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Abstract: *Do students see things the same as what mathematics teachers have anticipated? In a light-hearted and inquisitive mode, two fundamental mathematical tasks were administered to 147 pre-university students who newly joined a private college for enrolment into various programs. One task simply required the participants to state the variables in a linear equation and the other attempted to elicit what the participants would naturally perceive from a diagram featuring a straight line crossing the coordinate axes. This study aimed to examine the participants' perceptual flexibility with mathematical elements without explicit hints. The emergent textual responses were qualitatively classified and the frequencies of the various types of response evaluated. While the two tasks appear to be drearily ordinary to arouse excitement, the data startlingly revealed a wide range of valid and invalid responses from the participants, and some mathematical elements appeared to be more visually implicit or explicit than others to the participants. Specifically, few participants perceived composite functions embedded in the linear equation as variables and even fewer participants noticed a right-angled triangle emerging from the straight line crossing the coordinate axes. The results are discussed from the perceptual perspective of Gestalt psychology, which suggests the participants' lack of flexibility in mentally reconfiguring perceptual elements. The potential implications for mathematics learning associated with perceptual flexibility are discussed and the instructional efforts that may enhance students' perceptual flexibility recommended. In particular, we argue that a problem-solving process may require flexible perception of mathematical elements in order to gain access to learned concepts and hence particular solution choices.*

Keywords: Gestalt psychology, mathematics learning, perceptual flexibility, mental reconfiguration

Introduction

This study was intended to establish the fact that even simple mathematical objects (e.g., functions or graphs) may be perceived and interpreted by pre-university students rather differently. The study also aimed to illustrate that some mathematical elements within a

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mathematical object may be more implicit or explicit than others to perceptual inspection. It serves as a preliminary to a larger study which aims to investigate the possible influence of perceptual outcomes on students' accessibility of learned concepts and thus their nature and choices of mathematical solutions.

Past studies primarily focused on conceptual understanding and procedural knowledge in relation to mathematical learning and problem solving (Hiebert, 1986; Hiebert & Handa, 2004; Rittle-Johnson & Alibali, 1999; Rittle-Johnson & Koedinger, 2009; Rittle-Johnson, Siegler, & Alibali, 2001; Rittle-Johnson & Star, 2007; Selden & Selden, 2005; Skemp, 1976; Star, 2013). Conceptual understanding is embodied in the comprehension of mathematical concepts, operations, and relations (NRC, 2001) and the mathematical understanding which enables active building of new knowledge from experience and prior knowledge (NCTM. 2000). On the other hand, procedural knowledge refers to the ability to carry out procedures for solving a problem (Rittle-Johnson & Alibali, 1999; Rittle-Johnson & Koedinger, 2009). Hiebert and Lefevre (1986) refers to procedural knowledge as the knowledge of how-to, while conceptual understanding the knowledge of why. Such emphasis on the ability to explain and justify mathematical definitions, rules and relationships as an embodiment of having conceptual understanding has been echoed by many others (e.g. Star, Caronongan, Foegen, Furgeson, Keating, Larson, Lyskawa, McCallum, Porath, & Zbiek, 2015; Zuya, 2017) Earlier studies mainly viewed conceptual and procedural knowledge structures as distinguishable entities, such as Skemp's (1976) instrumental (i.e., rote learning without understanding) and relational (i.e., learning with understanding) knowledge. Latter studies however contended that it is the understanding of how the two knowledge structures intertwine which should receive higher priority. While some studies have reinforced the generic view that conceptual understanding has a greater influence on procedural knowledge rather than the reverse (i.e., Lauritzen, 2012; Rittle-Johnson & Alibali, 1999), others have suggested the important interplay and interactive nature of the two knowledge structures in support of their development (Rittle-Johnson & Koedinger, 2009; Rittle-Johnson et al., 2001; Rittle-Johnson & Star, 2007). However, relatively little effort has been directed toward exploring the underlying factors which enable access to the learned concepts in a problem-solving situation. Does knowledge of concepts sufficiently guarantee use of the concepts when solving a problem? What task features or mathematical elements would a problem solver perceive before some concepts could be brought to attention? Could the application of some concepts and solution choices be perceptually driven? In other words, could there be a possibility that failure to apply some learned concepts is due to the inability to notice the essential elements related to the concepts rather than a lack of the concepts? For instance, could the learned concept of Pythagorean Theorem surface before a problem solver notice a right-angled triangle embedded in a diagram? Is there a need "to study a man's perceptions before one can understand his behavior" (Gibson & Gibson, 1955)? All these questions have been greatly intriguing to us. We would argue that the sole focus on conceptual and procedural knowledge, without the consideration of perceptual processing and outcomes, may not be sufficient in understanding the nature of students' mathematical performance, in general, and their solution choices, in particular. This view is similarly espoused by Goldstone, Landy and Son (2010) who concluded following their study that sophisticated performance in science and mathematics is achieved not by ignoring perceptual features in favor of deep conceptual features, but rather by adapting perceptual processing so as to attain formally sanctioned responses.

