The Mathematics Enthusiast

Volume 16
Number 1 *Numbers 1, 2, & 3*

Article 17

2-2019

Revisiting Multiplication Area Models for Whole Numbers

Oh Hoon Kwon

Ji-Won Son

Ji Yeong I

Revisiting Multiplication Area Models for Whole Numbers

Oh Hoon Kwon[1]
University of Wisconsin Madison

Ji-Won Son
University at Buffalo

Ji Yeong I
Iowa State University

Abstract: We argue that there are two conceptually different area models for whole number multiplication: area-to-area model and lengths-to-area model depending on how multiplication operation is conceived: action on / change of an initial quantity, or operation / coordination of two quantities. These models complement each other to promote students' conceptual understanding of whole number multiplication and help students extend their understanding of whole number multiplication to fraction multiplication.

Keywords: Multiplication Area Model, Area Model, Whole Number Multiplication, Operation

Multiplication is a challenging operation in both calculation and justification (Flowers, Kline, & Rubenstein, 2006). Area models help students develop multiplicative reasoning embedded in multiplication and strategies for building number flexibility and the mathematical automaticity. As such, the Common Core State Standards for Mathematics (CCSSM, 2010) highlight the importance of developing students' abilities to representing and interpreting multiplication in various ways including arrays and area models. However, it is not clear yet what essential ideas are embedded in the use of area models in the context of multiplication. In addition, although there are two types of area models for whole number multiplication, these two models are often used without any distinction in current U.S. curricula, which may produce discontinuity of ideas in area models from whole number multiplication to fraction multiplication and to decimal multiplication. In this article, we introduce two area models for whole number multiplication and highlight the characteristics and core ideas of each area model to help teachers support students' conceptual understanding. We first describe the general use of the area models and then discuss each of two area models.

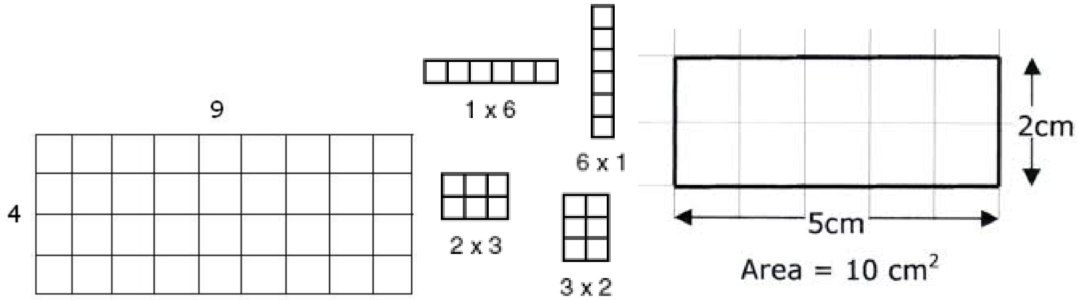
General Use of Area Models for Whole Number Multiplication

The area model is a pictorial way of representing multiplication. Typically in the area model, the length and width of a rectangle represent two factors in whole number multiplication, and the area of the rectangle represents their product. Hence, numbers are used to represent the size or

[1] kwon@math.wisc.edu

The Mathematics Enthusiast, **ISSN 1551-3440, vol. 16, nos.1,2&3**, pp. 359-368

magnitude of sides, and the area of the rectangle with or without the unit of measurement, as shown in Figure 1.



(a) (b) (c)

Figure 1. Typical area (array) models for whole number multiplication

Figure 1 shows three different appearances of area models for whole number multiplication commonly found in US curriculum materials. Although these three representations illustrate the product of each multiplication correctly, it is important to note the different pedagogical transparency of each model. For the size or magnitude of each side, “(number)[unit]” is a typical form. Similarly, “(number)[$\text{unit}_1 \times \text{unit}_2$]” is a typical form for the area of the rectangle. Numbers in (c) show the roles as measures of sides by including a measurement unit, but numbers in (a) and (b) have ambiguity. In (a) and (b), each number could be read as numbers of rows and columns, a number of squares in each side, or 1-dimensional measures of the sides. Interpreting each number as numbers of rows and columns, students can relate the area models (a) and (b) to the equal-group model by circling groups of rows or columns. For example, when the equal-group model is used for whole number multiplication, the first factor typically represents the number of groups (e.g., the number of rows in the area model) and the second factor the size in each group (e.g., the number of squres in a row). However, due to the different roles of two numbers in multiplication, it is unclear how the commutative property of multiplication works when the area model is connected to the equal-group model. It is more challenging when involving a context. For example, 3 plates of 4 apples and 4 plates of 3 apples are different situations unless you are counting the number of apples only, ignoring plates as a unit. This may confuse teachers and students and make it difficult to figure out the connection between the verbal and the pictorial representation of multiplication.

