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Abstract: Do we need to insist on real numbers? We present our discussion of this question in the style of duoethnography.
Keywords: Duoethnography; Dialogue; Real Numbers; Constructing Real Numbers

Duoethnography is a collaborative research methodology that takes the form of a dialogue or play script and juxtaposes two or more different perspectives (Norris, 2008). In its initial conception, duoethnography examined researchers' histories and identities related to social issues (Norris, Sawyer, & Lund, 2012). However, when adopted in mathematics education, duoethnography – whether or not it was called as such – was used to examine perspectives on various issues related to mathematical discovery and mathematical pedagogy in a scripted dialogue between a mathematician and a mathematics educator (Nardi, 2008; Rapke, 2014). We follow this avenue, by reconstructing a dialogue between a research mathematician (RM) and a mathematics teacher educator (MTE).

Part 1: Raising the question

RM: My students have a real difficulty with real numbers.

MTE: Many students have many difficulties. Why do you highlight real numbers?

RM: I was very disappointed with responses to one of the tasks students completed in my course. It is Algebra course, in which we work with real numbers. But as the students are secondary school teachers, I asked them to design a teaching sequence in which they approach and explore a property related to an operation with real numbers.

MTE: What property? What operation?

RM: It was their choice.

MTE: So, I sense in your disappointment that the students did not meet your expectation.

RM: Well, first, they didn't explore the diversity of properties.

MTE: What do you mean by diversity? When I think of properties of operations, there are associativity, commutativity, and distributivity. You may add identity and inverses… what else?

RM: There are so many properties, with so many details… Let me give you an example. The algebraic argument for the property of zero product in real numbers – that is, if

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$a \times b = 0$ *then* $a = 0$ *or* $b = 0$ – is not readily understood. It can be easily proved using field axioms, but in school we do not necessarily turn to a formal proof of this. However, we expect students to know this property, and therefore for teachers to develop means to teach this, developing intuition. So it is helpful to think of multiplication $a \times b$ not as a groups of b elements (or b groups of a elements), but as the area of a rectangle. Then it makes sense to infer that $a = 0$ *or* $b = 0$ because the area of the corresponding rectangle can be zero only when one of its sides is reduced to a point, that is, measures zero.

MTE: Now I am curious. I thought a mathematician would expect more emphasis on a formal approach in the teaching of mathematics, rather than appeal to intuition.

RM: Not exactly! As in general, every mathematics claim is supposed to be proved, or justified. But, before it, one must be convinced that the fact to be proved is in fact true! Heuristic process should also be part of one's mathematics education.

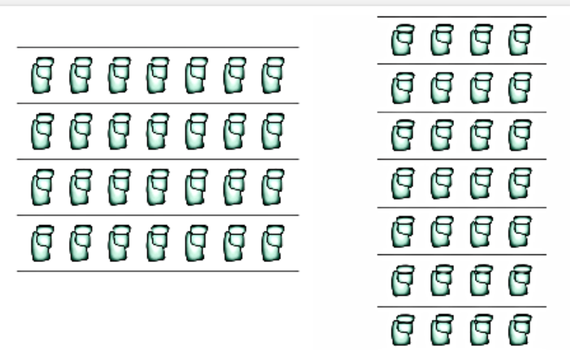
MTE: No disagreement here! But you mentioned being disappointed with your students' responses to the task. What did they do? What did they do wrong?

RM: See, most of their interpretations were based on either natural numbers or rational numbers. For example, here a student illustrates the commutative property for multiplication of real numbers, using 4×7.

Oi pessoal!

Respondendo a questão 1: "Propriedade comutativa da multiplicação de números reais".

A propriedade comutativa da multiplicação pode ser facilmente compreendida se considerarmos, por exemplo, duas possibilidades de dispor as carteiras de uma sala de aula. Como ilustrado na figura abaixo, pois não importa se formamos 4 fileiras com 7 carteiras ou 7 fileiras de 4 carteiras, o número total de carteiras será sempre 28, ou seja 4 · 7 = 7 · 4 = 28.



(a) 4 fileiras de 7 carteiras. (b) 7 fileiras de 4 carteiras.

MTE: And what would you rather have them do?

RM: The property concerns real numbers, so it should be illustrated by an intuitive model compatible with the continuous nature of the real numbers. You see, in the example, the student contextualized the explanation with objects of discrete nature.

MTE: Model like what? Would you be happy if this student drew a rectangle? So length × width results in the same area as width × length, so L×W = W×L ?

