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Abstract: *This study aims to exemplify a framework based on an empirical study and the empirical study is to explore how a mathematics expert teacher makes sense of multiplication. The study was conducted through interviewing a secondary school mathematics expert teacher with over 30 years of mathematics teaching experience. The theoretical framework used in this paper is proposed by Chin and Tall (2012) and further developed by Chin (2013) and Chin (2014). The collected data are anaylsed qualitatively using this framework in order to find out what are the supportive and problematic conceptions involved in making sense of multiplication and consequently investigate how do these conceptions affect the sense-making of multiplication across different contexts. The findings suggested that when changes of meaning occur, supportive or problematic conceptions might arise and these either support or inhibit learners from building a coherent understanding. This study also showed that supportive conceptions might contain problematic aspects and problematic conceptions might contain supportive aspects. The respondent strived to build a coherent understanding by removing the problematic aspect, however when the problematic aspect cannot be removed then he had no choice but to accept the new meaning of multiplication. It is the teachers' responsibility to guide students in realising their existing knowledge which may not be appropriate in a new context. Thus, this study offers a foundation for teachers to sense where, when and how the changes of meaning of multiplication take place which in turn can help them to facilitate students effectively in making sense of multiplication.*

Keywords: multiplication, supportive conception, problematic conception

Introduction

Mathematics is considered as a unique subject in a sense that the symbols used are ambiguous.

Take for instance, -5 can be conceived as the process of "subtract five" and the concept of "-5".

William Oughtred (1574-1660) is credited with using 150 different symbols in his work. His

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contributions were significant and widespread including the multiplication sign '×' that invented in the early seventeenth century and named as the Oughtred's cross. Using the multiplication sign, multiplication may be expressed as $p \times q = m$ where p is the multiplier, q is the multiplicand and m is the product. In general, it can be interpreted as p times of q.

It is believed that by examining how mathematics teachers make sense of multiplication, this may give important implications towards mathematics education. In mathematics, changes of meaning might occur when a context changes. This may lead to either coherent understanding or conflicting understanding. However, the issue is do teachers know exactly when the mathematical meanings of multiplication change in particular in the transitions of contexts from natural numbers to integers and to fractions? Can teachers sense the changes of context? How do they cope with these changes? In this paper, we aim to exemplify the framework proposed by Chin and Tall (2012), Chin (2013) and Chin (2014) that can help teachers to understand the complexities in making sense of multiplication across different contexts. In order to exemplify this framework, we demonstrate how this framework can be used to describe how a mathematics expert teacher makes sense of multiplication. There is no doubt that most mathematics teachers possess the procedural fluency in performing multiplication and for most of them, the teaching of the multiplication concept is primarily concerned with memorizing the multiplication table, which can then be applied to solve examination problems (Vohra, 2007). However it will be more fruitful to see how they make sense of multiplication, which eventually has the potential to guide their students in understanding multiplication.

The finding of this study is noteworthy as most of the past research studies (Whitacre & Nickerson, 2016; Lo, Grant, & Flowers, 2008; Menon, 2003; Southwell & Penglase, 2005; Thanheiser, 2010) have focused on the understanding of multiplication of students or prospective teachers and the research on expert teachers is still rare. On top of that, the framework that we want to exemplify in this paper has never been applied for the topic of multiplication and we believe that this framework has great potential to explain how humans make sense of multiplication over the longer term. It is hope that this paper can make readers especially teachers to realize the implications of changes of context in making sense of multiplication. Teachers who act as the facilitator of students' learning need to be able to sense when, where and how the changes occur in multiplication so that they can create the necessary experience for students to learn multiplication over the longer term. In this paper, first we review literature about the development of multiplication within the mathematics education research then we outline the framework that forms the basis of this study. This is followed by details of the methodology, the findings and discussion then finally the conclusion.