Despite the importance of using different multiplication problem type (CCSSM, 2010), there are few studies on corresponding representations and models and most US curricula present models by just reflecting the appearance (Set model vs. Number line model vs. Area model) or different characters of models themselves (Discrete model vs. Continuous model, or 1-dimensional model vs. 2-dimensional model etc.), not carefully connecting students’ action, thinking, and reasoning process to the nature of whole number multiplication.

Area Model 1: Area-to-area Model

The typical array model gives a 2-dimensional picture (see Figure 1) but focuses on the one or two dimensional *arrangement* of unit area based on the idea of equal groups. We call this traditional array model *the area-to-area model* for multiplication. In Figure 1, (a) and (b) fall into this category.

When students represent multiplication process with the area-to-area model, the model can be 1 or 2 dimensional depending on how they want to use directions of arrangement (see Figure 2). The area-to-area model emphasizes a change of the given initial quantity represented by 1-unit area (a unit area is a representation of 1 unit or 1 unit of measurement). In this model, multiplication is used to figure out how the multitude of the initial quantity, 1-unit area, changes by the whole number operator (multiplier) or scaling factor (see Figure 2).

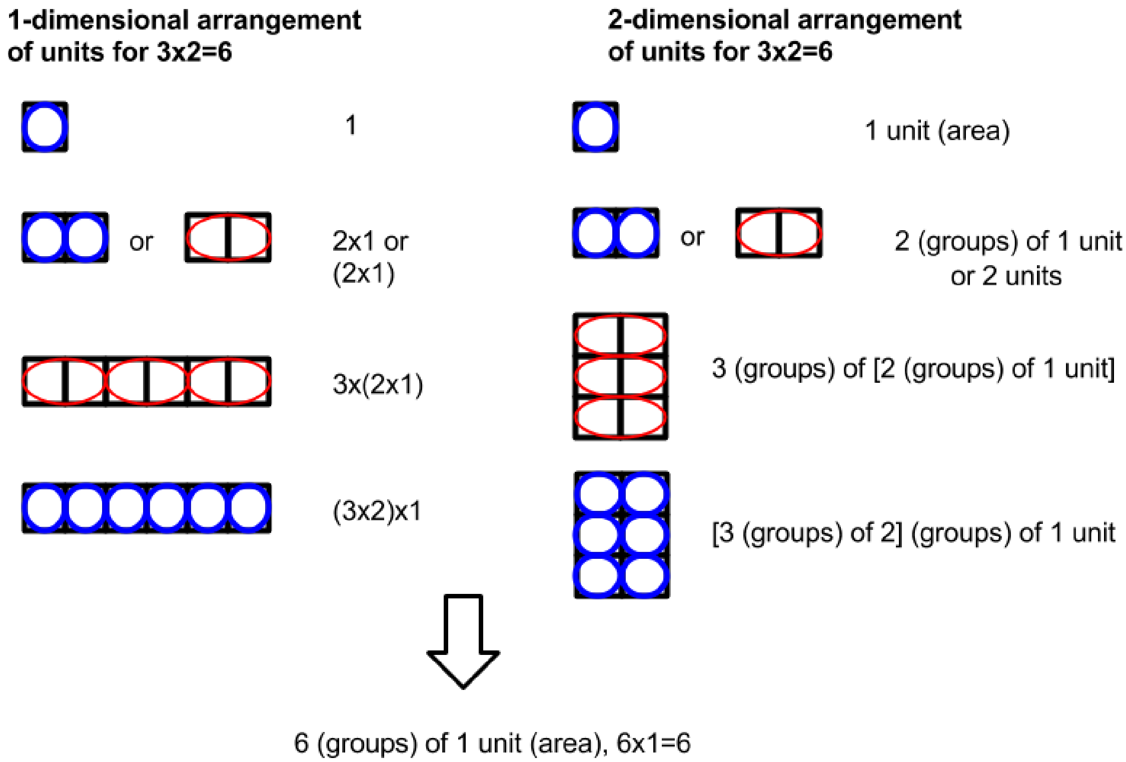


Figure 2. Construction of the area-to-area model focusing on arrangement of units

When constructing the area-to-area model for 3×2, students identify the size of one group, 2, and iterating it three times horizontally, which gives 1-dimensional arrangement of units for 3×2 (left in Figure 2), or iterating 2 three times vertically, which gives 2-dimentional arrangement of units (right in Figure 2).