In this study, we define perceptual flexibility as the ability to perceive mathematical elements flexibly, drawing on the Gestaltist perspectives (Murray, 1995), as well as the ideas of perceptual learning (Gibson, 1969; Kellman & Garrigan, 2009) and Rivera's (2011) visual orientation. Perceptual learning is the natural ability to extract invariant information across multiple learning experiences (Gibson, 1969). Both perceptual learning and Gestaltist perspectives hold that individuals' perceptions may vary via learning and experiences. We thus contend that perceiving sensory elements is a cognitive and dynamic experience, and thus individuals' perception may vary even with seemingly relatively fixed and simple sensory elements.

We would clarify that we do not perceive conceptual (or relational) understanding and perceptual flexibility as two competing knowledge structures. Conversely, we see perceptual flexibility as a possible feeder to establishing connection with one's conceptual knowledge. For instance, a problem solver may need to first observe a right-angled triangle (rather than simply three unrelated lines) in order to be able to connect with concepts that are pertinent to a right-angled triangle, e.g. Pythagoras Theorem and Trigonometric ratios. Similarly, if a problem solver does not perceive (2x2 + 3) in the equation (2x2 + 3)2 – 2(2x2 + 3) – 15 = 0 as one single variable, he may encounter a more complex Quartic equation rather than a simpler Quadratic equation in (2x2 + 3) which could be simply solved by factorization. We conjecture that a problem solver's conceptual or relational knowledge may remain latent without such perceptual trigger. This preliminary study set out to investigate the perceptual flexibility of Pre-university students with mathematical elements, presuming that such flexibility is vital for connecting with learned concepts for solving a problem. At this stage, we are not certain about the extent to which perceptual flexibility may influence students' ability in connecting with learned concepts critical for problem solving. We surmise that perceptual flexibility does play a significant role in making connection with learned concepts, which would only be triggered upon perceiving the relevant sensory elements. This aspect will be dealt with in a separate study.

Motivation

This study was initiated upon observation of A-Level students' consistent use of relatively fixed ideas and solution steps in mathematical problem solving, across participants, seemingly with little or no exploratory predisposition (Low, 2015; Low, 2016; Low & Chew, 2013). Furthermore, most of the students provided solutions which were algebraically-driven rather than geometrically-based, even with a problem that was geometrically presented. For instance, Low (2016) studied 32 A-Level students' mathematical flexibility, i.e. the ability to produce alternative conceptually-varied solutions to multiple-solution tasks, at a private college. At the pre-test, the participants were presented with a diagram featuring a line, in the form y = mx + c, crossing the coordinate axes. They were asked to find the shortest distance from the origin to the line. Disregarding those solutions from empty or premature attempts, or which attended to irrelevant task features (i.e., taking the x- or y-intercept as the shortest distance from the origin to the line), approximately 68% of the attempted solutions were found to be algebraically-driven (e.g., solving linear equations of two perpendicular lines, or differentiating and minimizing a distance function). Only the remaining few solutions were apparently geometrically-based (e.g., applying trigonometry or area of triangle, capitalizing on the right-angled triangle formed by the line and the coordinate axes). One explanation could be that the students generally did not notice the right-angled triangle□an inability to perceive mathematical elements flexibly and hence the inability to approach a mathematical task adaptively (Low, 2016). While most of the A-Level students joined the college under study with fabulous academic results from their secondary studies (ibid.), it is not out of place to reason that there may exist other factors (e.g., low perceptual flexibility), besides procedural and conceptual knowledge, which may hinder the students' access to their learned concepts resulting in failures in solving a task flexibly.