The Case of Multiplication

Fischbein, Deri, Nello and Marino (1985) claimed that the concept of multiplication is intuitively attached to a repeated addition model. However, under this intuitive interpretation, they showed that the operator must be a whole number. Take for instance, 2×3 can be written as 3+3 where we can interpret it as 2 lots of 3. In other words, the meaning of multiplication in this case is interpreted as repeated addition. Surprisingly, when the operator is a decimal or fraction, it has no intuitive meaning (Fischbein at el., 1985). Fischbein at el. (1985) also argued that for multiplication by a decimal or fraction, although the multiplier has no intuitive meaning, it is not to say that it has no mathematical meaning. This sparks an interesting question of what is the meaning of multiplication when it involves decimal or fraction?

In this regard, based on a survey conducted by Saleh, Saleh, Rahman and Mohamed (2010) on 202 Year Two primary school students, their findings also showed that the concept of multiplication as repeated addition is open to more than one interpretation. This led us to think what is/are the other interpretations of multiplication. This also made us to realise the subjective nature in making sense of multiplication. Although some pupils in the study of Saleh et al. (2010) can perform the standard operation of multiplication as they memorised the multiplication tables and multiplication facts but most of the participants could not solve real life problems related to multiplication. This suggested that a conceptual understanding is needed by pupils to excel in all contexts of multiplication. In this case, conceptual understanding is not limited to the understanding of concepts only but it also involves the understanding of meanings underlying the operations (Burris, 2005). In order to have a conceptual understanding, the pupils need to be able to make sense of multiplication. According to NCTM (2009), "sense making may be considered as developing understanding of a situation, context, or concept by connecting it with existing knowledge or previous experience" (p.4).

The development of multiplicative reasoning is a complex process (Kaput, 1985; Schwartz, 1988). A study done by Simon and Blume (1994) reported that many prospective elementary teachers do not have a well-developed concept about why the relationship of the length and width of a rectangle to its area is appropriately modeled by multiplication. Simon and Blume (1994) described the reasoning of one prospective teacher, Molly, who was making sense of the given rectangular area based on repeated addition. However, there is a subtle difference in using repeated addition for the purpose of computing area with those ordinary multiplications which do not involve unit of measurement. Even though the underlying operation of multiplication is the same for two instances but the mathematical meanings are different. Take for instance, you have two bags of apples and each bag contains 3 apples then altogether you will have 6 apples that can be worked out by 2×3. Another particular instance says you have a rectangle with length 2cm and width 3cm then the total area of this rectangle can be computed by $2\text{cm} \times 3\text{cm} = 6\ \text{cm}^2$. Obviously the same operation is used for these two instances but the mathematical meanings are different. Certainly the change of contexts is obvious for these two instances but the change of contexts might not be easily sensed when they involved different real numbers sets such as natural numbers, integers and fractions. This sparks a deeper question of how teachers cope with the changes of mathematical meanings which arise due to the changes of contexts. In school, the concept of Area is generally being taught as length times width without emphasizing the changes of contexts and meanings. Students almost always plug in the measurement of length and width into the Area formula without grasping the basic underlying idea which they can build upon on for future learning (Simon & Blume, 1994).

Webel and DeLeeuw (2015) performed a study on three fifth grade mathematics teachers and the data revealed that one of the teachers made explicit connections between the multiplication of whole numbers and the multiplication of fractions during mathematics lessons. The interpretation of multiplication as repeated addition has been extended from the notion of "times" to 'part of a part' in the multiplication of fractions. Larsson, Pettersson, and Andrews (2017) highlighted that consistent use of repeated addition or equal groups among students raise important questions about teacher's instruction. They argued that the way teachers introduce multiplication to students as repeated addition is a problematic instruction, particularly when multiplication is extended to multi-digits and decimals. The issue here is can teachers sense when exactly the mathematical meaning changes and how do they cope with it?

Multiplication is one of the four basic arithmetic operations in mathematics. Most students can perform multiplication to obtain an answer as this only involves the procedural skills of multiplication. Nevertheless, knowing solely the procedural skills of multiplication might not be able to assist students to solve non-routine mathematics problems successfully. This is supported by research studies which showed that students had insufficient aptitude to approach mathematical problems, especially non-routine mathematical problems in a successful way (Asman & Markowitz, 2001; Higgins, 1997). This is partly because students need to choose the correct arithmetic operations prior to reaching the correct answer. In this case, students must be able to interpret the meaning of multiplication in different contexts so that they know what arithmetic operations to choose in solving non-routine mathematical problems. This sparks an interesting question about arithmetic operations. Do arithmetic operations have different meanings in different contexts, in particular is this the case for multiplication? If there are different meanings for multiplication, then what are the consequences when changes of meaning occur?