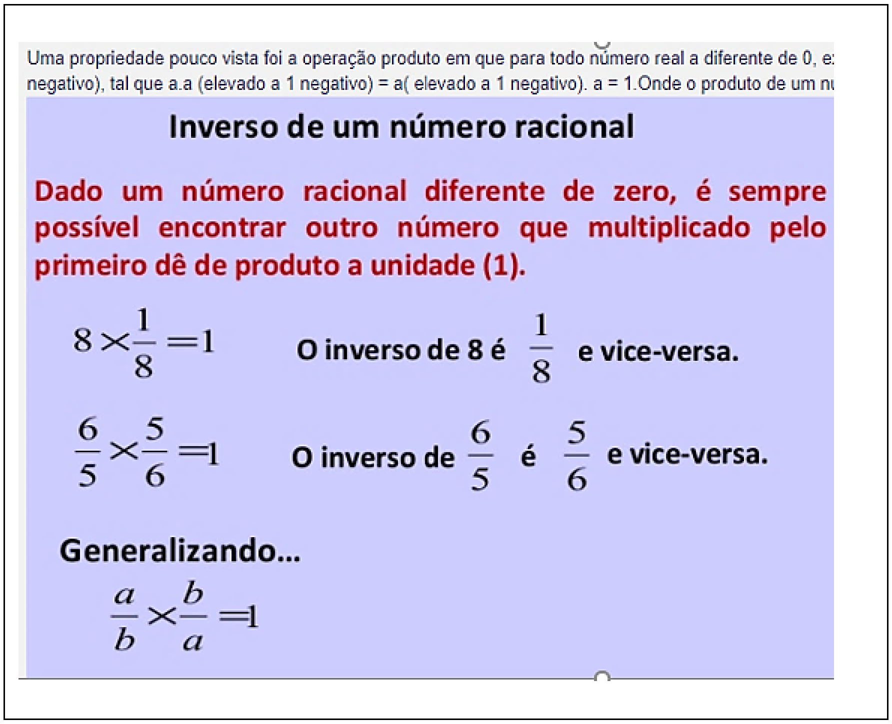
RM: It is one possible approach, as long as L and W represent positive real numbers.

MTE: But who decides what they represent? I can write "*a*" and think of 7.

RM: Indeed, this is not a problem, as long as you don't restrict the discussion to a figure of a rectangle over a grid given by the unit square. If one represents two rectangles 7×3 and 3×7 without making references to integer marks on the number line or to a grid of unit squares, that is, if one doesn't count, but uses other process that doesn't make reference to the integer nature of the chosen numbers, it is not so bad.

MTE: OK, obviously you can use a geometric construction, I mean construct a segment of length XY, given X and Y, but you task was about teaching sequence… Teaching who?

RM: This was also open for a choice, but my students had in mind middle or secondary school. Here is another example. The property that is being demonstrated is the existence of inverse element, multiplicative inverse, for real numbers other than zero. But – again – it is demonstrated using rational numbers.



MTE: I see your point. Your students used rational numbers or natural numbers to demonstrate properties of real numbers. But what's the harm? If I think of properties of real numbers, and all I imagine are rational numbers, where will I go wrong?

RM: Real numbers are very different.

MTE: Different – obviously. But does it really matter?

RM: There are many differences.

MTE: Sure. But what's the harm? If I think of properties of real numbers, and all I imagine are rational numbers, where will I go wrong?

RM: Here are a few of the differences:

- We cannot count the real numbers (The set is uncountable). But we can count rational numbers. (For example, we can prove facts about rational numbers by mathematical induction.)
- In general we cannot give a set of real numbers by listing its elements.

- Every open and closed intervals in $\mathbb{R}$ are always different sets. But in $\mathbb{Q}$ open and closed intervals can be the same thing.
- If $f : I \subset \mathbb{R} \to \mathbb{R}$ is continuous, $x, y \in I$ and $f(x) < a < f(y)$ then there is $z \in I$ such that $f(z) = a$. This general result can be used with particular expressions as $\sqrt[n]{x}$, $\log x$ and $\text{tg } x$.
- If $a > 0$ the $x^2 - a$ can always be factored in $\mathbb{R}$.
- Every odd degree polynomial can be factored in $\mathbb{R}$.
- If a polynomial $p(X)$ has integer number coefficients and no prime number divides all the coefficients at once then $p(x)$ factors in $\mathbb{Q}$ if and only if it factors in $\mathbb{Z}$. (This is the Gauss lemma for polynomials and we cannot use it with $\mathbb{R}$.) This result can be used to explain why there is no rational numbers root for $x^n - a$ for many $a \in \mathbb{N}$.