Similarly, Low (2016) also consistently noticed his A-Level participants' uniform solutions to particular problem types with little knowledge of alternative solutions, let alone a conscious predilection for efficient solutions. For instance, the A-Level participants were found to consistently demonstrate an immense tendency to expand such expression as (2x – 5)2 into $4x^2 - 20x + 25$, seemingly viewing $(2x - 5)^2$ as a squaring process. As a result, they mostly missed the opportunity to solve such equation as $(2x - 5)^2 = 1$ or inequality as (2x – 5)2 < 1 efficiently by perceiving (2x – 5) as a product, e.g. 2x – 5 = ±1. Similarly, only when (2x – 5) is perceived as a variable, would a student aim at the values that are within 1 unit from 0, namely $-1 \leq 2x - 5 \leq 1$, such that $(2x - 5)^2 < 1$. And such perceptual inflexibility probably explains why students fail to generalize the concept of variables and perceive (2x – 5) as a variable (Wagner & Parker, 1993). In fact, mathematics learning should involve flexibility, the ability to select appropriate solutions and modify solutions when conditions warrant (National Research Council, 2001). Certainly, by rearranging the above equation and inequality to $(2x - 5)^2 - 1 = 0$ and $(2x - 5)^2 - 1 < 0$, and perceiving the left-hand-side as a difference of two squares, a different solution with direct factorization could be obtained. Do pre-university students actually possess such perceptual flexibility in perceiving a composite function as a variable? The curiosity about this question shaped one of the reasons for this study.

Similarly, students have been found to view geometrical elements as separate entities rather than an interactive, collective whole, where possible (Low, 2016). We believe that such inflexibility in restructuring perceptual stimuli would limit the potential to solve problems flexibly and creatively. We conjecture that when students are shown the line graph of a linear function crossing the coordinate axes, most would inexorably attend to such textbook details as the gradient of the line and the x- and y-intercepts. Few students would quickly notice the interactive effect of the line with the coordinate axes in forming a right-angled triangle. Consequently, the power for accessing any concepts pertinent to a right-angled triangle during a problem-solving process may lose strength. This conjecture shaped the second reason for this study.

A student's failure in solving a mathematical task is commonly attributed to gaps in one or more of declarative, procedural, and conceptual knowledge structures. While such premise to a high degree could be true, we however would meticulously argue that students may not perceive mathematical elements the way we educators or expert mathematicians do and that perceptual outcomes may facilitate or thwart access to exiting knowledge structures, be it procedural or conceptual, which in turn may influence the quality and choice of solution. In order to establish the validity of this belief, we began in this preliminary study by first examining the kinds of mathematical elements that would enter students' perceptual fields from both symbolic and geometric representations. In particular, we investigated the kinds of mathematical elements which would readily capture students' attention and enter their perceptual fields when they were exposed to a linear equation and a line crossing the coordinate axes.

Underpinning Theories

Perceptual Issues

Mathematical learning and problem solving are greatly a perceptual, not only conceptual or procedural, issue (Rivera, 2011). Visualization skills help optimize use of working memory and thus problem-solving performance (Carden & Cline, 2015). An effective exploitation of visual reasoning in tandem with algebraic representations via digital tool has been claimed to aid the teaching and learning of mathematics (Healy & Hoyles, 1999). Effort has been seen to improve visual selective attention, which enables effective selection and processing of task-relevant information without much influence of distracting contextual elements (Libera & Chelazzi, 2006). Unfortunately, the need for visualization has not been widely emphasized and there were relatively few studies on visualization in mathematics education (Presmeg, 2006). College students' reluctance toward visual thinking and processing has ever been a concern in the study of visualization and spatial thinking (Rivera, 2011).

However, as opposed to what most people might have thought of, "visualizing facts and images does not necessarily imply the use of pictures alone. They could also be routed propositionally, that is, in either linguistic or algebraic form." (Rivera, 2011, p. 59). Furthermore, it is not the formation of visual image that is important, but the active selection and integration of visual information in problem solving (ibid.).