Tall et al. (2001) showed that many discontinuities in the expansion of number systems, whether from whole numbers to integers, from integers to rational numbers, the use of whole number powers, fractional powers, negative powers, infinite decimals, infinite limit processes and so on. With these discontinuities, it is definitely reasonable to expect that students might get confused with the different meanings of multiplication in different contexts. Students might know the different meanings of multiplication but may not know when the meaning changes. As the common interpretation for multiplication is repeated addition, the mathematics textbook of Malaysia introduces multiplication immediately after the learning of addition (Wong, Wong, & Poh, 2011). In this case, multiplication is introduced as repeated addition in Year Two Mathematics textbook (Wong et al., 2011). This is ideal, as the curriculum designer was trying to build the meaning of multiplication based on students' prior learning which is related to addition.

But Devlin (2007) claimed that multiplication is not repeated addition. He wrote a Mathematical Association of America column titled, "It Ain't No Repeated Addition" and urged teachers to stop telling students multiplication is repeated addition. Based on Devlin, multiplication of natural numbers certainly gives the same result as repeated addition, but soon a different story is needed when multiplication involves fractions or arbitrary real numbers. Byers (2007) stated that a learner must learn how to deal with multiplication not only as repeated addition in order for them to be successful in arithmetic. The curriculum and instruction which emphasize the memorization of multiplication facts have produced students with less understanding of the basic concepts of multiplication than a standards-based curriculum, and instruction which emphasizes the construction of number sense and meaning for operations (Smith & Smith, 2006). Based on their study, Smith and Smith (2006) also found that most of the teaching methods had narrowed students' focus and had given the wrong impression about the concept of multiplication. This invites the question as to what are the other impressions and meanings of multiplication besides repeated addition that cannot be clearly seen by students and teachers?

There are many research studies that focused on examining the understanding and reasoning of multiplication of primary school students by using certain objects such as array representation in particular Barmby, Harries, Higgins and Suggate (2009); Young-Loveridge (2005); Izsak (2003); Steinbring, (1997) and so on. There are also many past research studies which have focused on the understanding of multiplication of prospective teachers such as Whitacre and Nickerson (2016); Lo et al. (2008); Menon (2003); Southwell and Penglase (2005); Thanheiser (2010) but the research on expert teachers is still rare. Whitacre and Nickerson (2016) pointed out that prospective teachers need to be able to make sense of mathematics and use their prior knowledge as a resource in learning. They investigated how prospective teachers made sense of multi-digit multiplication that involved positive integers. Our study is different with other past research studies in the sense that we focus on an expert teacher. On top of that, our study also focuses on the changes of meaning in the transition of natural numbers to negative integers and then to fractions. We wish to explore how an expert teacher uses his knowledge structure in mathematics to embody the operation of multiplication in the above mentioned different real number sets. Whitacre and Nickerson (2016) stated that prospective teachers came into classrooms with undesirable ideas that need to be fixed and removed. A deeper question is how these ideas can be fixed? Thus building on these arguments, we want to exemplify a framework that can be used to explain how teachers fix these undesirable ideas in a new context.

