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Pre-service teachers' mathematical reasoning – how can it be developed?

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Abstract: The focus of this study is the mathematical reasoning of pre-service teachers. The author of this paper was the instructor of a class of pre-service teachers preparing to teach in lower secondary school. The instructor divided the class in small groups and gave each group some mathematical exercises to work on. The unit of analysis of this study is the video and audio-recorded dialogue of each group. The aim of this study is to characterize different aspects of pre-service teachers' mathematical reasoning, as well as to indicate ways to develop their reasoning further. The basic idea of the study is that rote learning reasoning is imitative, and the opposite kind of reasoning is creative. However, the study indicates that the reasoning of the participating students sometimes could be neither imitative nor creative. Thus, some of the reasoning was in a grey zone somewhere between imitative and creative reasoning.
Keywords: Pre-service teachers, imitative reasoning, non-imitative reasoning, creative mathematical reasoning.

Introduction

Freudenthal (1991) is concerned with mathematics as an activity. To describe the main aspects of this activity he uses the term mathematising. Referring to Treffers (1986), Freudenthal (1991) makes the distinction between horizontal and vertical mathematising. According to Freudenthal horizontal mathematising leads from the world of life to the world of mathematics to which we only have access through symbolic representations. In this process, the mathematical representations get their meaning through their relations to real life situations. The mathematical concepts and methods are gaining their meaning as abstract constructs in many different contexts. When symbols are shaped, reshaped, manipulated, and connected to the representation of other mathematical concepts or methods the mathematising is vertical.

However, the distinction between horizontal and vertical mathematising is vague. Mathematical objects may be very different for the expert mathematician and the novice. The mathematical activity of novices may be restricted to procedures or rules, and to particular contexts and situations. Skemp (1978) has labeled this instrumental understanding. Expert's mathematical activity is on the other hand normally conceptually based and relational understanding (Skemp, 1978). In addition, expert reasoning may be less restricted to procedures. Thus, the distinction between horizontal and vertical mathematising depends on the situation and the persons involved. According to Freudenthal (1991), the best way to explain this distinction is to give some examples. One of his examples concerns the figurate numbers. For instance, the sum of the n first odd integers equals the n'th square integer. When

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represented geometrically by arranging figurate numbers as dots (units) in a square this may be a matter of horizontal mathematising. Formulating this relation with number symbols and proving it using an integer variable and mathematical induction would be vertical mathematising.

This study is concerned with the way pre-service teachers' reason mathematically and how to develop their reasoning through teacher education. Freudenthals (1991) general idea or maybe even philosophy about mathematical activity as mathematisation seems to be a fruitful basis for such an endeavor. Accordingly, it is particularly interesting to understand the pre-service teachers reasoning when mathematising.

Pre-service teachers are hardly expert mathematicians. Therefore, one may expect that their mathematical reasoning is different from the reasoning of the expert mathematician. Based on teaching experience one may have some idea of what these differences might be. One of these differences might be the role of creativity in mathematical reasoning. Of course, this is a question of what creativity means in the mathematical field. Sriraman (2009) conducted a study where he interviewed expert mathematicians. In this study, he also discusses the concept of creativity in the mathematical field. His conclusion is that it is sufficient to define creativity as the ability to produce novel or original work. Sriramans study indicates that in general, the interviewed mathematicians' creative process followed the Gestalt model (Wallas, 1926). The Gestalt model has four stages. The first stage of this model, called *preparation* consist of hard work in order to understand the problem. The second stage is *incubation* where one puts the problem aside for some time. However, probably without forgetting it completely. The third stage is *illumination* when perhaps doing something else the solution suddenly appears. One has of course no guarantee that this will happen. The fourth stage is *verification* where one works out the details of the solution to verify it.

In an essay, Hadamard (1954) discusses the role of invention in the mathematical field using his own and other's research as examples. The reading of Hadamard's essay indicates that what he has in mind when discussing invention in the mathematical field is similar to what the expert mathematicians have in mind when interviewed by Sriraman (2009). However, the question would be if pre-service teachers see it the same way. Does mathematics as an activity (Freudenthal, 1991) involve creativity in any way for pre-service teachers? The motivation for this study is that the way pre-service teachers see this question would probably have an impact on their future teaching. It is reasonable to assume that if pre-service teachers do not see creativity as part of any mathematical activity, then their teaching would probably reflect this. In turn, their students would probably experience mathematics as not having anything to do with creativity. On the other hand, if pre-service teachers do see creativity as part of mathematical activity, then their future teaching would probably reflect this as well, and hence their students would experience creativity as an integral part of mathematical activity. By making creativity an element of teaching pre-service teachers mathematics, one could make them aware of creativity as an element of mathematics. This would probably improve their mathematical reasoning, as well as have an impact on their future teaching. It is reasonable to expect that students, who have experienced creativity as an element of working with mathematics will make it and element of their future work with mathematics. There are no reasons why this should not apply to pre-service teachers as well. The pre-service teachers participating in this study were working with certain exercises involving number sequences. Hence, the research question of the study is as follows:

RQ: How to characterize and develop further pre-service teachers' mathematical reasoning in problem solving activities related to number sequences?