Students' persistent failures in mathematical attempts, despite sufficient emphasis on declarative, procedural and conceptual knowledge, could be due to the lack of instructional focus on the need for perception and visualization use in mathematical problem solving. It has been empirically proven that students may gain mathematical improvements by perceptual learning, defined as the experience in extracting crucial and relevant information (Kellman, Massey, Roth, Burke, Zucker, Saw, Aguero, & Wise, 2008; Kellman, Massey, & Son, 2010). Kellman et al. (2010) developed and applied perceptual learning modules to mathematics learning. The results revealed that perceptual learning led to better accuracy, transfer, and efficiency in students' problem solving. For instance, in the Algebraic Transformations perceptual learning module, practice in seeing equation structure across transformations, even without solving equations, led to dramatic improvement in the speed of equation solving. In addition, Kellman et al. (2010) developed Multi-Rep perceptual learning modules and applied them to 68 ninth and tenth grade students. The students were required to map linear functions in different representations, such as graphs, equations and word problems, by recognizing patterns and structures on computer. Such short, interactive classification tasks were found to have facilitated fluent extraction of important features and patterns. It was claimed that such perceptual learning differs greatly from traditional mathematics learning whereby students are commonly asked to work out solutions numerically. In contrast, perceptual learning, such as practice in mapping across representations, focuses on the ability to extract particular structural attributes (e.g., knowing where to look in an equation to obtain the slope) and the development of intuition about the way equivalent structures relate across representational types (e.g., learning the graphical consequences of various slopes or intercepts of various signs).

In addition to conceptual understanding, or relational knowledge in Skemp's term, Goldstone et al. (2010) contended that proficiency in mathematics also involves executing spatially explicit transformations to notational elements. They argued that perceptual processing should not be ignored in favor of conceptual features but be adapted to support learning and formally acceptable responses to attain sophisticated performance in science and mathematics. Kellman and Massey (2013) contended that perceptual learning has been largely downplayed and neglected in formal instruction. They stressed the important role of perceptual learning in information extraction, i.e. for relevant features and structural relations, and argued that perceptual mechanisms would engender representations which are not limited to merely low-level sensory features but dynamic changes and adaptation in providing complex and abstract description of reality, interacting deeply with higher cognitive functions. Most importantly, perceptual acuity can be nurtured and attained by training with perceptual learning aimed at more efficient extraction of crucial and relevant information (Kellman & Massey, 2013).

Gestalt Psychology

Interestingly, in approximately the same era as the advent and development of behaviorism, there emerged a cognitive-based school of thought—the Gestalt psychology—posing a challenge to the behaviorists. According to Gestalt psychology, a perceptual field consists of part-whole structures or perceptual configurations (e.g., figure-ground configuration) called Gestalten (Wagemans, Feldman, Gepshtein, Kimchi, Pomerantz, van der Helm, & van Leeuwen, 2012), and a gestalt refers to an integrated coherent structure or form, contributing to a whole that is different from the sum of the parts. Most critically, gestalts emerge spontaneously from self-organization, subjecting to the simplest possible organization, or minimum solution, to the stimulation. When a stimulus is contextually modified, gestalts as emergent features would surface (Pomerantz & Portillo, 2011), increasing the likelihood of some sensory structures or elements to be perceived. However, prior knowledge about what is to expect in a sensory environment may, to some extent, also determine what gestalt a person would perceive (Summerfield & Egner, 2009). In relation to this study, it will be interesting to explore the kinds of gestalts the pre-university students would perceive as variables in a linear equation and with a line crossing the coordinate axes.

The Gestalt perspective argues that the arousal of behavior is not simply a natural, involuntary response to the environmental elements as stimuli, as characterized by the behaviorist stimulus-response model. Instead, it contends that qualities exist in perception (i.e., a cognitive perspective) in addition to the separate sensory elements; and it is the stimulus configuration which elicits an individual's perception as an experienced whole. In other words, an individual does not respond to local stimuli as separate elements or sum of the elements, but react to the constellation (or organization) of the stimuli as segregated sensory whole. This idea has led to such well-known remarks as "the whole is more than the sum of its parts" (Sills, Fish, & Lapworth, 1996, p. 3) and "the whole determines the parts" (Perls, Hefferline, & Goodman, 1976, p. 19). Furthermore, an individual actively and dynamically organizes rather than statically reacts to the sensory environment whereby the field elements are organized into a meaningful, complete, and simple whole, in accordance with the principles of perception, the law of Prägnanz, which is the German word for "essence" (Olson & Hergenhahn, 2009). In this case, the dynamic interaction of forces within a (visual) field (i.e., the sensory environment) actually influences the individual's perception. The Gestaltist view of learning is thus cognitive.