The Framework of Supportive and Problematic Conceptions

Prior knowledge and previous experiences shape our conceptions as stated by Tall (2013) and Chin (2013), constructivists believe that humans construct knowledge and meaning from their past experiences and this needs to be guided by a mentor, rather than only being transmitted. Lima and Tall (2008) proposed the term met-before and refer it as a mental construct that an individual uses at a particular time based on experiences he/she has met before. However, reasoning based on met-before and prior knowledge may create problems in a new context. Hence, Chin (2013) and Chin (2014) formulated the framework of supportive and problematic conceptions in order to illuminate how the effect of personal conceptions that were developed through met-before and prior knowledge affects the sense-making of humans in a new context. Two types of conceptions are involved in making sense of mathematics which known as supportive conception and problematic conception (Chin, 2013). In this case, a conception that works in an old context and continues to work in a new context is known as supportive conception. We can think of a supportive conception as an existing conception that can fit into a new context of learning perfectly (see Fig. 1). On the other hand, a conception that works in an old context and doesn't work in a new context is known as a problematic conception. Here, we can notice that the existing conception cannot fit into the new learning context nicely (see Fig. 1). As an illustration, the conception of multiplication makes bigger can be considered as a supportive conception in the context of positive integers (i.e. new context) and this conception might have arisen from the context of natural numbers (i.e. old context). On the other hand, if we move to the context of negative integers then the conception of multiplication makes bigger will be regarded as a problematic conception because this conception does not work in this new context. In short, a supportive conception supports generalisation in a new context whereas problematic conception impedes generalisation. Additionally, a supportive conception might consist of some problematic aspects whereas a problematic conception might contain some supportive aspects.

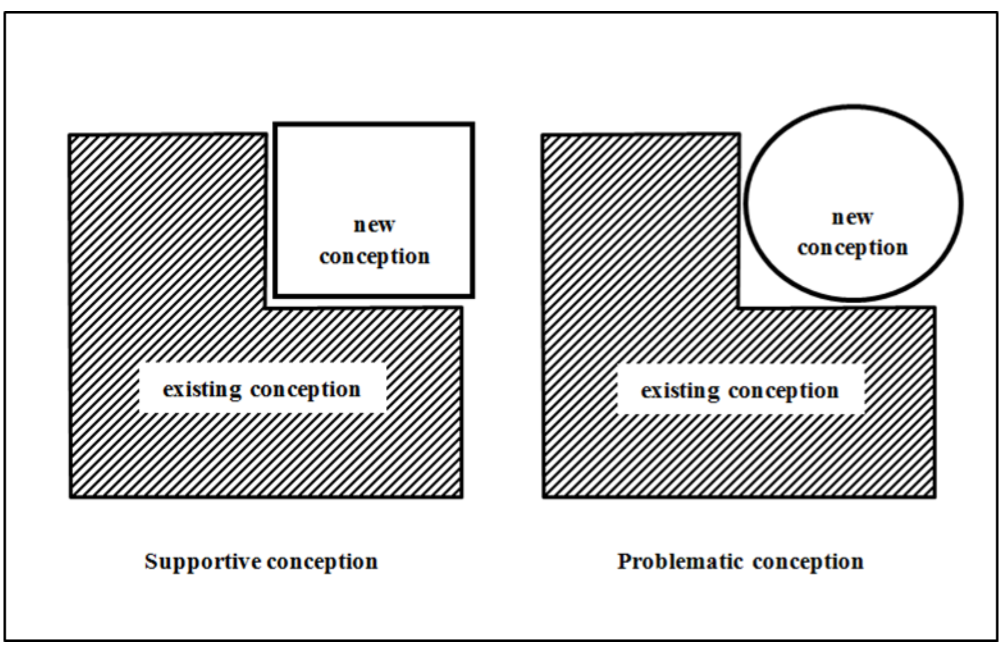

Figure 1. Supportive and problematic conceptions.

Smith, diSessa, and Roschelle (1993) argued that viewing learners' prior knowledge as the raw material for building more advanced knowledge was more beneficial than recognizing it as misconceptions and this was consistent with the view of Chin (2013). Many research studies concerning mathematical thinking on students have supported this view however it was less often applied to prospective teachers (Whitacre, 2013). What about expert teachers? This invites the question as to how expert teachers use their conceptions in making sense of mathematics? This framework is potentially useful in making sense of varies topics in mathematics. Take for instance the work of Taliban and Chin (2015) has demonstrated how this framework can be applied for the understanding of Complex Numbers among teachers. In this case, three teachers were asked whether $2+3i<4+3i$ was correct or not. Two of the teachers responded correct because they conceived 3i as a pronumeral that can be ignored as both sides of the given inequality had 3i. Hence they just compared 2 and 4 of the inequality. Another teacher responded this item by calculating the magnitudes of the given complex numbers. This shows how these teachers made sense of complex numbers based on their personal conceptions that have developed through the learning of real numbers. Additionally, Chin (2013) and Chin (2014) have also used this framework to describe how a group of prospective teachers made sense of trigonometry.