MTE: Sure, I am definitely familiar with this countable-uncountable stuff, I learned about roots of polynomials… But we are talking about middle or secondary school here. All these properties you listed are learned at the University. They are out of scope of school curriculum. But let us focus on school mathematics for a moment. If my thinking is limited, if my concept image[2], or example space[3] is limited (MTE elaborated here on the terms used in mathematics education) to rational numbers, where will I make a mistake discussing real numbers?

RM: There are many more examples. When asked to work with real numbers intervals, students are using rational numbers as endpoints…

MTE: Let us call this "rational number bias". There is research that uses the construct "natural/whole number bias"[4]. That is, students think of properties of natural numbers, such as order, when they work with rational numbers. There are many examples related to density or fraction comparison, such as students believe that 0.3 is smaller than 0.23 because 3 is smaller than 23. For a similar reason 1/5 is erroneously considered smaller than 1/6. These are examples of the natural number bias. So you exemplify a similar tendency: students are using rational numbers when the issue concerns real numbers. I get it…

RM: There is more…

MTE: More examples, OK, but what's the harm? Again, if I think of real numbers, and all I imagine are rational numbers, where will I go wrong? If I talk about flowers and all I have the image of are roses, what's the harm? You also said that heuristic process should be part of the teaching and learning of mathematics. In particular, you could accept the inductive reasoning: properties that work for the set of rational numbers also work for the set of real numbers.

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[2] Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational studies in mathematics*, *12*(2), 151-169.

[3] Watson, A., & Mason, J. (2006). *Mathematics as a constructive activity: Learners generating examples*. Routledge.

[4] Van Hoof, J., Verschaffel, L., & Van Dooren, W. (2015). Inappropriately applying natural number properties in rational number tasks: Characterizing the development of the natural number bias through primary and secondary education. *Educational Studies in Mathematics*, *90*(1), 39-56.

RM: Hmm, I guess you got me!

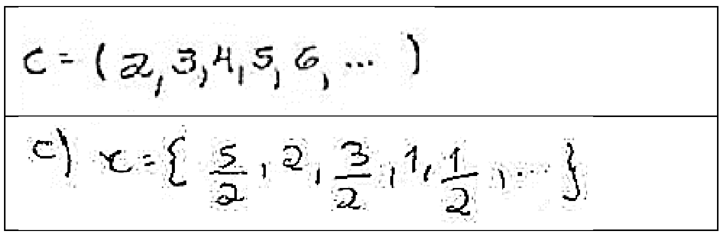
Here the conversation was interrupted and MTE was left without an example –where thinking of rational, rater than real numbers, is "wrong" – that she considered as "relevant" at a school level. The conversation continued several weeks later, initially unrelated to real numbers.

Part 2: Here is the answer

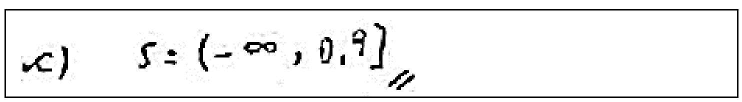
RM: Here is the task on inequalities. It concerned students' ability to interpret graphical information, the answer should have been $x > 1$.

MTE: Was it a hard task?

RM: The problem is with how the solution was represented. Here are a few examples:



And here a student's interpretation of $x < 1$, note the upper limit. It is very telling on how this student thinks of numbers.

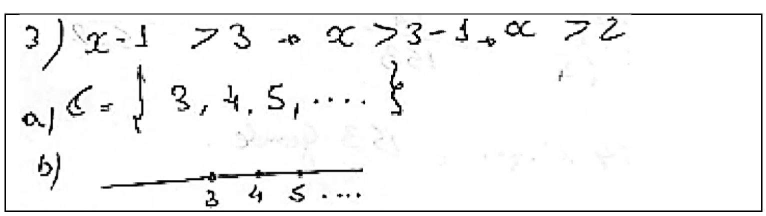


MTE: That's it!

RM: What's it?

MTE: That's it! This is an example I was looking for. I asked you, where one's thinking of rational numbers rather than real numbers could hurt. And it is here. You have just shown me how this thinking generates incorrect solutions, at a school level mathematics.