Wertheimer's gestaltist view of teaching and learning emphasizes meaningful, insightful learning with true understanding, rather than the establishment of stimulus-response connection based on the doctrine of associationism (Olson & Hergenhahn, 2009). He claimed that two teaching approaches would inhibit understanding and thus insightful learning, namely instruction which over-emphasizes rule-bound logic and which is basically driven by associationist beliefs. Wertheimer (1938) identified four primary characteristics of the visual field that may influence an individual's perception. They are proximity (i.e., nearness of elements to each other), similarity (common features), open direction (the tendency of elements that complete a pattern) and simplicity (stimulus configuration which contributes to a total simple structure). However, the Gestalt theory does not point to any specific instructional principle or method. Its popularity mainly lies with the insightful “AHA” or euphoria experience on arriving at a successful solution to a problem which is deemed to happen when the problem is viewed differently in a new light (Murray, 1995).

The semantics of form, however, is deemed to be an insufficient synonym to gestalt, but “one variety of form” (Simons, 1988, p. 160), which suggests a greater emphasis on one’s possibility of and ability in switching among forms. The key idea being suggested from the instructional viewpoint is that learners should be provided with information that will assist them in visual field reconfiguration (i.e., reorganization or restructuring of sensory environment) to arrive at different gestalts so as to enhance the chances of successful attempts in problem solving. One classical example is Wertheimer’s transformation of a parallelogram into a rectangle in teaching the area of a parallelogram. In other words, Gestalt principles assert that learners should play an active role in reorganizing and restructuring field elements, and be perceptually flexible for successful problem solving (Murray, 1995). Insightful moments leading to plausible solutions are said to occur abruptly when the field elements are viewed in a new light.

That is with this perceptual flexibility from the perspective of Gestalt psychology, we champion the importance of cultivating in students a predisposition toward highly dynamic reconfiguration of mental field elements which we believe is an essential step to enhancing students’ problem-solving capability. It is noteworthy that such educational effort does not only emphasize the accuracy aspect of a solution (i.e., a for-the-test mentality), but also transcend into the realm of such other qualities as efficiency and elegance of a solution (Posamentier & Krulik, 1998). Relating to mathematics learning, in particular, we postulate that it is the perceptual flexibility in perceiving $(2x – 5)$ or $(2x– 5)^2$ as a (collective whole) variable by mentally configuring the field elements in $(2x – 5)$ or $(2x – 5)^2$ into a product, rather than a process, gestalt that would give rise to not only successful but also more elegant mathematical solutions. The ability of human switching a perceived symbol between product and process, termed as procept, has been addressed by Tall (2006). By perceiving $(2x – 5)^2$ as a process, a student is more likely to expand the expression into $4x^2 – 20x + 25$. On the contrary, by flexibly perceiving $(2x – 5)^2$ as one single variable (i.e. a product), only then would a student possibly think of what values $(2x – 5)$ can be for some condition, such as $(2x – 5)^2 = 1$ without the need to expand $(2x – 5)^2$.

The Purpose of the Study

In short, the purpose of this study was to investigate pre-university students' perception of mathematical elements, i.e. to collect evidence of the participants' perceptual outcomes from visualizing mathematical elements without explicit hint, such as in a real problem-solving scenario. In particular, the study aimed to (1) assess the algebraic expressions in a linear equation pre-university students would naturally perceive as variables, and (2) elicit what the students would naturally perceive from a diagram featuring a line crossing the coordinate axes. The study thus sought to answer the questions: (1) What would pre-university students perceive as variables in the linear equation $2x + y + 5 = 0$? and (2) What would pre-university students perceive in a line crossing the coordinate axes?