Note that we do not need to start from 1 unit (area). For example, students can see 3×2 as 3 (groups) of 2, which is 3 as a multiplier, 2 as a multiplicand as shown in Figure 3. Then, this multiplication investigates how the multiplicand 2 changes when multiplied by the multiplier 3, so 3 is the only operator or scaling factor. This model is useful for both equal group and scaling factor perspective of multiplication because it focuses on the change of the initial quantity, given as a multiplicand. However, students can also assume 1 unit (area) to be the starting quantity or the whole and see the 2 as an operator to 1 unit (area), and the 3 as another operator to the result as Figure 2 shows. Thus, in this context, multiplication 3×2 is a combined multiplier of two multipliers, 2 and 3, for an invisible multiplicand 1 unit (area) representing the whole, which emphasizes the associative property of multiplication, such as $3 \times (2 \times 1) = (3 \times 2) \times 1 = 3 \times 2$.

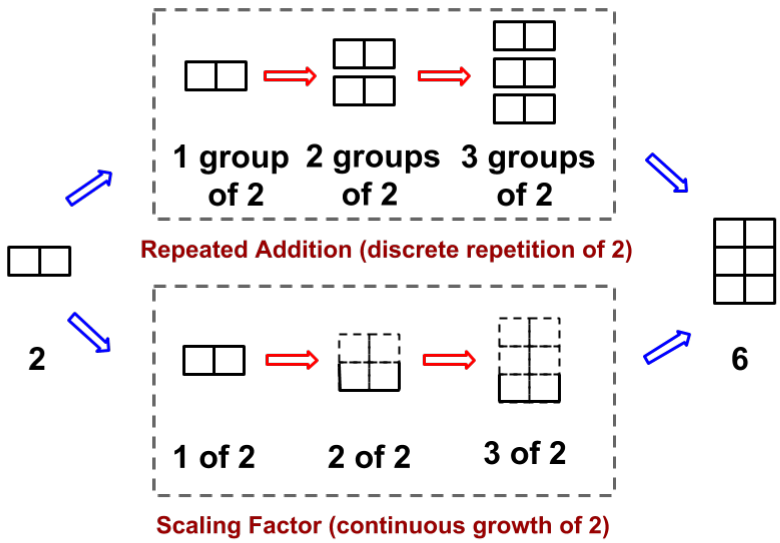


Figure 3. Construction of the area-to-area model reflecting change of initial quantity

How the whole or the initial quantity is changing in whole number multiplication is the core idea of the typical area models for fraction multiplication prevalently used in U.S. curricula (see Figure 4).

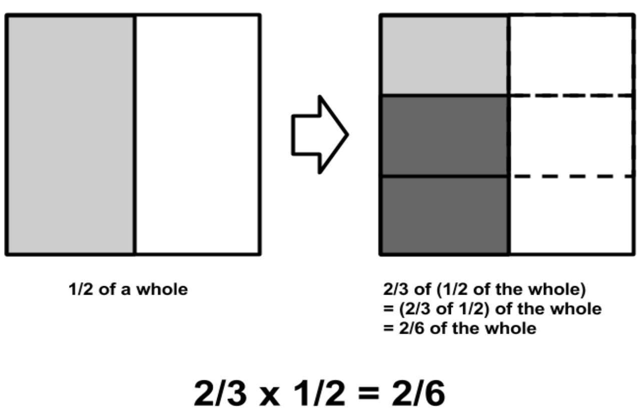


Figure 4. Area-to-area model for fraction multiplication

Typically, the multiplication of two fractions, $\frac{2}{3} \times \frac{1}{2}$ is introduced as $\frac{2}{3}$ "of" $\frac{1}{2}$, but the actual area model shows $\frac{2}{3}$ "of" $\frac{1}{2}$ "of" a whole (see Figure 4). A whole, the initial quantity, is represented by the area of the outer square. The fraction $\frac{1}{2}$ is an operator applied to the whole first and changed to a partial quantity $\frac{1}{2}$ that a new operator $\frac{2}{3}$ is applied to. Eventually, students need to figure out the fraction operator $\frac{2}{6}$ or $\frac{1}{3}$, the combination of two previous operators $\frac{2}{3}$ and $\frac{1}{2}$, that would be applied to the whole, the initial quantity. The actual multiplication can be described as, $\frac{2}{3} \times \left(\frac{1}{2} \times 1\right) \rightarrow \left(\frac{2}{3} \times \frac{1}{2}\right) \times 1$ if we assume the whole as area 1.