RM: There are more examples. Here is the task where students had to represent a solution on a number line

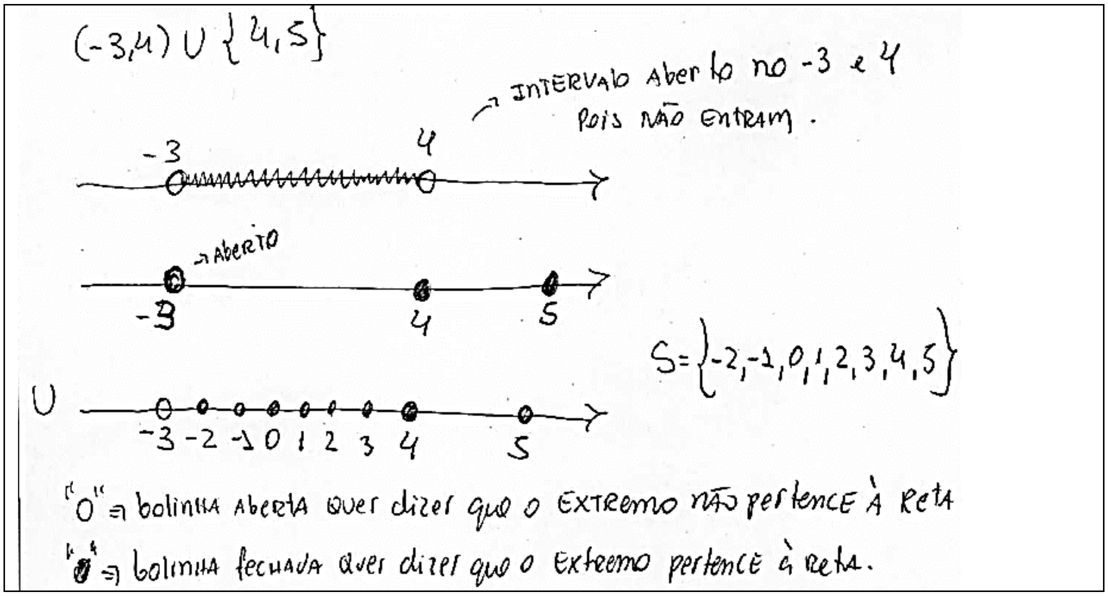


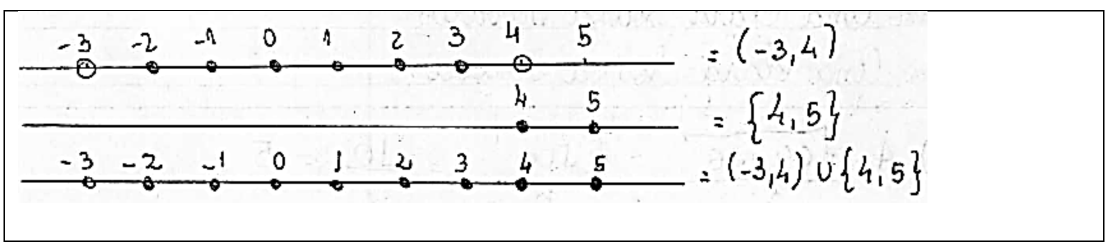
And here is an example in which a student found intersection of 2 sets:

$A = \{x \in \mathbb{R} : 1 \leq x \leq 6 \}$ and $B = \{x \in \mathbb{R} : x > 3\}$

A student's solution was {4, 5, 6}, that is limited to natural numbers in the interval.

And here are 2 examples in which students represented the set $(-3, 4) \cup \{4, 5\}$ on the number line. Note that $(-3, 4)$ relates to an open segment $\{x \in \mathbb{R} : -3 < x < 4 \}$, and my students were familiar with moving back and forth between these representations





MTE: So we have examples for natural number bias, which is not new, there are many examples in the literature, but also these solutions demonstrate rational numbers bias, which I have not seen reported in research.

RM: Maybe there is more than that, it is *numerical representation bias*.

MTE: What do you mean?

RM: When working with real numbers, one eventually will work with general symbols instead of numerical representations, but students always seek numerical representation.

MTE: Maybe the issue is of non-existence of "transparent" representation. At some point (in Zazkis, 2005) I claimed that one of the problems with understanding irrational numbers is a lack of representation that clearly points to the property of being irrational. When you represent a number as $2n$, for a natural n, you have an even number. The "evenness is embedded in the representation. When you represent a number as n^2, for a natural n, you know it is a square number, just by considerting the representation. When we have $\frac{a}{b}$, with some constraints, it is a rational number. But there is no representation that points to irrationality. Similarly, there is no representation to point to a real number.