The participants of this study were students in a class of pre-service teachers. The author did not consider any kind of mathematical giftedness or excellence when selecting the participants of the study. All student of the class were invited. The students were preparing to teach mathematics in lower secondary school. They were specializing in mathematics by taking a course on number theory. The course involved figurate numbers, sequences, divisibility, greatest common divisor with the Euclidean algorithm, linear Diophantine equations, congruences and some cryptology.

Most of the students of the class with a few exceptions did take part in the study. The students worked in small groups. A colleague of the instructor recorded the students on video while they were working. Each group was also audio recorded. The instructor answered questions and gave some help if the students asked for it. In each group, there was a discussion among the students on how they could solve the problems. These discussions or rather the transcripts of them constitute the unit of analysis of this study. Excerpts of these transcripts will follow below.

The students worked with certain exercises on number sequences. The exercises gave the students the first few terms of a sequence of numbers, and asked them to find the next term as well as a general expression for each term of the sequence. Thus, the students were working with numbers, finding patterns and generalizing to find the general term of a given sequence. One could characterize this as vertical mathematising (Freudenthal, 1991). To work with number sequences does probably not require calculations that are very complicated for the students. The students only needed some basic algebra knowledge. The concepts involved such as numbers and sequences should not be very difficult for the students. Thus, the students could concentrate on the problem solving aspect of it and work with their own ideas, some of which possibly might have some novelty for the students. In addition, some sequences such as the Fibonacci numbers have appeared as topics of exercises in lower secondary school in Norway. Thus, the topic of number sequences is relevant for their future teaching. The students experienced this relevance for their future teaching, and this was clearly motivating the students to work with this particular topic.

Literature review

The first part of the research question is to characterize pre-service teachers' mathematical reasoning as the first step to develop ideas about how to develop their reasoning. Therefore, as part for the research behind this paper, the author conducted a limited and systematic search for literature dealing with the characterization of mathematical reasoning. This review will mention some of the most relevant literature. The study uses the research framework of Lithner (2008) and consequently Lithner is fundamentally important. Individuals are using the notion of creativity in many different contexts. This makes it necessary to discuss the meaning of creativity in the context of mathematics. The purpose of this literature review is to draw attention to what literature has to say about the meaning of creativity not in general but in the field of mathematics. Since this study is looking for ways and ideas to develop pre-service teachers' mathematical reasoning, it might be useful to consider the differences between expert mathematicians and novices.

Sriraman (2009) investigated how mathematicians create mathematics. In a qualitative study, he interviewed five creative mathematicians. The results indicate that in general, the mathematicians' creative process followed the four-stage Gestalt model. Sriraman also thoroughly discussed the meaning of creativity in the mathematical field. In his definition, the novelty of the work is essential. Since pre-service teachers are not expert mathematicians, the Gestalt model may be less relevant for this study. The Gestalt model describes a creative process where during the period of incubation one may leave the problem for a while and do something else. This is something expert mathematicians can do, but it would not have been easy for the participating students of this study to do so within the time schedule of the lesson where the recording took place. For the Gestalt model to be relevant, it would require the possibility to leave the problem and come back to it after a period of incubation. Possibly this does not happen very often in upper secondary school, perhaps even less so in lower secondary school. What is required for this study is to relate the notion of creativity to the mathematical activity of students rather than expert mathematicians.

Haylock (1987) proposes a framework for fostering and rewarding mathematical creativity in schoolchildren. Two key aspects form the basis of this framework. One aspect is the ability to overcome fixations in mathematical problem solving. The other aspect is the ability for divergent production within mathematical situations. Haylock makes the point that thinking flexibly and divergently are qualities of mathematical thinking, which might justify the description "creative". According to Haylock these qualities of mathematical thinking are sadly neglected in school mathematics. Haylock makes the distinction between *algorithmic fixation* and *content universe fixation*. To explain these ideas, Haylock makes it clear that a pupil may show fixation in mathematics by continued use of an initially successful algorithm even when this becomes inappropriate or less than optimal. This would be algorithmic fixation. Secondly, the fixation may be some sort of self-restriction related to the content universe of the problem. The pupil may restrict inappropriately or unnecessarily the range of elements which may be used or related to the given problem. This would be content universe fixation. To explain the notion of divergent production, Haylock gives as an example the task to find all possible ways to use a brick. The task has many solutions and the challenge is to find many solutions. Divergent production is contrasting convergent thinking where the task has only one solution. The research framework of Haylock (1987) is concerned with schoolchildren. However, the present study focuses on pre-service teachers.

