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Journaling to Support Student Learning: The Case of an Elementary Number Theory Course

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Journaling to Support Student Learning: The Case of an Elementary Number Theory Course

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Abstract: The use of journals in supporting student learning in elementary number theory is explored. Implications are made for the use of reflective writing for the teaching and learning of proofs and for undergraduate mathematics education.

Keywords: reflective writing; deductive reasoning; elementary number theory; teaching and learning proofs; undergraduate mathematics education

1. INTRODUCTION

Many mathematics students profess an interest in investigating patterns and solving puzzles, yet emphatically dislike writing proofs (Salazar, 2012). Furthermore, numerous studies highlight undergraduate mathematics students' difficulties with proof and proving, suggesting that learning to prove presents a great challenge (Moore, 1994; Weber, 2001; Hoyles & Healy, 1999; Harel & Rabin, 2010; Salazar, 2012; Dreyfus, 1999). Number theory as a topic and as a course presents rich opportunities for students to explore patterns and develop mathematical thinking, with many concepts accessible to mathematics learners with a range of abilities. Number theory affords opportunities for students to explore and foster their intuition while formalizing their thoughts using mathematical language (Campbell, 2006; Manouchehri & Sriraman, 2018). Moreover, proof is a central aspect of number theory and the field of mathematics at large, and thus it is critical to investigate students' learning of proof and how it can be developed (Harel & Sowder, 2009).

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1.1 Background

In order to address the struggles students have with learning to read and write proofs, and to help students develop more comprehensive notions of proof, researchers are increasingly calling for shifts away from traditional proof instruction, in which students are presented complete proofs and must reproduce them on exams, to instruction that makes students more active in the proving process (Yoo, 2008; Jones, 2000; Blanton & Stylianou, 2014; Weber, 2001). This includes providing opportunities for students to learn to write proofs of their own and to look back and reflect on what they have done. Within the literature on proof writing, researchers (Weber, 2001; Raman, 2003) have suggested that there is more to proof writing than merely possessing a conceptual understanding of the theorem to be proved and knowledge of proof methods. The prover must also have ideas about how to connect these two related, but distinct types of knowledge. Raman (2003) categorizes three types of ideas about proof writing as *heuristic* (based on informal understandings that a theorem or assertion ought to be true, but with little or no ideas about how to turn the argument into a formal proof), *procedural* (based on general known proof strategies, logic and formal manipulation that can lead to a proof, however that lacks a link to informal understandings), and *key* (based on an idea that gives a sense of understanding and knowledge of why a certain claim is true, and recognition of how that understanding can be translated into a formal proof; the bridge between *procedural* and *heuristic*). Raman (2003) found that mathematics professors tend to have *key* ideas, while undergraduate mathematics students overwhelmingly possess *procedural* or *heuristic* ideas. It is therefore important to investigate how *key* ideas are developed.

Research is needed to investigate pedagogical approaches to teaching proof and how students' thinking about proof develops (Harel & Sowder, 2007). There are few studies that examine how students' proof understanding develops over time. The studies that do exist overwhelmingly provide snapshots into students' understandings at a given time but offer little insight into how those understandings were developed.

1.2 Writing to Learn Framework

One possible technique that may be useful to help students develop a deeper understanding of ideas about proof writing and capture their longitudinal growth is *writing to learn. Writing to learn [WTL]* in the form of reflective journaling has been shown to be a unique and valuable tool for supporting students' learning and provides insight into students' thinking in other mathematical domains (Borasi and Rose, 1989; Clark, Waywood, & Stephens, 1993). *WTL* emerged in the 1970s as a pedagogical tool founded on the theories on learning of Emig (1977), Vygotsky (1962), and Bruner (1971). In these philosophies, learning is connective and selective, active, and personal and can be defined as reorganizing or confirming a cognitive theme as the result of an experience.