Methodology

In this paper, we report the data regarding multiplication that was collected by interviewing a mathematics expert teacher with extensive experience in teaching secondary school mathematics. He graduated with a second upper class degree in Mathematics and had 32 years of experience in teaching mathematics at the time of the study. He is currently teaching mathematics in a secondary school and participated in this study on a voluntary basis. This is a case and we have chosen a mathematics expert teacher because we think that this teacher has a better grasp of the mathematics content knowledge and the pedagogical content knowledge than many other teachers, and therefore we believe that he should be able to provide robust data on how he makes sense of multiplication. Furthermore, the expert teacher can give us insights on how he overcomes any problematic conceptions. Only one expert teacher is involved in this study because there are not many expert teachers in our region moreover this is the only expert teacher who is willing to participate our study. In previous sections, we notice that there are many past research studies on the understanding of multiplication of pupils and prospective teachers but not expert teachers. Thus this study can fill in the gap of knowledge in this area. An in-depth interview was conducted by using an interview protocol which was formulated and validated by two experts in mathematics education. The interview was conducted for an hour and intended to explore how the respondent made sense of multiplication which involved natural numbers, integers and fractions. The respondent was also required to give an example of real life problem for every instance. The collected data was analysed based on the framework of supportive and problematic conceptions as discussed in the previous section.

Results and Discussions

In this paper, we will represent the expert teacher as “T” and the researcher as “R” in reporting our data from the interview. Relevant excerpts will be presented first prior to the analysis of data. The data will show how the respondent interpreted the meaning of multiplication in different contexts.

Multiplication as "Repeated Addition"

R: What is 2 times 3?

T: It's 6.

R: Why it is 6?

T: Interesting questions. It is because there are 2 three. 2 three add together we get 6. Let me show you how it works (he was writing on a piece of paper (see Fig. 2 below)). When we multiply 2 and 3, we can simply add 3 two times to get 6. Or if 3 times 2, we may write as 2 plus 2 plus 2, which is equals to 6.

R: Why 2 times 3 is 3 plus 3 but not 2 plus 2 plus 2?

T: No, you see. What is 2 times x? There are two x right? It's not x number of 2. So it's x plus x. Although they provide the same answer but they are with different concept of explanation and different meaning.

R: What is multiplication then?

T: It is the process of combining all the quantities of certain object.

R: Can you please explain more?

T: It is a mathematical operation that involves the process of adding continuously the same quantity. For example your question just now, 2 times 3 is adding 3 continuously 2 times to get the final product as 6. There is a close relationship between addition and multiplication. Students have to learn addition before they learn multiplication. All this because mathematics is build out from pervious concept, it's not independent, they are all related.

Excerpt 1. Teacher's explanation for 2×3

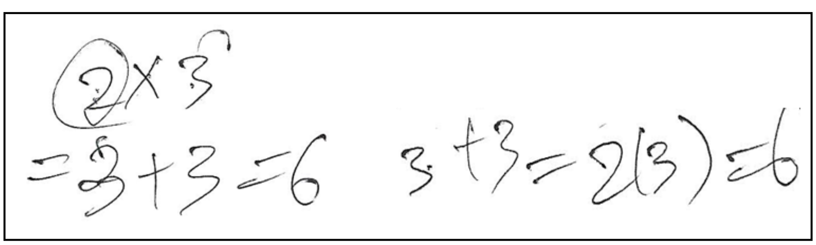

Figure 2. Teacher's writing for $\mathbf{2 \times 3}$.

Multiplication as "Repeated Subtraction"

R: Alright. Now what about -2 *times 3?*

T: This is a bit difficult. Quite hard to explain this. Make it this way. I owe you 3 dollars, I owe you two times of 3 dollars, so I owe you 6 dollars (he was writing on a piece of paper (see Fig. 3 below)).