A research framework that is concerned with students beyond upper secondary school, such as beginning under graduates is the research framework of Lithner (2008). This framework might be more relevant for the present study. The purpose of Lithner's (2008) research framework is to characterize mathematical reasoning, and explain the origins and consequences of different reasoning types. A basic idea is that rote learning reasoning is *imitative*, while the opposite type of reasoning is *creative*. Lithner says that *reasoning* is the line of thought adopted to produce assertions and reach conclusions in task solving, thus not restricted to proof, and may even be incorrect. Suppose the task is to find the maxima or minima of a given function $y = f(x)$. Some students may try to solve this task by finding the derivative of the function and solving the equation $f'(x) = 0$. This strategy may work if the function is a second-degree polynomial. However, in general it is insufficient. It may fail even for a third-degree polynomial such as $f(x) = x^3$. This function has a critical point at the origin, but this point is not a maximum nor a minimum. The reason for this strategy choice may be an earlier example. The student may have experienced that the strategy has worked before. To consider the similarity with an earlier example would be a *surface property consideration,* when the mathematical content involved is less important for the strategy choice. Sometimes surface property considerations only may be the basis of the strategy choice or reasoning (Lithner, 2008). If so, the reasoning would be imitative. The other type of reasoning called *creative mathematically founded reasoning* fulfils all of the following criteria.

1. Novelty. A new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created.
2. Plausibility. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible.
3. Mathematical foundation. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning.

Polya (1954) makes the distinction between *demonstrative reasoning* and *plausible reasoning*. The framework of Lithner (2008) proposes a concept of mathematical reasoning inspired by Polya's (1954) notion of plausible reasoning. In his preface, Polya makes the point that "the result of the mathematician's creative work is demonstrative reasoning, a proof; but the proof is discovered by plausible reasoning, by guessing" (Polya, 1954, p. iv).

The intrinsic mathematical properties of the components involved, refers to the properties of the numbers, functions, matrixes or other mathematical components involved in the reasoning.

Lithner (2008) summarizes by explaining that in creative reasoning the epistemic value lies in the plausibility and logical value of the reasoning, whereas in imitative reasoning it is determined by the authority of the source of the imitated information.

The research framework of Lithner (2008) is concerned with the mathematical reasoning of students. However, in a literature review Leikin and Pitta-Pantazi (2013) makes the point that some research studies focus on the creative person, some on the creative process, some on the creative product and some on the creative environment. To study the environment such as the class would have required a different approach. One would presumably look at such phenomena as group dynamics. The present study focuses on the mathematical reasoning of pre-service teachers. Thus, the study focuses on the creative individual. Leikin and Pitta-Pantazi also makes the useful distinction between relative and absolute creativity. Absolute creativity refers to the kind of creativity the professional community evaluates as high and significant achievements. Such as work rewarded by international prizes. Relative creativity refers to the work of an individual such as a student when the work is not new to the professional community but has novelty to the individual.

Methodology

The motivation for the study was my own teaching experience. This includes teaching classes in upper secondary school as well as teaching pre-service teachers mathematics. Mostly students preparing to teach in lower secondary school. My own teaching experience gave me the impression that students reasoning is largely imitative and restricted to procedures. The idea was that by setting up a teaching experiment it might be possible to understand the reasoning of the students more deeply. By dividing the students into smaller groups, the students could work together and solve some exercises. The idea was that to understand their reasoning I could analyze the dialogue of each group. It might be that one could understand more by analyzing the dialogue of each group than by analyzing the students' written works. I divided the students into groups from two to four students and recorded their work on video and audio. The students constituted their own group as they wished. This may perhaps have helped the dialogues to run smoothly. However, there may also have been certain group dynamical aspects that could have had an impact on the dialogues within a group. It remains for a later study to look at what kind of impact group dynamical aspects may have. The only restriction was the size of the group. To avoid passive group members it was important that the groups were relatively small. The assumption was that with small groups every group member would have to contribute to the dialogue. I did not look for excellence or any kind of mathematical giftedness when asking the students to participate. This means that the students probably were more or less on the same level with regard to mathematical ability. However, they were pre-service teachers and perhaps more willing than other students, to share and explain what they had understood to their fellow students. In any case, the dialogues were running smoothly in each group. I informed the students that the exercises would be an integral part of the course, and thus that they were working as normal and preparing for their exam. This may have contributed to the authenticity of the study. Sfard (2008) introduced the term *commognition* to make the point that communication and cognition are like the two sides of the same coin. This means that thinking is an individualized version of interpersonal communication. Thinking is the communication we do with our selves. The point of view in this paper is that thinking includes mathematical reasoning and any mathematical dialogue is a form of communication. Thus, it makes sense to analyze the mathematical dialogues of the students to understand some of their individual mathematical reasoning. The participating students of this study were asking each other questions within each group on how to understand the problems and how to solve them. They also suggested to each other what to do at each step of the solution process. They shared both questions and mathematical ideas. This means that at least some of the individual reasoning of each student became part of each group dialogue as they communicated it to each other. I did not give any complete method of solution to the exercises. However, I did give the students a hint. The hint was to write down the differences of consecutive terms of the sequence to get certain equations. Then to add these equations. The students were supposed to experience how the terms of the sequence cancel out when adding the equations, and how this makes it possible to solve the problem. One might argue that giving this hint to the students, I was guiding the steps of the learning process of each student. If so, this would be what Freudenthal (1991) label as *guided reinvention.* I also introduced some notation through an initial example. This included writing a sequence as an infinite row of numbers $a_1, a_2, a_3, \ldots$ using three dots to indicate infinity. I also introduced the notation (a_n) for the sequence itself and a_n for the general expression of each term of the sequence.