RM: But if I follow your reasoning, there is also no representation to point to a natural number either.

MTE: Agree, but this has never been a problem, as using our "natural number bias" we think of a natural number, unless instructed to think differently. When you think of a number between 5 and 10, you think 7, not 2π .

RM: Indeed, but numerical representation 7 "declares" that here we have a natural number. Numerical representation of $\frac{2}{7}$ or $2.\overline{7}$ clearly points to a rational number.

MTE: I see your point. But some representations do not point to a property clearly. Even numerical representation of algebraic numbers is problematic. We know that $\sqrt{2}$ or $\sqrt{5}$ or $\sqrt{17}$ are irrational, but you have no idea about $\sqrt{18769}$ … (which is in fact 137).

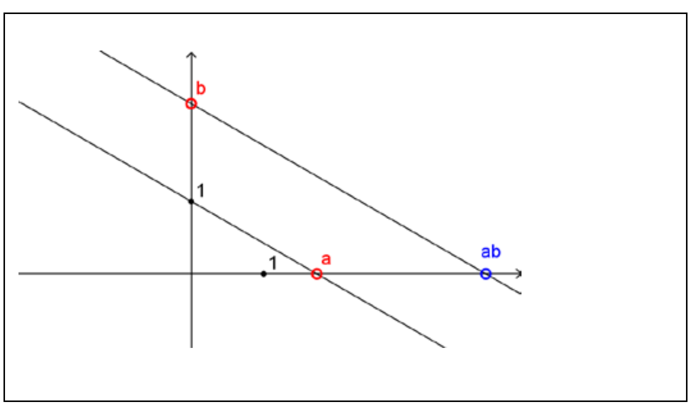
RM: Indeed. Wait! This concept of transparent representation called my attention to other issue. The representation 1/*a*, when *a* is a real number, is interesting if we want to say something about the multiplicative inverse of *a*, for $a \neq 0$. However, it is not transparent to other usual interpretations related to the symbol. I mean, 1/Pi doesn't represent the unit divided in Pi parts!

MTE: So, what do you do when there is no "convenient" transparent representation?

RM: There IS a representation, convenient, but not numerical. A number line provides such representation. A point on a number line represents a "general" real number.

MTE: And you said previously, when we talked about rectangle area as representation of multiplication, that a segment represents a "general" real number. So, is it a point on a number line or a segment length?

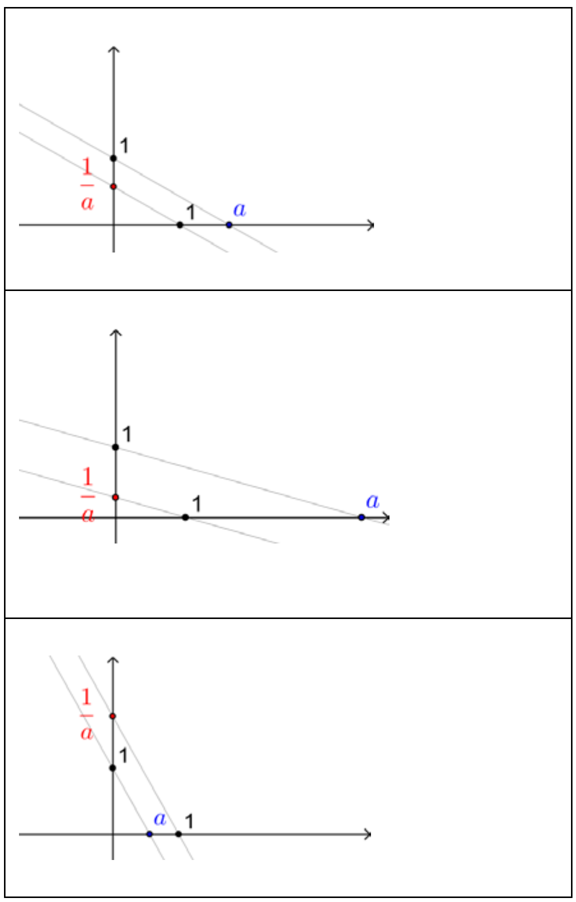
RM: Be flexible on this, but consider the following. Number line is a great tool for recognizing and demonstrating a variety of properties of real numbers. We can represent the addition and multiplication operations, also inverse elements and the relation of order. For example, we can represent pairs of real numbers and its product as in this figure:



We usually call this construction Descartes' multiplication. This kind of representation is very interesting and we don't need to make reference to numerical representations to explore several properties.