R: What is the meaning of multiplication in this case?

T: Well, it's still addition where we add -3 *two times.*

R: Why is the negative sign now follow 3 not 2? Check back the question just now, it is -2 *times 3, not 2 times* -3*. What can you say about this?*

T: Actually the negative sign denotes the notion of owing. Owing you 2 times of 3 dollars.

Excerpt 2. Teacher's explanation for -2 × 3

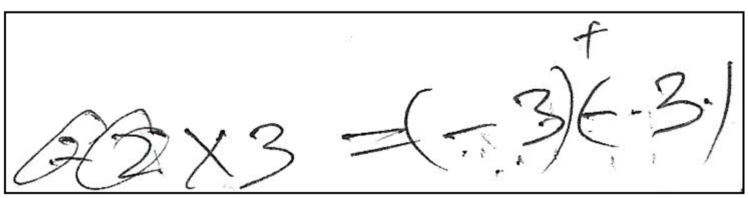

Figure 3. Teacher's writing for 2 × 3.

For -2 × 3, he transferred the minus sign to 3 and conceived it as owing 3 dollars then he further explained it as owing two times of 3 dollars. Based on his responses for 2 × 3, we generalise this concept to mean x times y can be conceived as x of quantity y. Building from this generalisation, for the case of -2 × 3, he transferred the minus sign to 3 so that he could interpret this as two times of owing 3 dollars. If he did not transfer the minus sign to 3 then he will have to interpret -2 × 3 as -2 of quantity 3 which will be problematic for him to explain using the notion of repeated addition. In order to build a consistent meaning for multiplication as repeated addition, the first number in the multiplication of two numbers must be a counting number. Repeated addition is conceived as a supportive conception with a problematic aspect for -2×3 because he could not explain it by using the notion of repeated addition without transferring the minus sign to 3. The problematic aspect arises because the multiplication is not a counting number. In this case, repeated addition was a supportive concept with a problematic aspect for him. He therefore transferred the minus to 3 in order to avoid the problematic aspect. At the end, he managed to interpret it as repeated addition by transferring the minus sign to 3.

If we reflect deeply on -2×3 and 2×-3, we will notice the subtle difference between them. -2×3 can be interpreted as $- (3) - (3)$ which means taking away 3 twice. On the other hand, 2×-3 can be interpreted as $(-3) + (-3)$ which means owing 3 for 2 times. Both answers are the same but the interpretations are different in real life situation. Multiplication in the first instance is repeated subtraction whereas in the second instance is repeated addition. Despite that, some people might conceive -2×3 and 2×-3 are the same and interchangeable due to the associative property and commutative property of multiplication. Take for instance,$-2 \times 3= -1 \times 2 \times 3 = 2 \times -1\times3 = 2 \times -3$. Based on Tall (2013), contextualisation gives us the meaning whereas symbolism gives us the power of computation. Given a real life context such as 2 bags of 3 marbles, we will write it as 2×3. On the contrary, 3×2 is interpreted as 3 bags of 2 marbles even though both situations would generate the same answer. However they are different in a real life situation. For the respondent, repeated addition is a supportive conception with a problematic aspect for the context of -2×3 and the problematic aspect can be removed by transferring the minus sign to 3.

R: What is -2 *times* -3*?*

T: (…) negative times negative become positive. It is based on the multiplication rule given (…)

R: In this case of negative times negative, is it still related to addition?

T: Ya, still. You see… we see -3 *as any object, let's represent it as "A". Again, we will have owing two times of "A" which is owing two times of* -3. *Eventually we get 6.*

Excerpt 3. Teacher's explanation for -2×-3 symbolically.