Moreover, as shown in Figure 5, we can combine two area-to-area models, one from whole number multiplication and the other from fraction multiplication, to create a consistent area-to-area model. In Figure 5, an initial quantity or the whole is represented by a unit square. 2×3 is described a growth of the unit square: a unit square to 3 groups of the unit squares (iterated/enlarged horizontally), i.e., 3 unit squares, then to 2 groups of 3 unit squares (iterated/enlarged vertically), i.e., 6 unit squares. On the other hand, $\frac{1}{3} \times \frac{1}{2}$ is described as a reduction of the unit square: a unit square to $\frac{1}{2}$ of the unit square (partitioned/reduced horizontally), i.e., a half unit square, then to $\frac{1}{3}$ of a half unit square (partitioned/reduced vertically), i.e., $\frac{1}{6}$ of the unit square. Even if the multipliers are fractions greater than 1, once the focus is how the change of one whole occurs, it is manageable because the area-to-area model allows both scaling up (from whole number multiplication cases) and scaling down (from fraction–less than one–multiplication cases) of the whole. Combining whole number multiplication and fraction multiplication models together implies they are connected.

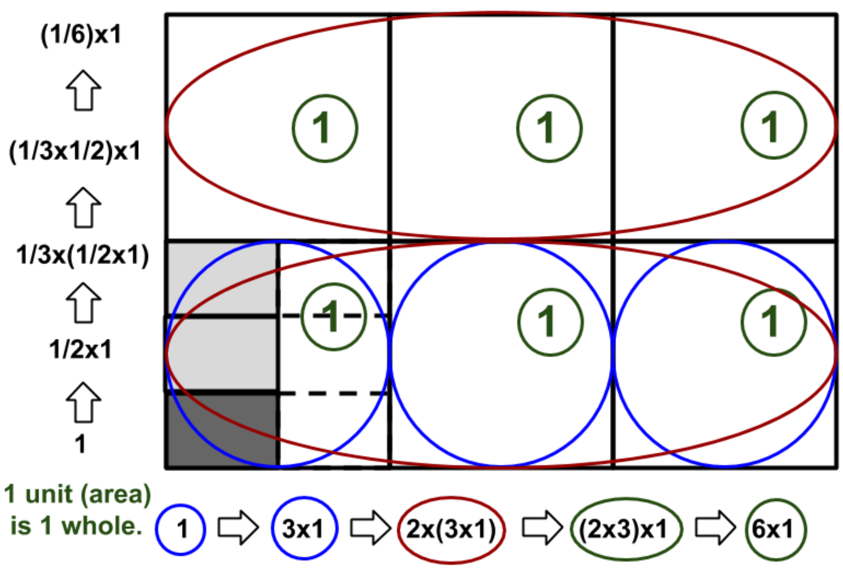


Figure 5. How “1 unit (area)” is changing over with multiplicative operators (multipliers) 2 and 3 in $2 \times 3 = 6$, as well as $\frac{1}{3}$ and $\frac{1}{2}$ in $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

Teachers usually guide students using the array model with grids (closed array) such as Figure 1(a), and later use the array model without grids (open array), which is described in Figure 6. However, there are many confusions between the array model without grids and the (lengths-to-) area model for multiplication.