Analysis

To have a closer look at the students reasoning let us start with one group with two students. Called Sara and Tom in this paper. They were working with the following exercise:

Episode 1

Exercise. Given the sequence starting with the terms:

$$0, 4, 10, 18, 28, 40, \ldots$$

Find the next term of the sequence. Then, find an expression for the general term of the sequence.

When the video recording starts, the two students have written down the following sequence of numbers:

$$0, 4, 10, 18, 28, 40, 54, \dots$$

Obviously, they have found the next term of the sequence to be 54. The video recording also reveals that the students have written down the following equations:

$$a_2 - a_1 = 4 = 2 \times 2$$

$$a_3 - a_2 = 6 = 2 \times 3$$

$$a_4 - a_3 = 8 = 2 \times 4$$

$$a_5 - a_4 = 10 = 2 \times 5$$

$$a_6 - a_5 = 12 = 2 \times 6$$

$$a_n - a_{n-1} = 2n$$

$$a_n - 0 = 2 \times 2 + 2 \times 3 + 2 \times 4 + 2 \times 5 + 2n$$

$$a_n = 2(2 + 3 + 4 + 5 + \cdots + n)$$

The two students obviously starts by using the idea explained to them by the instructor. They also realize that $4 = 2 \times 2$, that $6 = 2 \times 3$, that $8 = 2 \times 4$ etc. and in general that the difference between consecutive terms is $2n$. This must have been the students own reasoning as the instructor did not help them with this. However, the video reveals that the students have a problem with the sum in the brackets. Let us look at the dialogue when they discuss the situation.

1. Sara: Must find the number one.
2. Tom: It is not for sure that there should be a number one.

The students were familiar with the triangular numbers and thus they knew that

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$$

However, the sum in the brackets does not start with the number 1. It appears that this constitute a problem for the students. They did not simply rewrite and use the equation

$$2 + 3 + 4 + \cdots + n = \frac{1}{2}n(n + 1) - 1$$

This would of course have solved the problem. The question is why they did not do so. If the students' reasoning were imitative, then they would probably look for the way they think one is supposed to work it out, rather than to try finding a solution by themselves. This would mean that whenever the students find themselves in an unfamiliar situation they may not know what to do. The question of Tom in line 2 might indicate imitative reasoning. If one were considering surface properties rather than intrinsic ones, it might be natural to ask whether there should be a number one or not. However, if one were considering intrinsic mathematical properties, it would perhaps be more natural to ask what one could do in the situation. Therefore, when he says that there should perhaps not be a number one, he might perhaps be considering surface properties rather than intrinsic mathematical properties. The video shows that Tom is turning the pages in his notebook and studying in particular a certain page. The video reveals that Tom has written down the following equations in his notebook:

$$a_2 - a_1 = 8 = 4 \times 2$$

$$a_3 - a_2 = 12 = 4 \times 3$$

$$a_4 - a_3 = 16 = 4 \times 4$$

$$a_5 - a_4 = 20 = 4 \times 5$$

$$a_n - a_{n-1} = 4n$$

Let us go back to the dialogue.

4. Tom: Here we have, here we have started with four times two, four times three,
5. Sara: Yes… and then we have put four outside, but we have a one there, so that we have…
6. Tom: It must be the same.

To solve the problem they obviously look at similar example they have found in Tom's notebook. This further indicates that the two students are into surface property considerations rather than intrinsic mathematical properties and thus that their reasoning is imitative. The video reveals that the example they look at is the sequence (a_n) beginning with the terms

4, 12, 24, 40, 60, …

Writing down the differences between consecutive terms of the sequence one finds the above equations. Adding these equations gives the equation

$$a_n - a_1 = 4 \times 2 + 4 \times 3 + \cdots + 4n$$

Since $a_1 = 4$ we find that

$$a_n = 4 \times 1 + 4 \times 2 + 4 \times 3 + \cdots + 4n$$

Hence, we have

$$a_n = 4(1 + 2 + 3 + \cdots + n).$$

As Sara says in line 5, they have put a four outside and then they have a one inside the brackets. The problem with the missing number 1 does not seem to exist in this example. The video does not show whether the two students have done this by themselves or not. They may for instance have written it down from the blackboard. Let us continue with the dialogue.