This study was aimed at examining pre-university students' perception of fundamental mathematical elements under a no-hint condition. In a no-hint condition, the participants were not given such clues in the tasks as there could be many possible variables in a linear equation, nor were they hinted to observe any geometrical shapes which may appear in a line crossing the coordinate axes. Such no-hint condition corresponds to most real problem-solving scenarios which do not explicitly point to particular solution choices for attaining the required outcomes (Posamentier & Krulik, 1998). In particular, this study attempted to assess the kinds of expressions the participants would naturally perceive as variables in the linear equation $2x + y + 5 = 0$ (e.g., x, y, 2x, 2x + 5, etc.) and the elements they would naturally perceive from a geometrical diagram featuring a line crossing the two coordinate axes (e.g., slope and the y-intercept of the line, the perpendicularity of the coordinate axes, etc.).

We conjectured that there could be a great variety of emergent sensory elements even with such simple mathematical objects and that most students would perceive only x and y as variables, disregarding other expressions in x or y, or in both x and y, which are equally variables. We also conjectured that students would characterize the straight line crossing the coordinate axes by such common attributes as gradient, x-intercept, and y-intercept. However, the extent to which such conjectures were true was initially beyond our knowledge.

Being able to perceive sensory elements flexibly would certainly increase the likelihood of attaining more efficient and elegant solutions to tasks (Posamentier & Krulik, 1998). The significance of this study does not lie merely with students' ability to perceive possible variables in a linear equation and elicit multiple (particularly geometric) elements from a diagram featuring a line crossing the coordinate axes. More importantly, this study brings to attention possible gap in the current mathematics curriculum which may have largely neglected the likely influence of students' perceptual outcomes on their mathematical performance and flexibility. Furthermore, we believe that mathematical skills are transferable both in time and across problems. It is more of the nurturance of habits of mind in helping students to be more flexible in perceiving mathematical elements and in problem solving (Leikin, 2007) and learn and adopt some of the ways that mathematicians think about problems (Cuoco, Goldenberg, & Mark, 1996). As asserted by Cuoco et al. (1996), it is the algorithmic thinking behind the learning of difference equations, and "the habit of using geometric language to describe algebraic phenomena (and vice versa)" that should be infused in the mathematics curricula. We similarly champion the cultivation of habits of mind with perceptual flexibility in dealing with mathematical elements and believe in its likely influence on students' ability to relate to learned concepts for immediate application.

Method

Research Design

This study adopted a descriptive approach to qualitative data analysis, capitalizing on a convenient sample of 147 pre-university students who joined a private college in the same intake. The instrument consisted of two fundamental questions which aimed to explore the students' perceptual flexibility with variables in a linear equation and a geometric diagram featuring the line of a linear function crossing the coordinate axes. The emergent textual responses from the participants were categorized and the frequencies of the various types of response quantified. In other words, the study assessed the number of occurrence for each similar response in an attempt to investigate what variables in a linear equation and what kinds of mathematical elements with a line crossing the coordinate axes the participants would naturally capture in their perceptual fields.

The first and the second authors independently examined the participants' textual responses numerous times and categorized the responses into various response types. The results were subjected to an assessment of inter-rater levels of agreement using IBM SPSS version 12.

Participants

The participants comprised 147 pre-university students (55% male, 45% female) of ages from 17 to 29 years, with 88% of them falling in the 17–19 age range. The race compositions were 73% Chinese, 7% Indian, 5% Malay, 3% others and the remaining non-disclosure. Among those 101 participants who had completed the SPM examination (Sijil Pelajaran Malaysia, or Malaysian Certificate of Education Examination, equivalent to O Level) and had disclosed their SPM Mathematics results, 34% of them scored an ‘A+’, 33% an ‘A’, 6% an ‘A-’, and 13 % at least a ‘B’ on Mathematics. On the other hand, among the 90 of these 101 participants who had also sat for the Additional Mathematics paper, 18% of them scored an ‘A+’, 24% an ‘A’, 6% an ‘A-’,and 23% at least a ‘B-’ on Additional Mathematics.

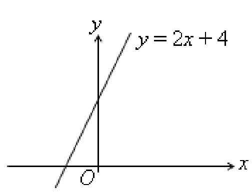
According to the secondary school mathematics syllabuses, the participants should have learned about variables and functions being variables. Similarly, the various geometrical shapes such as squares, rectangles, triangles, parallelograms, etc. have been addressed in the syllabuses. However, the extent to which the participants would perceive the learned mathematical elements embedded in other contexts, such as equations or graphs, were unknown prior to this study.

Measure

The measure of this study consisted of two fundamental, open-ended mathematics tasks as shown:

1) State the variables in the equation: $2x + y + 5 = 0$.

