\infty} \ln \frac{p_{n+1}}{p_n} = 1$$

and the hypothesis (2) is therefore proved.

Further, let's have a look at the problem of how to express p_{n+1} approximately knowing the value p_n. It follows from (6) that

$$p_n \cdot e^{1-\frac{1}{p_n}} < p_{n+1} < p_n \cdot e^{1+\frac{1}{p_n}}.$$

What can we say about the size of the interval $\left(p_n \cdot e^{1-\frac{1}{p_n}}, p_n \cdot e^{1+\frac{1}{p_n}}\right)$? We derive, using l'Hospital's rule,

$$\begin{aligned}\lim_{x\to+\infty} x \cdot \left(e^{1+\frac{1}{x}} - e^{1-\frac{1}{x}}\right) &= \lim_{x\to+\infty} \frac{\left(e^{1+\frac{1}{x}} - e^{1-\frac{1}{x}}\right)}{\frac{1}{x}} \\ &= \lim_{x\to+\infty} \left(e^{1+\frac{1}{x}} + e^{1-\frac{1}{x}}\right) \\ &= 2e.\end{aligned}$$

Thus

$$\lim_{n\to\infty} \left(p_n \cdot e^{1+\frac{1}{p_n}} - p_n \cdot e^{1-\frac{1}{p_n}}\right) = 2e.$$

Let us present an example:

$$p_{21} = 740461601,\ \ p_{21} \cdot e^{1-\frac{1}{p_{21}}} = 2012783311.9\ldots,\ \ p_{21} \cdot e^{1+\frac{1}{p_{21}}} = 2012783317.3\ldots$$

Consequently, $2012783312 \leq p_{22} \leq 2012783317$. The exact value of p_{22} is 2012783315. We found the values p_{21} and p_{22} in (The On-Line Encyclopedia of Integer Sequences, 2010, Sequence A004080).

At the end we would like to mention an interesting result obtained by Sierpiński (1956):

Every positive integer n can be expressed in the form

$$n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_r},$$

where $1 < 2 < 3 < \cdots < k < k_1 < k_2 < \cdots < k_r$ and $\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k} < n$, $n \leq \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k+1}$ (we see that $k + 1 = p_n$).

For example, $p_3 = 11$,

$$3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{15} + \frac{1}{230} + \frac{1}{57960}.$$

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