7. Sara: But why have they 4 times 1, plus 4 times 2, from where do we have that one… it wasn't 4 first, it was 8… in the sequence. Here we have zero first in the sequence in the ordinary sequence before we have found the derivative sequence, there is the derivative sequence.

The terminology used by Sara requires some explanations. Given any sequence (a_n), the difference $a'_n = a_{n+1} - a_n$ defines a new sequence (a'_n). The instructor introduced the notion of the derivative sequence for this new sequence. When Sara in line 7 speaks of the derivative sequence, she probably has in mind the derivative sequence of the first sequence. In line 7, Sara is comparing the two sequences. The first part of what she says is about the sequence beginning with 4, 12, 24, 40, 60, … The second part is obviously about the sequence beginning with 0, 4, 10, 18, 28, 40, …

8. Tom: Yes… but four times one is again four, there in the derivative sequence
9. Sara: right? here is the derivative sequence
10. Tom: Don't know if one can say that between zero and four there is four. We start it at four.
11. Sara: Okay, it is here, it is only four in between. It starts at four, that one starts at four times eight that one.
12. Tom: yes, yes, but one can put zero in front and put four times one here. This one is four times two, times three, dat, dat, four times n plus one
13. Sara: Now you get one from there.
14. Tom: mhm.
15. Sara: Thus, you get 4 times 4 here
16. Tom: and 4n there, then you take one of those and move it forward
17. Sara: but we do not have that number one here

What Sara says in line 17, indicates that they still have a problem with the missing number one in the sequence they started with. In addition to that, if one were considering surface properties rather than intrinsic mathematical properties it might be natural to ask the way Sara does in line 17, indicating imitative reasoning.

18. Tom: but don't just say one
19. Sara: That is why I think, must be, therefore it must plus one, that is first and then you must have…

Tom calls for the instructor by raising his hand.

20. Instructor: Yes
21. Sara: We don't know where to find the number one to get a triangular number.

The instructor tried to guide the two students by making them see the possibility that one could write

$$2 + 3 + 4 + \cdots + n = \frac{1}{2}n(n+1) - 1$$

The students eventually realized that this would be a way out.

To contrast the first group let us look at another group with three students, Mary, Jane and Johanna.

Episode 2
Exercise. The given sequence (a_n) starts with the terms:

3, 8, 15, 24, 35, 48, …

The dialogue opens as follows:

1. Mary: a-n minus a-one will be

2. Jane: a-n minus a-one equals, or we don't take that one. Two times two plus one, then we see that
3. Johanna: so there, in a way you have done the "zipper".

The terminology used here by Johanna needs explaining. "The zipper" was the method of writing down the difference between consecutive terms to get certain equations and then add these equations.

4. Jane: yes, should we do something about the number one first?
5. Mary: two-n plus one
6. Jane: should we begin by picking up that number one? We have, a-one is three, should we use that to get it into the sequence in some way?
7. Mary: What do you mean?

So far, the dialogue indicates that the students are considering several options. In line 4, Jane asks if they should do something with the number one first. Thus indicating that there are other possibilities. In line 6, Jane suggests they should do a certain operation first, also indicating that they could as well do something else. It might be that they are aware of the possibility of choosing between different mathematical ideas. At this point, the video recording reveals that Mary has written down the following equations:

$$a_2 - a_1 = 5 = 2 \times 2 + 1$$

$$a_3 - a_2 = 7 = 2 \times 3 + 1$$

$$a_4 - a_3 = 9 = 2 \times 4 + 1$$

$$a_5 - a_4 = 11 = 2 \times 5 + 1$$

$$\vdots$$

$$a_n - a_{n-1} = 2n + 1$$

$$a_n - a_1 = (2 \times 2 + 1) + (2 \times 3 + 1) + (2 \times 4 + 1) + \cdots + (2n + 1)$$

The three students obviously follow the hint the instructor gave them, that they could write down the differences between consecutive terms of the sequence to get certain equations and then add these equations. Thus, this part of their reasoning would be imitative. However, it was their own idea to write $5 = 2 \times 2 + 1$, $7 = 2 \times 3 + 1$, etc. The interesting part of their reasoning is perhaps the second part. They have found the difference $a_n - a_1$ expressed as a sum and the problem is to find this sum. Let us return to the dialogue.