MTE: Of course, I mentioned the possibility of such construction previously. But in order to appreciate this representation, one has to know something about similar triangles. And this topic is usually approached in school much later than multiplication. It seems that we are returning to the beginning of our conversation.

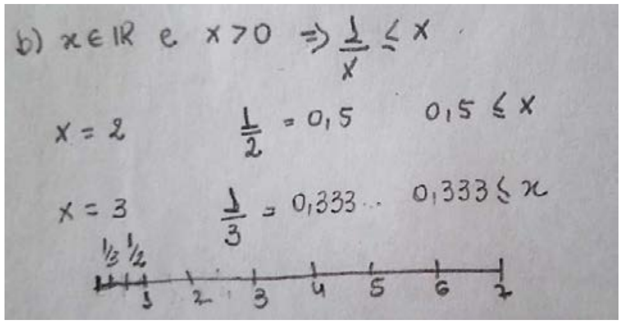
RM: NO! In fact, you are helping me to reinforce the didactical aims of my task.
Let me explain. We can use Dynamic Geometry to consider Descartes' representation of multiplication to explore the property of inverse element. This construction represents the multiplicative inverse of a nonzero real number. The figures illustrate some positions of a relative to 1.



RM: See what happens when I drag the point a. We can see that $1/a$ is between 0 and 1, when a is larger than 1, but it is larger than 1, when a is between 0 and 1 (figures below illustrate the dragging action). I guess we wouldn't find natural number bias in an approach like that, we wouldn't find many students claiming that $1/a$ is always smaller than 1.

MTE: Do they?

RM: More than you would believe. Let me show you this answer of a student of mine, another in-service mathematical teacher. Even having the number line in mind, the analysis was restricted to numerical symbols.



Using the Dynamic Geometry construction, the condition "$x > 0$" could be considered in a different way, and the student probably would find examples where $1/x > x$.

MTE: Well, it seems to be an interesting exploration activity with Dynamic Geometry resources. Very interesting! I explored these and alike constructions in an Abstract Algebra course. We mentioned "constructible numbers" as an introduction to classical proofs of what is not constructible. But I have never considered these constructions for exploring properties of operations and focusing on real numbers.

RM: That is where some knowledge from advanced mathematics courses can be useful in teaching school mathematics.

MTE: In fact, NOW I see more didactical potential in your task. For example, using the Descartes' construction of $a \times b$ I can see the issue of interpreting the expression $a \times b = 0$ in a different way. I can imagine a student trying to make $a \times b = 0$, by dragging a point for a, or for b, and finding that the only way to have $a \times b = 0$ is when a or b is placed at zero.

RM: In an environment like this, once the student knows how the algebraic expression for an operation is represented, s/he can search for a meaning for it. And that's how most mathematicians work. Most of the time we resort to personal arguments that help us remember a known fact or to convince ourselves of its truth. We save the formal proofs for the articles or advanced Algebra courses.

MTE: I also see now how this geometric representation of multiplication can be useful when a or b are negative real numbers.

RM: Absolutely. And what is important is the continuity of representation of real numbers that you get by dragging a point.

MTE: However, let us return to our conversation about rational numbers and real numbers. Well, you gave me some interesting examples showing that we can explore properties of real numbers using geometric representation. But are you concerned with a systematic study on the subject? Do you think the geometric representations, together with resources of the Dynamic Geometry, are appropriate for a systematic study?

RM: Yes, I do! At least I have produced several scenarios that allow the representation of several properties for the real numbers (see Moutinho, 2013).

MTE: Aren't you regressing now to the Euclidean era? In the *Elements*, numbers are represented by line segments. And we can find in the *Elements* constructions similar to Descartes' multiplication.

RM: Piaget said that we understand objects by operating on them. He probably said it better, but this is the idea. I believe that geometric dynamic representation of real numbers provides a fruitful venue to explore properties of real numbers by operating on them. So, I'm not regressing, but standing in the shoulders of giants, and with some help from modern technology I try to help my students appreciate real numbers, making what is "real" their reality.

"Duoethnographers do not end with conclusions. Rather, they continue to be written by those who read them." (Norris & Sawyer, 2012, p. 21). As such, we invite readers/ colleagues to provide their perspectives on whether, and especially when, we should insist on real numbers.

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