2)



State what the above diagram shows.

The first task was to determine what expressions in the linear equation pre-university students would perceive as variables, while the second task was aimed at eliciting the elements the students would notice from a line crossing the coordinate axes. In particular, we were interested to see if the right-angled triangle (a gestalt) formed by the line interacting with the coordinate axes was apparent to them. It is further clarified that the two tasks were not intended to be a test and it was not the intention to assess if the participants obtained the right or wrong answers. The tasks were mainly aimed at exploring the mathematical elements which the participants would naturally perceived without explicit clues. Thus, no marks were allocated for the participants' responses.

Procedure

This study capitalized on a convenient sample of 147 pre-university students who joined a college for enrolment into various programs for Diploma or A-Level certification. Before the study, permission to conduct the study was obtained from the college management for gaining access to the participants during an orientation session.

During the study, the participants were led into a hall in which they were seated in rows. The seating arrangement was not strictly in an examination style so as not to infuse the atmosphere with unnecessary fear of being assessed. Before administering the two tasks, the participants were briefed on the purpose of the study. Each participant was requested to provide information of the program enrolled, age, gender, race, as well as SPM Mathematics and Additional Mathematics results. Both anonymity and confidentiality were ensured. The participants were required to respond to the task individually without discussion with adjacent others under full supervision by the first author with the assistance of two staff members from the college. The participants were however allowed to ask questions or seek assistance if deemed necessary. No time limit was strictly imposed. The entire session took about 40 minutes.

Coding and Data Analysis

In the coding process, the emergent textual responses of the participants to the administered tasks were read multiple times and coded independently by the two authors. The textual responses, classified as mathematically valid or invalid, were tallied to determine the frequencies of the various response types.

For the first task, variables in the linear equation could possibly be x, y, 2x, x + y, x + 5, 2x + 5, y + 5, x + y + 5, and any valid expressions should the equation have been appropriately transformed (e.g., by multiplying both sides of the equation by x). Here, the expression 2x + y is logically not considered a variable because it gives the constant -5.

For the second task, possible responses could be any statements in relation to the diagram, such as “gradient = 2”, “y-intercept = 4”, “ a line”, “a line equation”, “x-axis”, “y-axis”, “a triangle”, etc.

The inter-rater Kappa measure of agreement for Task 1 and Task 2 are 0.897 and 0.895, respectively, indicating very good levels of agreement between the two raters.

Results

Table 1 shows the participants’ responses and the corresponding frequencies. Apparently, majority of the participants have perceived only x and y as variables, disregarding the fact that any expression in terms of one or more variables could be a variable. It is equally startling as to how the huge number of various (invalid) expressions could have been perceived as variables. However, one participant and seven others had perceived x + y and 2x as variables, respectively.

Table 1

Response items to "State the variables in the equation: 2x + y + 5 = 0"

Response Item	Frequency	Response Item	Frequency
Valid		*Invalid*	
y	67	$y = -2x - 5$	41
x	59	A restructured form of the equation	15
$2x$	7	Constant or coefficient (i.e., 0, 1, 2 and/or 5)	14
$x + y$	1	No response	14
$x + 5$	0	$x = ½(-y - 5)$	10
$y + 5$	0	x found given a y value	7
$2x + 5$	0	x-intercept found	5
$x + y + 5$	0	Gradient	5
		'y-intercept' stated	5
		y-intercept found	4
		y found given an x value	2
		$2x + y$	1
		'x-intercept' stated	1

In the diagram featuring a line crossing the coordinate axes, the mathematical elements which have captured the participants' attention are illustrated in Table 2. While the information in Task 2 was basically a geometrical line crossing the coordinate axes, it is interesting to note that most participants had noticed and provided more symbolic, rather than geometric, mathematical features pertinent to the line and its equation, such as the gradient of the line, the x- and y-intercepts, and the nature of the graph/equation (i.e., linear graph/equation). Twenty participants just simply restated the given equation! There were relatively fewer responses in geometric nature, such as 'a line crossing the axes' and 'a triangle'. Few participants had noticed the right-angled triangle formed by the line and the coordinate axes. There were only four participants who explicitly related the diagram to triangle, and merely one of the four participants was specific enough to pinpoint the existence of a right-angled triangle.