At this particular moment, the respondent was trying to build a coherent understanding for multiplication as repeated addition across different contexts. He elaborated -2 times -3 as owing two times of "A" where "A" is -3. He conceived "A" as an object because this will enable him to perform the repeated addition operation to this object. Likewise, at this particular instant, repeated addition was a supportive conception with problematic aspect. Again, the problematic aspect was caused by -2. Based on the excerpt above, his interpretation of -2×-3 can be written as [-A] + [-A] symbolically. For him, -2×-3 = [-(-3)] + [-(-3)] because A=-3. Just like the previous case, it was most likely that he did transfer the minus sign to A and subsequently arrived at a conclusion of "owing two times of A". This chronology can be illustrated symbolically as below:

$$
\begin{aligned}
-2 \times -3 &= -2 \times A \quad (\text{Let } A = -3)\\
&= 2 \times (-A)\\
&= (-A) + (-A)
\end{aligned}
$$

Indeed his interpretation was not correct because -2×-3 should be conceived as - (-3) - (-3) which means taking away (-3) for twice. In this case, it is a repeated subtraction. Our claim is supported by Kilham (2011) as follow, The number of times a multiplicand is either added or subtracted could be mapped onto a signed number, so that a positive multiplier means an iterated addition and a negative multiplier means an iterated subtraction. (Kilham, 2011, p. 103).

Multiplication as "Of"

R: Ok. Let us see $\frac{1}{2}$ multiply 2. Why is it 1?

T: This is easy. Means half of 2. We divide 2 into 2 equal half, we get 1 and 1. So half of 2 is 1.

R: What is the meaning of multiplication in this case?

T: (…) Half of 2. One plus one is two. But you are asking the opposite side, half of 2. Let's see how we can relate it to addition. We are going to add 2 half time. How am I going to write half time ya? Haha…

R: Haha…Maybe you can try to write it on a paper.

T: No… Oh! $\frac{1}{2}$ times 2 is actually the same as 2 times $\frac{1}{2}$ right? So we repeat half two times! Then we can get 1! Half plus half is one (he was drawing on a piece paper (see Figure 4 below)).

Excerpt 4. Teacher's explanation for $\frac{1}{2} \times 2$.

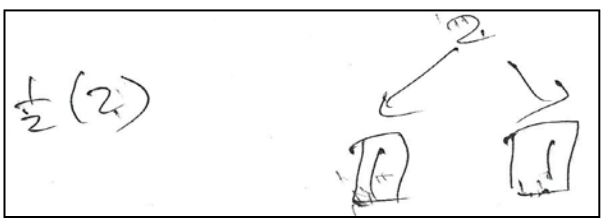

Figure 4. Teacher's drawing for $\frac{1}{2} \times 2$.

According to the excerpt above, again he was trying to build a coherent understanding of multiplication as repeated addition. It was problematic for him to explain $\frac{1}{2} \times 2$ as repeated addition because $\frac{1}{2}$ is not a counting number. He changed $\frac{1}{2} \times 2$ to $2 \times \frac{1}{2}$ so that he can conceive it as adding half for two times. He was trying to make sense of multiplication based on his supportive conception that was repeated

addition.

R: What if we have ½ multiply with $\frac{1}{3}$?

T: Haha… both fractions (he drew a picture (see Fig. 5))! Here we half a bar, I divide it into six equal parts. I cut it into half, so half of this is 3 over 6, and one third of this 3 over 6 is 1 over 6 (he was explaining his drawing. (see Fig. 5)), Right? But I guess you want to ask me about the meaning of multiplication here, so the original meaning of multiplication in this case is missing, I guess. $\frac{1}{2}$ times $\frac{1}{3}$, hmm… It's $\frac{1}{3}$ of half. Can we add half one third time? What a ridiculous explanation. Hmm… Well, just a portion of a particular amount concerned. No addition can involve.