8. Jane: I mean that, if we begin by adding the number of times we have the number 1,
9. Mary: We have the number one n plus one times.
10. Jane: we have the number one n minus one times, we have that many equations there, we have the number one that many times there.
11. Mary: no, we have deleted that one
12. Jane: yes but, now I am only counting the number of equations we have here
13. Mary: yes, but when we going to add up to n plus one
14. Jane: Yes, we agree so far, but then I thought that now when we move on and think about how many times we have the number one, yes and we have the number one, as

many times as there are equations here, we have that many times the number one, and there are that many equations, n minus one equations. I learned that last lesson.

15. Mary: Yes, we have the number one n minus one times yes.

Obviously, Jane has understood the idea of the method. Mary responds to this in an affirmative way indicating that she has probably understood the idea as well. Since Johanna does not protest to this, one may perhaps assume that she has understood as well. However, Johanna does not say as much as Mary and Jane so it is more difficult to know how she thinks.

16. Jane: So let us write it up first. a-n minus 3 equals n minus 1 plus and then we may write,
17. Mary: Okay, writing one, one really puts one outside like that. We have now added all the number ones, and then we may put two outside.

The video reveals that when Mary is saying this she has written $a_n - 3 = 1(n - 1)$ and that she is pointing her index finger at the expression on the right side of this equation. By doing so, she is probably asking the two other fellow students if they agree with what she is doing.

18. Jane: Yes, but should we not move over the number 3 first, for it is really 2 times 1

Jane has realized that to write $3 = 2 \times 1 + 1$ is useful.

19. Mary: It says plus between them
20. Jane: then we have the number three, we may write it as one plus one times two. It is not a triangular number
21. Mary: Not? Plus the number three over,
22. Jane: yes, yes,
23. Mary: three, dat, dat, dat.

The students are quite happy with what they are doing.

24. Jane: If we take the number three over, we could write it as one plus two time's one. Do you agree? I don't have any rubber so it is going to be messy.
25. Mary: three equals two times one plus one, yes.
26. Jane: and then we may use it, in the sequence,
27. Mary: but do we then have another number one?
28. Jane: yes, we had two times two only, right? Here we have two times two, we don't have two times one, but now we have.
29. Mary: Yes, but a number one like that more, more than n minus one times the number one.
30. Jane: yes, we have a number one an extra time, but we simply put it outside the triangular number there, formula.
31. Mary: But then we only have n right, now we get plus two times one plus one. Then we have a number one, one extra time, this is the number of ones.

At this point, the video revels that Mary has written $a_n = 1(n - 1) + 2 \times 1 + 1$ still pointing her index finger at the same spot.

32. Jane: so when we add this up, n minus one plus one, then that one disappears.

33. Mary: For now we have that plus that, then we have plus two times two plus one, no the plus one we have removed.
34. Johanna: But if we start by move over the number three, it will not be so messy.
35. Jane: Isn't that what we have done?
36. Johanna: Yes, we started out by making this expression using the number of ones. If only we start out correctly by moving over.

What Johanna is saying in line 36 may indicate that she has understood what has been going on even though she has not been very talkative.

37. Mary: so a-n, it will be two times one plus one, plus two times two plus one.

Here the video reveals that Mary is writing the expression

$$a_n = (2 \times 1 + 1) + (2 \times 2 + 1) + (2 \times 3 + 1) + \cdots + (2n + 1)$$

38. Jane: Okay, you haven't taken the number one yet. Must see if we end up with something similar.

Perhaps Jane thinks that Mary is too slow.

39. Mary: But do we have the number one n minus one times now?
40. Jane: This is what you should do now. You put the number ones outside. Now there is one extra, so there is one times n terms.
41. Mary: Yes, for there we had n minus one times the number one, but here we have n times the number one because we have moved one over. Then we have one n plus two times one plus two plus three plus, dat, dat, dat, what do we hav now? Plus n?
42. Jane: Yes
43. Mary: plus 2n, I am not so sure about this.

The video shows that at this point Mary has written

$$a_n = 1(n) + 2(1 + 2 + 3 + \cdots + 2n)$$

The use of parentheses is somewhat inadequate. However, Jane has written

$$a_n = n - 1 + 1 + 2(1 + 2 + 3 + \cdots + n)$$

44. Jane: Yes, but I have calculated like we did and I have the same that a-n equals n plus 2, then we have that one, whatever it was.

The video show that Jane is writing

$$a_n = n + 2 \times \frac{1}{2} n(n + 1)$$

45. Johanna: two-n plus
46. Jane: n plus one
47. Mary: n plus one to the second.
48. Jane: that is what we use I guess
49. Mary: then you have n plus, what was the result, 2, what did we get?
50. Jane: two times T-n, triangular number n, right, isn't that the way you are supposed to write?