In light of promising results of *WTL* in English and reading, researchers in the 1980s began to explore how *WTL* could be used in mathematics, investigating the influence of *WTL* on students' vocabulary, conceptual understanding, performance, and views about mathematics. Numerous studies suggest that *WTL*, particularly in the form of reflective journals, increases students' performance in mathematics and promotes favorable attitudes and views about mathematics in secondary and lower-level undergraduate mathematics courses (Clarke, Waywood, & Stephens, 1993; Santos and Semana, 2014; Powell, 1997; Borasi and Rose, 1989; Hari, 2002; Loud, 1999).

1.3 Mathematics Journaling Frameworks

In their 1989 study, Borasi and Rose asked freshman algebra students to keep weekly journals, which the instructor read and then replied with written responses which the researchers subsequently analyzed. Borasi and Rose's analysis revealed the following taxonomy of benefits to the students' mathematics learning as a result of the journals:

<u>Potential benefits as the students write their journal</u>:

> 1.1 A *therapeutic effect* on the emotional components of learning mathematics can result as students express and reflect on their feelings about the course, mathematics and schooling;
>
> 1.2 An *increased knowledge of mathematical content* can be gained as writing about the material covered in the course provides a better and more personal understanding of the same, as well as the stimulus for new inquiry;
>
> 1.3 An *improvement in learning and problem-solving skills* can result from the articulation of and reflection on their process of doing mathematics;
>
> 1.4 Steps towards *achieving a more appropriate view of mathematics* can be taken, as one's beliefs on the nature of the discipline are made explicit and consequently reevaluated.
>
> (Borasi & Rose, 1989, p. 352)

Borasi and Rose noted a strong relationship between what the students wrote and the benefits they received. They recommended using a combination of structured and unstructured writing prompts to help students see all benefits of journaling. This relationship between the students' writing and the outcome of the journals was also noted by Clarke, Waywood, and Stephens (1993) who categorized student's journal writing into three types in order of sophistication: *recount, summary,* and *dialogue*. In recount, students list events from class or feelings towards class and mathematics, without discussing particular content and making connections between problems or events class. When summarizing, students discuss content and major ideas from class, summarizing what they learned. Finally, in dialogue, students summarize topics from class but also pose questions about the topics and discuss their understandings of the topics and how they are connected. Clarke et al. (1993) found that the more sophisticated the mode of journaling, the higher the appreciation, frequency of writing, and length of entries. Therefore, when implementing *WTL*, it is important to carefully craft prompts that will facilitate students' progression to dialogue.

Although it has shown promise in mathematics education, there has been little research into *writing to learn* in advanced mathematics (Starkey, 2016). In order to explore *writing to learn* in the context of learning to prove, this study investigated the questions:

- How do reflective journals support students' learning to prove in an undergraduate elementary number theory course?
- How do reflective journals provide insight into the development of students' thinking about proof in an undergraduate elementary number theory course?

2. METHODOLOGY

We implemented an embedded case study methodology (Yin, 2014) to explore how undergraduates' use of journals in a mathematics class supported students' learning of number theory and proof writing. In particular, we used journal entries as units of analysis within the larger unit of students to identify and describe our research questions.

2.1 Context

The study was conducted in 2014 over a 14-week period at a large state university in the southern United States. We examined the use of journal writing by 17 undergraduates in an Honor's Number Theory course taught by a teacher with over 30 years of teaching experience and who also participated in this research as the third author of this paper. The participants ranged from mathematics majors taking their first course in proof writing as preparation for more advanced mathematics course, to liberal arts majors who were taking the course to fulfill their basic mathematics requirement but wanted the challenge of learning about how mathematics might be related to other disciplines.