Excerpt 5. Teacher's explanation for $\frac{1}{2} \times \frac{1}{3}$

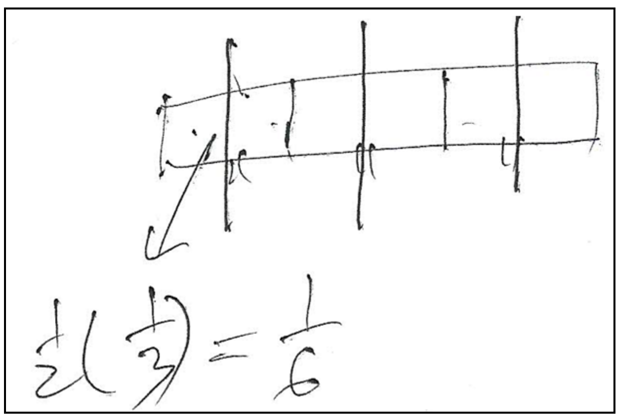

Figure. 5. Teacher's drawing for $\frac{1}{2} \times \frac{1}{3}$

He realised the changes of meaning in the multiplication of fractions and mentioned explicitly that the meaning of multiplication is "of". Obviously, the respondent did realise that repeated addition was a problematic conception in the multiplication of fractions. A summary on the meanings of multiplication is provided in Table 1.

Table 1

Meanings of Multiplication across Different Contexts

Contexts	**Meaning of Multiplication**
Multiplication of natural numbers	Repeated addition
Multiplication of integers	Repeated addition (when the multiplier is a positive integer) or, Repeated subtraction (when the multiplier is a negative integer)
Multiplication of fractions	The notion ‘of’

Implications for Teaching and Learning

This study has shown the complexity in making sense of multiplication due to the supportive and problematic conceptions that arise from other contexts. As we can see from the data analysis, it was not easy for the respondent to make sense of multiplication. The main issue was the changes of meaning for the multiplication symbol across different contexts. This suggests the need to develop a whole new course in teacher preparation that addresses not only the supportive conceptions but also the problematic conceptions that impede student learning. We need to educate teachers so that they know when the meaning of a particular symbol changes. Furthermore they need to be aware of the possible problematic conceptions that arise from previous contexts so that they can help students to make sense of their learning efficiently.

Conclusions

As the results outline, the changes of meaning in multiplication in different contexts might lead to the formation of supportive conception or problematic conception. Based on Table

1, we can see that when the numbers involved in multiplication are expanded to other different real number sets such as natural numbers, integers and fractions, the mathematical meaning of the expressions will change. On top of that, it should be noted that when the multiplier is changed to a negative integer or a fraction then the mathematical meaning is changed unless some manipulations are performed in order to maintain the meaning of repeated addition. Hence it is difficult for learners to build a coherent understanding. This is not the whole story. A supportive conception might contain problematic aspects whereas a problematic conception might contain supportive aspects. The previous example shows that the respondent had a supportive conception with a problematic aspect for the case of -2×3 and he tried to remove this problematic aspect by transferring the negative sign to 3 so that he can continue to interpret multiplication as repeated addition. He used his knowledge on commutative property of multiplication to transfer the minus sign, realising the fact that $a\times b=b\times a$ when a and b are real numbers. This indicates that one of the possible ways to remove a problematic aspect is to draw upon on the knowledge of mathematical structure. In contrast, a problematic conception with a supportive aspect can be conceived as the situation where the respondent could no longer interpret multiplication as repeated addition. For instance, the respondent couldn't manipulate $1/2\times1/3$ so that he could interpret it as repeated addition. However there are some supportive aspects in this problematic conception such as the associative and the commutative properties of multiplication still valid in this context.

In this study, the respondent did sense the difficulty in making sense of multiplication in different contexts. Furthermore, he did make some mistakes in interpreting the meaning of multiplication when he was trying to apply the concept of repeated addition for different contexts. This shows the complexities of making sense of multiplication. According to the

National Council of Teacher of Mathematics (NCTM, 2000), students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge. However, this is not always achievable as the meaning of arithmetic operations or symbols changes according to context. As an effective mathematics teacher, he/she should guide the students to make sense of mathematics and make them realize their own supportive and problematic conceptions in different contexts. It is always easy to build on supportive conception whereas teachers and curriculum designer should not take it for granted about the issue of problematic conception. Therefore, the framework of supportive and problematic conceptions should offer a useful starting point for teachers who are looking to help students to understand multiplication over the longer term. It will be fruitful to use this framework for future research studies which involve different topics of mathematics, because the results can make us aware of all the possible problematic conceptions that might be arise from other contexts. Even though only one expert teacher is involved in this study but this study does show that the framework of supportive and problematic conceptions is able to explain the data well in this study.

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