According to the video Mary completes the calculation and writes

$$a_n = n^2 + 2n$$

Having done that she concludes with

51. Mary: should be n to the second plus two-n?
52. Jane: let us find out if it is correct. We take term number four or something, n equals 4, n to the second plus two-n is 16 plus 8, which is 24. Is it correct? n equals 5, 5 to the second plus two times 5 is 35
53. Mary: 24, yes it is correct.

Discussion

As their instructor, I selected the exercises for the students to work with. Thus, the students did not pose the problems themselves. Neither did they develop any further questions to the exercises. It is quite possible that the form of exercisers did not invite for any further questions. I also provided the students with a hint to get started. The idea with the hint was to avoid that some groups of students were totally stuck. From the point of view of the instructor, this was simply a way to administrate the situation when several groups of students were working on some mathematical problems at the same time. From the researcher's point of view, this was also a way to improve the chances to have some interesting dialogues to analyze the students reasoning.

However, as a possible further development of the teaching experiment it could be interesting to see what happens if the participating students have to work with such problems without any hints.

Imitative reasoning

The first episode shows two students who probably are largely reasoning imitatively. They look at an earlier example to solve the exercise. Probably what the students are doing is to look for similar examples. Having found an example which they consider as similar and which they trust, the idea obviously is that they can do likewise with the problem they have as with the similar example. One might think that to look at an earlier example would complicate matters. Not make it easier. However, to find an earlier problem, might not be a bad idea. In fact, to recall a formerly solved problem is what Polya (1945) suggests one should do. However, the similarities the students find are surface similarities. It is of course not sufficient to consider surface properties of the components involved. When Polya says that if you have solved a related problem before, you should try to exploit that, he has of course in mind the intrinsic mathematical properties of the components involved. The problem for the students is perhaps not using an earlier problem but rather that they are largely considering surface properties. There is a lot more to learn from a former example by looking at the intrinsic properties of the components involved. The question why students choose to consider surface properties of the components involved rather than intrinsic properties is complicated. There are probably no simple answers to that question.

Flexible reasoning

The second episode shows three student who are considering different options. Thus, their reasoning is to a certain extent flexible (Haylock, 1987). The three students individually suggests what they can do. They discuss these suggestions and try to agree on what to do.

They also ask each other questions. This means that they chose between different ideas in a flexible way. It is reasonably clear that one difference between the two episodes is that in the first episode the two students try to find out what they are supposed to do. Whereas, in the second episode, the three students try to come up with ideas of what they can do about the situation. However, nothing indicates that the ideas they come up with have any novelty to them. They simply choose between familiar mathematical ideas. The flexibility of their reasoning indicates that they are not reasoning imitatively. Lithner makes the distinction between imitative reasoning and creative reasoning. Novelty is required for creative reasoning. If their reasoning does not have any novelty, their reasoning is not creative either (Lithner, 2008). To understand the reasoning of the pre-service teachers of this study it is required with a further development of these categories of reasoning.

Non-imitative reasoning

The first episode shows two students who are probably largely reasoning imitatively although not totally. They started out by writing: $4 = 2 \times 2$, $6 = 2 \times 3$ etc. To do this was not part of the hint I gave them. Therefore, it is possible to argue that this part of their reasoning was not imitative. The second episode shows three students who reason more flexibly (Haylock, 1987). However, they began their reasoning by following the hint I gave them. Therefore, this part of their reasoning was imitative. Having finished the first part of their reasoning, they discuss among them certain ideas to solve the problem. However, if these ideas were new to the students, this would probably show in the dialogue. The dialogue of the second episode shows three students discussing ideas that are probably already familiar to them. Therefore, nothing indicates that the ideas they discuss have any novelty to them. This means that their reasoning is not imitative nor creative. Mathematical reasoning that is not imitative we might label as non-imitative reasoning. This paper will make the distinction between imitative reasoning and non-imitative reasoning. Thus, non-imitative reasoning would include reasoning which is creative as well as not creative.

Mathematical justification

The question of what a mathematical justification or verification is for students as opposed to expert mathematicians might be very different. By the end of the second episode, the three students Jane, Mary and Johanna want to examine if the result is correct. They do so by looking at examples. They choose $n = 4$ and calculate that $4^2 + 2 \times 4 = 16 + 8 = 24$ which equals a_4 of the given sequence. They choose another examples which is $n = 5$. Again, they calculate and find that the result is in accordance with the term a_5 of the given sequence. Having done that they conclude that what they have found is correct. Nothing indicates that they have in mind to look at their own reasoning critically to see if it is valid. Freudenthal (1991) makes the point that even though mathematics is rooted in common sense, mathematics is very different. Freudenthal argues that the most striking example of mathematics rooted in common sense is whole numbers (Freudenthal, 1991). Childrens acquisition with whole numbers comes with their normal activities. Freudenthal also states that the use of numerals in spoken language supports the acquisition of number. However, mathematics is different from common sense. Perhaps a very early example in the history of mathematics beyond common sense is the discovery of incommensurable quantities such as the side and diagonal of a unit square. We now instead have the notion of irrational numbers. It would be common sense to examine a result by checking a few examples. However, that is not sufficient in mathematics. Perhaps this is one reason why students struggle with the logic of mathematics. The logic of mathematics is not as simple as the logic of common sense. Mathematical logic is more abstract and contains variables.