The course allows students at different levels of mathematical maturity to participate and work together. It provides a context for students to learn how to explore problems deeply and give careful, rigorous mathematical proofs. Students learn to explain their ideas both orally and in writing, and how to apply the mathematics learned to different types of problems. The class met twice a week for 80 minutes and had one required book, *The 5 Elements of Effective Thinking* (Burger & Starbird, 2012). These 5 elements include earth (thinking deeply about problems), fire (learning the importance of not being afraid of mistakes but rather viewing mistakes as a natural part of making new discoveries, wind (the value of developing questions out of thin air and becoming one's own Socrates), water (following the flow of ideas and building on what we know), and change, the quintessential element (being willing to change one's attitudes about learning and develop new ways of thinking). Class notes were written on an overhead by the instructor during class or handed out by the instructor.

One of the main differences between this course and the traditional mathematics course is that students were asked to investigate new ideas independently before they were discussed in class. For example, the class was asked to think about the basic assumptions they wanted as axioms, and then developed proofs using this foundation. They were asked to explore solving systems of congruences in examples before discussing techniques such as the Chinese Remainder theorem; and they explored complete and reduced residue systems in problems sets before using these to prove Fermat's Little theorem. Another difference between this and a typical mathematics course was that the problems were to prove or disprove and salvage if possible. So rather than being told what was true, the students were asked to come up with their own conjectures and to see that wrong guesses and mistakes were a natural part of making new discoveries.

This way of learning to prove at first gave students a certain amount of discomfort. One of the ways that was used to address the discomfort in being asked to discover and prove things for themselves was that students were required to submit a weekly journal entry that described problems they might be having and how they were addressing them. These journals provided a way for the students to get feedback from the instructor about how they were approaching proofs, as well as a way for the students to reflect on what they were doing in the course.

2.2 Data Collection

The 17 undergraduates in this course submitted weekly journal entries online to their instructor and reflected on their mathematical learning. The journal assignments consisted of both structured and unstructured prompts. The instructor provided comments in response to each of the students' journal submissions that informed him of each student's successes, challenges, issues, and questions.

We conducted pre-post surveys to examine students' views on mathematics, attitudes towards mathematics, proofs, journal writing, and course expectations, all of which were completed by 13 students. The pre-post surveys were open-ended (Appendix A and B). The students completed the pre-survey online, while they wrote their responses to the post-survey on paper.

The third source of data consisted of individual interviews conducted in the last few weeks of the semester. The 5 students interviewed were chosen out of the 17 students to represent categories of students who (a) wrote extensively in their journals and did well on their midterm exam; (b) wrote in their journals but did not do as well on their midterm exam; (c) did not write very much in their journals but did well on their journals; and (d) did not write very much in their journals and did not do as well on their midterm. We accessed the students' midterm exams to determine the students' grades and examined students journal submissions to identify students in each of these categories.

The interviews consisted of two parts:

1. The task-based portion consisted of the student talking aloud as he/she thought about and wrote out a proof provided by the researchers. The students were asked to prove or modify appropriately the following statement: For numbers A, B, and C, if A < B then CA < CB.

2. The semi-structured portion of the interview asked about proving, journaling, and the course.

The interview was intended to give additional insight into the students' experiences in the course and to triangulate the surveys and journal data.

2.3 Data Analysis

The journal component was coded using the Borasi and Rose framework (1989). We analyzed the 11 unstructured weekly journals by coding for evidence of the following four components.

1. *Therapeutic value*
2. *Increased learning of mathematical content*
3. *Improvements in learning and problem-solving skills*
4. *Reconceiving one's conception of mathematics*

This framework was used to examine how the journals were supporting the students' learning to prove in the course. We also used this framework to examine the students' development of students' thinking about proof as evidenced in their pre-post surveys and in the semi-structured portion of their interviews. The two structured journal entries were coded using content analysis (Zhang & Wildemuth, 2009) to investigate what themes were present in the students' writings. The authors compared their independent codings of the journal entries of 5 of the 17 students and reached 100 % agreement.

3. FINDINGS

We report our findings for the two research questions that we investigated in our study.