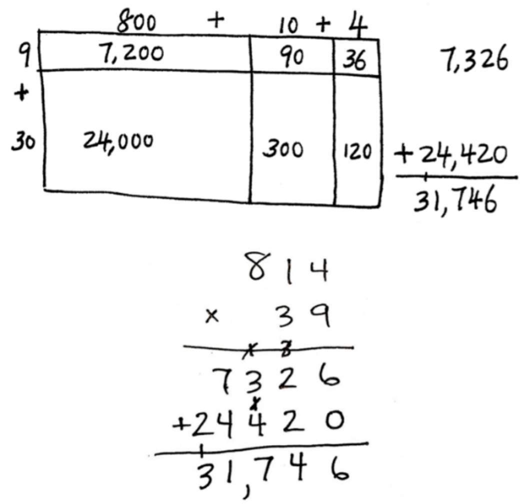


Figure 6. Array without grids or Area Model for partial product (EngageNY Grade 5 Module 2 Lesson 6, p. 84)

For example, as shown in Figure 6, teachers and students may not be sure if numbers used for factors represent the lengths of sides of a given rectangle or the numbers of rows and columns in the model. Different from the area-to-area model, the lengths-to-area model helps students avoid the aforementioned confusions by the open array model.

Area Model 2: Lengths-to-area Model

The fundamental idea of the area of a rectangle relies on counting the number of the unit areas inside the rectangle. The area can also be calculated from the product of two lengths, the length and the width. When two lengths are presented as factors and students are asked to find the product using the area concept, we will call this area model as the lengths-to-area model. In this model, numbers—used as factors in the whole number multiplication—can be represented as 1-dimensional lengths (or numbers of iteration of a unit area in two different directions), and their product—the result of multiplication—is represented as a 2-dimensional area.

The lengths-to-area model requires two length units of measurement reflecting multiplication as a factor times a factor. Figure 7 illustrates how 2×3 can be represented using the lengths-to-area model. Two factors have the same roles while in the area-to-area model, they have different roles as a multiplier and a multiplicand connected with “of”.

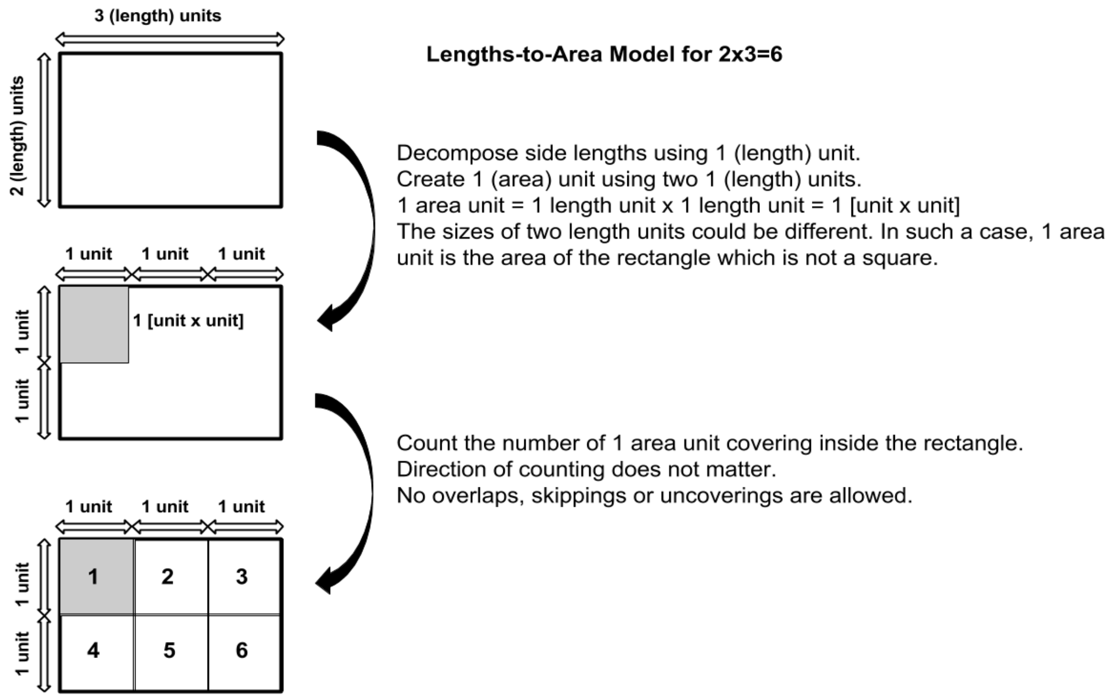


Figure 7. Construction of the lengths-to-area model

While the area-to-area model considers the process of multiplication as the change of the initial quantity, this lengths-to-area model focuses on the multiplication operation itself involving two quantities simultaneously to find the product. Due to such a simple and balanced structure of "a factor times a factor" in the lengths-to-area model, it is relatively easy to show the commutative property of multiplication.