The derivative of a sequence

The students worked with sequences given to them by me the instructor. I gave the students the first few terms of a sequence

$$a_1, a_2, a_3, \ldots$$

The students were supposed to find a general expression for each term a_n of the given sequence (a_n). I gave the students an idea to get them started. The idea was to write down the differences between consecutive terms.

$$a_2 - a_1 = a_1'$$

$$a_3 - a_2 = a_2'$$

$$a_4 - a_3 = a_3'$$

$$\vdots$$

$$a_{n+1} - a_n = a_n'$$

Adding together, we find that

$$a_{n+1} = a_1 + \sum_{k=1}^{n} a_k' \qquad (1)$$

The problem was to find the sum on the right hand side of equation (1). There is a clear analogy here with the fundamental theorem of calculus when stated as

$$f(b) = f(a) + \int_a^b f'(x)\,dx. \qquad (2)$$

In equation (2) f is a function with a continuous derivative on the closed interval $[a, b]$. This analogy might make it reasonable to speak of the derivative of a sequence and to write it as (a_n') where $a_n' = a_{n+1} - a_n$. My introduction of the notion of the derivative sequence in this teaching experience was justified by this analogy.

Inquiry based mathematics education

To get started the students had a hint, or an idea called "the zipper". It was to write down the differences between consecutive terms of the sequence to get certain equations, and then add these equations. All groups followed this hint. Obviously, they all experienced how the terms canceled out when they added the equations. Thus, this part of their reasoning was imitative. However, they did not get a complete solution. They had to work out the sum on the right side of the above equation (1). Each group of the class did this somewhat differently. In episode 1, we see two students who are largely reasoning imitatively. They eventually ask for some more guidance and then solve the problem. In episode 2, we see three students who reason more flexibly. Their reasoning is non-imitative and they solve the problem without any guidance except for the initial one I gave them. This indicates that it was not possible for the participating students to solve the problems using only imitative reasoning. Hence, the exercises were problem solving and not routine for them. A part of this problem solving was probably also to analyze the sequences and write for instance that $4 = 2 \times 2$, $6 = 2 \times 3$ etc. Problem solving is one aspect of inquiry-based learning (Artigue & Blomhøj, 2013).

According to Artigue and Blomhøj (2013) inquiry-based pedagogy can be defined loosely as a way of teaching in which students are invited to work in ways similar to how mathematicians and scientists work. The problem solving aspect of the students' work with the sequences makes it reasonable to characterize the students' learning as inquiry-based. The two groups, one reasoning largely imitatively, the other non-imitatively, must have experienced the teaching experience rather differently. None of the groups reasoning had any novelty. Thus, none of the groups reasoning was creative. However, some of the students' reasoning was not imitative either.

Conclusion

There is no reason to characterize the mathematical reasoning found in this study as creative, because nothing indicates that it had any novelty for the students. This does of course not exclude the possibility of creative mathematically based reasoning among pre-service teachers (Lithner, 2008). It remains for a later study to see what happens if the students have to work with such problems without any hints. It might be that the students would have had to come up with more ideas themselves. Using the research framework of Lithner, I characterized some of the students' reasoning as imitative. However, some of the students' reasoning had flexibility (Haylock, 1987). Thus, some of the reasoning found in this study is neither imitative nor creative. Hence, there is a grey zone between imitative reasoning and creative reasoning. In this paper, we call mathematical reasoning, which is not imitative, non-imitative. This means that non-imitative reasoning includes creative reasoning as well as reasoning that is not creative.

The reasoning of expert mathematicians and scientists is not imitative. Otherwise, it would not be research. The reasoning of the two students in episode 1 was largely imitative. They needed some extra guiding to solve the problem. I argued that in episode 2 some of the reasoning of the three students was non-imitative. The two episodes indicate that the students were probably unable to solve the exercises, using imitative reasoning only. This indicates that the teaching experiment was a kind of inquiry-based pedagogy. If students mostly work with exercises which they can solve using imitative reasoning, they may perhaps not develop their reasoning beyond imitative reasoning. Thus, to develop students' mathematical reasoning, it would be required to let them work with exercises they cannot solve using imitative reasoning. Obviously, there may be several ways to accomplish that students reasoning is non-imitative. However, one way indicated in this paper is to let students work be inquiry based (Artigue & Blomhøj, 2013).

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