3.1 Support Students' Learning to Prove

We began by coding the students' unstructured journals to see what benefits appeared to be present. For each student (17 total), the two most common benefits reflected in their journals were identified and counted (See Table 1). One student had a low engagement with the journals and only displayed one benefit throughout their journals, and thus only was counted once, and therefore the counts in the table below sum to 33 instead of 34.

Table 1: Value of Journals

Therapeutic Effect	Content	Problem Solving	Views
15	9	9	0
Example: "when you figure things out on your own, it's so trying that what you do stays with you permanently."	Example: "this problem set stood out for me as being particularly enjoyable…how the Euler's totient function plays a role in RSA encryption."	Example: "…need to study up on both the axioms and the theorems…list of assumptions that I would never have considered."	

The unstructured journals were predominantly therapeutic, with students generally discussing content and problem solving topics less often than reflecting on their feelings about the course. The survey responses were also examined to determine the students' perceptions of how the journals affected their learning. When asked, "Would you recommend keeping a journal to a friend about to take a proofs course?" 9 students said yes (two gave the condition that the instructor should read and respond to them), and 4 said no. Of the students who said no, they mentioned that they preferred to ask their questions and get feedback from the instructor in person during office hours.

The students were also asked, "What do you feel are the benefits of journal writing in a proof-based mathematics course?" in both the post survey and interviews. Examples of students' responses to this question include:

> Student J: "…it helped me identify areas that I needed to work on more by putting it down on paper and it allowed me to kind of go back and think through the week what I struggled with, what I didn't struggle with…"
>
> Student E: "…learning to prove…well I guess whenever I had specific issues, I would talk about them and then he would go over them…not just in class, but in the comment section whenever he would reply to my journal, so that was helpful."
>
> Student E: "…well, writing in general is a process of thinking. When writing journals I had to think what I was going to write. And so it helped me to become patient when writing proofs. It helped me a lot. When writing proofs, I had to sit down and think on what I was going to write to prove it."

Analysis of the post-surveys and interviews suggest the following factors about the journals supported students' learning to prove: (a) Provide an avenue for communication with the instructor (helps instructor make changes; allows for asking questions); (b) Keep Record of progress and problems in a timely manner so that they can be addressed immediately; (c) Force reflection on learning so that students are encouraged to follow the flow of ideas (from prior knowledge to proof); (d) Help with memorization by writing terms and concepts; and (e) Encourage creativity in writing about proving. In the post survey, we found the factor regarding communication with the instructor, as the most common benefit described by the students, suggests that the students placed great value on knowing that their journals were not only being read, but that their instructor was genuinely interested in what they were writing.

3.2 Demonstrate the development of students' thinking about proof

In order to examine the development of students' thinking about proof, we focused on the journal entries to structured prompts because the unstructured journal responses were predominantly therapeutic. Structured prompts used in this study included the following:

> *1. Although an example is not a proof, many mathematicians use them to help with proof writing. What are your thoughts or experiences on how examples can be used to aid proof writing?*
>
> *2. Discuss the role that definitions play in writing proofs. How are definitions important? How do you use definitions when writing proofs?*
>
> *3. When assigned to prove a theorem, what is your proving strategy? Pick a proof or problem that you recently completed and copy this into your journal. What did you think about and what was your process for solving that problem?*
>
> *4. Why do you think mathematicians place so much emphasis on the importance of being precise with language?*

These prompts focused the students' writings on aspects of proof and provided a window into their understandings (Tall & Viner, 1983; Selden & Selden, 1995; De Villiers, 1990).

Analysis of the first prompt regarding thoughts or experiences with use of examples to aid proof writing revealed that the students had widely differing ideas about how they used examples. We identified five types of views that emerged from our analysis regarding the use of examples and we indicate the number of students with those views in parentheses. Examples could be used to: Understand ideas (10); Disprove a statement by providing a counterexample (4); Help structure their proofs (4); Identify patterns (2); and Check their own proof for correctness (1). Student E talked about examples as, "a way to confirm that what you're trying to prove is possible. It helps me understand how the numbers fit together in a real, mathematical situation."