How Two Area Models are Connected to Multiplication Problem Types

The CCSSM (2010) introduce Multiplication Problem types. Students begin word problems with different multiplication situations in grade 3 and continue through grade 5. We combined our multiplication models with these multiplication problem types. Table 1 shows how two area models can be used for teaching multiplications when combined with mathematics problems. Students initially learn and solve the word problems with whole numbers and progress into word problems involving all fractions and rational numbers. However, studies such as Predinger (2008) have shown discontinuity of the mental models of multiplication when students having transitions from whole number multiplication to fraction multiplication. Considering two different multiplication area models helps students make those transitions by investigating the connection shown in the example in Figure 5.

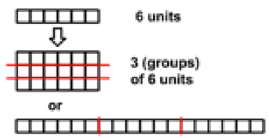
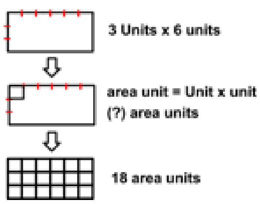
	Problem Types	**Equal Groups**	**Arrays / Areas**	**Compare**
	Word Problems	A. There are 3 bags with 6 plums in each bag. How many plums are there in all? B. You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	A. There are 3 rows of apples with 6 apples in each row. How many apples are there? B. What is the area of a 3 cm by 6 cm rectangle?	A. A blue hat costs \$6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? B. A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?
Area-to-Area Model (1 unit of measurement) 	**Action on/Change of an initial quantity**	A. 3 groups of 6 plums B. 3 (groups) of 6 inches	A. 3 groups of 6 apples	A. 3 groups of \$6 B. 3 of 6cm
Lengths-to-Area Model (2 units of measurement) 	**Operation/coordination of two quantities**	A. 3 bags times 6 plums/bag B. 3 lengths times 6 inches/length	A. 3 rows times 6 apples/row B. 3 cm times 6 cm	A. 6 \$/blue hat times 3 blue hat/red hat

Table 1. Multiplication Problem Types (CCSSM, 2010) and Area Models

Some problem situations can be interpreted in two different ways as in Table 1. For example, In the equal groups problem, "There are 3 bags with 6 plums in each bag. How many plums are there in all?," the word *bag* can be a unit of a quantity or a placeholder for iteration of 6 plums similar to *groups*. If it is a unit, two units of measurement—*bag* and *plums per bag*—are involved in the multiplication, and no different roles are imposed to them. The multiplication operation is 3 bags times 6 plums per bag, or 6 plums per bag times 3 bags, and the result of the operation is 18 bags times plums per bag (or plums per bag times bag). The newly produced (area) unit can be "plums." Now students should understand what the new (area) unit produced by the product of two units is and the situation that contains *bags* and *plums per bag* or *plums in each bag* together is equivalent to the situation that contains just *plums*. Coordination of two units to produce a new (area) unit is essential but not an easy task because it involves dimensional analysis. However, the difficulty will be made up later when thinking of the commutative property of multiplication of two quantities because interpreting "3 (bags or groups) of 6 plums is 18 plums" is easy while "6 plums of 3 (bags or groups)" does not make any sense.

Conclusion

Although teachers and students may think the typical area models work well to explain and represent whole numbers multiplication, we found that further exploration and subdivision of the area models are needed to produce continuity of ideas from whole number to rational numbers and algebraic expressions. We have shown two different area models for whole number multiplication with some variation of fractions: area-to-area model and lengths-to-area model depending on how multiplication operation is conceived: action on / change of an initial quantity, or operation / coordination of two quantities. The two models, area-to-are model and lenghs-to-area model, invite a careful investigation of effective use of the area models to help students develop the conceptual understanding of multiplication. Reconciling two models and having students make connections between two models also give a better understanding of quantities, operations of quantities, and their representations for future development of algebraic thinking.

References

Common Core State Standards Initiative.(2010) .Common Core State Standards for Mathematics (CCSSM). Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

EngageNY Grade 5 Mathematics Module 2 Lesson 6, p. 84

Flowers, J., Krebs,A.S., and Rubenstein, R. N (2006). Problems to Deepen Teachers' Mathematical Understanding: Examples in Multiplication. *Teaching Children Mathematics* 12(9), 478-484.

Prediger, S. (2008). Discontinuities for mental models: A source for difficulties with the multiplication of fractions. *Proceedings of ICME-11–Topic Study Group 10* : 29-37.