In our analysis of the second prompt pertaining to the use of definitions in proofs and mathematics, we found three types of responses in the student journal entries related to the development of students' thinking about using definitions in proofs to: (a) Understand ideas; (b) Communicate ideas; and (c) Provide mechanics for constructing a proof.

Table 2: Thinking about Proof

Understand ideas	Communicate ideas	Provide mechanics for constructing a proof
3 students	6 students	9 students
Example: "[without] definitions we cannot have proofs. If you are trying to prove that a < b and <u>you don't even know what that means</u>, who {how] can you go about proving it. It's like going on a scavenger hunt but you don't know what you're looking for." - M	Example: "In writing proofs. Definitions are important because having the definition is the equivalent of citing your source in your English class. If you have the definition, then it is evidence you use to back up a claim you're making. So even if the answer is not 100% accurate, <u>the grader can still see your logic behind your attempt."</u> -R	Example: "Take $a>b$ for example. By definition of greater than, either $a = b+n$ for some natural number (n) or $b = a$-n for some natural number (n). You can then take this definition to write a proof such as $a*b > b*c$ when c > or = 1. We can now go back and use our definitions to help prove this problem by making some of the factors similar. " -J"

After reading and analyzing these prompts, we recognized that the students predominantly believed definitions are useful in the mechanics of constructing proofs or communicating ideas. However, three students explicitly commented on using definitions to help them understand the ideas they were trying to prove, which the instructor could address in class. Without the journal entries, the instructor might have been missing this insight, as it is unlikely to be revealed in students' turned-in assignments.

In the third prompt, the students again provided a wide variety of responses pertaining to the strategies that they used to write proofs including: Listing all theorems or definitions that might be relevant (6); Working backwards from the conclusion (5); Making sense of the theorem (5); Starting with the assumption (4); Trying to identify the axiom or theorem that would lead them to their conclusion (4); and Breaking the elements of the theorem down into specific steps (3). When compared with Raman's (2003) framework about proof ideas, it appears the students predominantly held procedural ideas about proof writing, because only 5 students seemed to focus on heuristic ideas that is making sense of the theorem before beginning the proof.

This procedural focus also appeared in the interviews. A portion of the interviews consisted of students being asked to prove the following: For numbers A, B, and C, if $A < B$ then $CA < CB$. In most cases, the students immediately began to write a proof of the statement without taking time to consider whether it was true or not. It was only after they had worked on the proof for a few minutes that they realized the statement was not true unless $C > 0$.

For example, in the following interview, Student M quickly noticed, "So we want CA plus something to be equal to CB. And so if we have that, we can multiply it by C. So we get BC equals AC plus something, right?...I'm guessing that's it….yeah. I don't know where to go from here." She reread her proof and then said, "Something tells me that's wrong." However, rather than end the interview, she continued to persist with the problem. When prompted by the interviewer, "Are we sure that this is going to be true all of the time?", she decided to test some examples and successfully concluded that C must be greater than zero, exclaiming, "I'm done, I think I'm done with examples…wow." Had she taken the time in the beginning to heuristically consider the truth of the statement, she may not have gotten so stuck with her proof. However, her perseverance is commendable, and showed she placed high worth on the proving process and experienced positive, therapeutic value from completing a proof.

In the final prompt, students believe mathematicians place emphasis on precision with language in order to: Write proofs (5); Provide a means for others to follow their logic (4); Remove assumptions or interpretations and focus on truth (4); Help in writing proofs because math itself is so precise (4); and Create universal principles (2).

In their interviews and post-surveys, students were not explicitly asked to compare the structured and unstructured journals; however, numerous students discussed an appreciation for the structured prompts. For example, M said, "There were a few journals that required more than just feedback, and I feel more of those would encourage more actual reflection and pointed examination of progress." When asked how journal writing could be changed to be more effective, J said "More specific entry topics." In total, seven students suggested using more structured topics to promote reflection and to focus the writing on proof, and two suggested having students write proofs directly in their journals.

4. DISCUSSION AND IMPLICATIONS

4.1 Support students in learning to prove

Our study provided a lens for examining the role of journaling in the development of learning to prove in an elementary number theory course. This course had a broader goal of providing a foundation for effective thinking and inquiry using number theory as a setting for developing students' perseverance and ability to think critically.

The required book, *The 5 Elements of Effective Thinking* (Burger & Starbird, 2012), that the students read in the first half of the semester appeared to have supported teaching inquiry into proving and aided the students by providing them with a language and reference to which they could refer to for the challenges they encountered as they confronted difficulties in learning to prove. For example, making mistakes was considered a part of learning and from them one could build a better understanding of the mathematics and writing proofs in particular. Using a flow of ideas provided a way for students to try ideas and see where it led them in better understanding their arguments

The students used the journals to reflect and recap the ideas covered during the week. The metacognitive value of writing in their journals was an opportunity for students to reflect and reexamine what was secure in their understanding and others that “required revisiting” as Student G mentioned. Other students mentioned becoming aware of areas to work because they had to “put it down on paper” suggesting it was the act of writing that prompted the student to think about their mathematics more deeply and reflectively. Student D talked about proofs as akin to journal writing and the benefit gained was in that they both required thinking and connecting ideas in a written format. In this way, the journals allowed students to recognize and write a narrative for the course as they described their struggles and triumphs. The course narrative also connected to the narratives they created about their proof writing.

Instructor feedback and communication between student and instructor supported students in the course in general but also in their development of and appreciation for the value of mathematical proof. For example, Student J talked about “free space to come to him with ideas...and a good chance to kind of get ideas out there without judgment.” The instructor’s positive feedback was a frequent comment, among them J who mentions that it “motivated me and kept me going”. The students felt that there was a “direct communication” with the instructor who would reply personally to the student and then go over issues in class for the benefit of all the students. In addition, students valued having another person, in this case the instructor, comment specifically about their thoughts and ideas and not just about the finished proof.

Generally, when students turn in their completed proof homework, they submit what they consider to be their finished product: the proofs themselves. Further, the feedback they receive only considers the finished proof. However, the process of writing a proof is often not a direct or linear process. There is considerable background thinking and scratch work that goes into preparing the proof. The task then becomes to connect the background thinking into writing the formal proof, displayed as Raman's (2003) key ideas. Turning in completed proofs alone does not demonstrate the key ideas students may possess. The journals opened an avenue of communication with the instructor about students' proving process, in which students felt like their entire proof knowledge, which includes their supporting ideas, was valued.

4.2 Demonstrate the development of students' thinking about proof

The structured prompts focused students on their thought processes *during* the writing of their proof and gave them opportunities to reflect on their process of structuring their proofs. For students who are learning to prove, the proof and proof writing process are often thought of as separate (Raman, 2003). By writing about their process for writing proofs, students were able to think about how the formal proof and informal understandings were connected. This supported the development of the *key* ideas in proof writing referred to by Raman (2003) to connect their intuitive ideas about proof to their formal structuring of it. Using structured journals helped the students *connect* the journal writing with their learning of proof. The students appreciated the prompted journals because the prompts gave their instructor an opportunity to comment on their proof-related thinking, helping them to refine and develop their understanding of proof. Each prompt provided a unique look into how the students thought about proofs during the course.

The kinds of structured prompts students are given may impact the support and development of student's proof writing to different degrees and is a topic for future research. Furthermore, it was evident from the journal entries that journaling played an important role in creating an active learning environment and learning community. Future studies are needed to better understand the best types of prompts that can be used to promote student learning. There are various models for using the journals, including providing a forum for the instructor to share student responses with the class. However, having this type of open forum might also discourage the willingness of the students to share their problems, which was one of the most important aspects of the journals themselves. Again, that is a topic for future investigations.

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