Table 2

Response Items to What the Participants Perceived in a Line Crossing the Coordinate Axes

Response Item	Frequency	Response Item	Frequency
Valid		*Invalid*	
y-intercept	78	No response	26
Gradient stated	53	*y* (or *x*) found given an *x* (or a *y*) value	8
x-intercept	52	Equation restructured	7
'straight/linear line/graph/equation' stated	47	Inexplicable remark	2
$y = 2x + 4$	20	Math equation	2
line through axes	7	Quadratic equation	2
area of triangle	3	Directly proportion	1
x-axis/*y*-axis	3	Math/Add. Math	1
y increasing linearly with *x*	2	*x* and *y* given values	1
one-one function	1	$x > 0$	1
'equation' stated	1	'Too hard'	1
right-angled triangle	1		
Origin	1		
unknown *x* and *y*	1		
Graph	1		
Cartesian plane	1		

The perpendicularity nature of the two coordinate axes and other higher-order observations, such as tan tan $\theta = 2$ for θ being the angle the line makes with the horizontal, were completely absent in all the participants' responses.

Discussion

This study set out to assess pre-university students' perceptual flexibility with mathematical elements, with particular respect to what the participants would perceive as variables in a given linear equation, and what they would notice when they were shown a geometrical line crossing the coordinate axes.

The results both interestingly and startlingly imply that perceptual outcomes may vary significantly, even with simple visual stimuli, as shown in this study. In addition, some mathematical elements appeared to be more perceptually prominent than others (i.e., x , compared with x + y, perceived as a variable). As opposed to what most mathematics educators might think, mathematical elements may not simply be static entities, but highly dynamic features being influenced by subjective interpretations of students as propounded by the Gestaltist perspective. There appears to be a complex interplay between visual stimuli and cognitive subjectivity which co-determine the perceived gestalts. The cognitive process of perceiving visual stimuli could be more complex than what most mathematics educators have expected and thus should not be taken lightly. This study shares similar understandings to others that mathematical elements are by no means self-evident and self-explanatory. Rather, students actively interpret mathematical elements to make sense of them (Steinbring, 2005; von Glasersfeld, 1987). The observation from this study is also in line with that by Steenpaß and Steinbring (2014) who contended that mathematical diagrams are by no means self-evident but contingent on subjective interpretations. Steenpaß and Steinbring (2014) conducted clinical interviews with young children which led to the characterization of children's subjective frames between "object-oriented" (i.e., focus on the diagram's visible elements) and "system-oriented" (i.e., focus on relation between those elements).

Students' low perceptual flexibility and prevalent focus on explicit, more prominent elements could be a social-cultural issue (Rivera, 2011). As expressed in Rivera (2011), "… visualizing mathematically is an interpretive reproduction of particular cultural practice." (p. 243) It could be that the perceptual or visual aspects of mathematics has not been culturally valued in classrooms. Empirical evidence is available pointing to undergraduate students' "reluctance to visualize" in mathematics learning in favor of algebraic manipulation of symbolic forms (Eisenberg & Dreyfus, 1991). However, such visualization skills, the ability to recognize and reorganize visual stimuli and interpret relationships visually, have been found to be critical in students' mathematical achievements (Booth & Thomas, 1999). In particular, Arcavi (2003) called it "seeing the unseen", referring to the human "technologies" which he claimed might develop visual means to better see mathematical concepts and ideas (p. 56).

In the following, we further discuss the findings in relation to the two tasks, provide implications for instruction, and raise further questions for future studies.

Perception of Variables

The findings of the study suggest that students generally lack perceptual flexibility which could have prevented them from attaining and generalizing the true understanding of variables. They fail to perceive a variable expression, i.e. $2x + 5$, as a collective whole or Gestalt, which is conceptually a variable. Such shortcoming may point to numerous educational implications. While it requires further investigations to be certain, we believe from an a posteriori standpoint that perceptual learning (i.e., experience in extracting crucial information such as by perceiving variables in various gestalts) had not received sufficient emphasis in the participants' past learning experience.

Furthermore, the extensive invalid expressions provided by the participants justify the claims that the concept of variable, which is ostensibly straightforward and simple, is indeed elusive and challenging for attaining full understanding (Rosnick, 1981; Schoenfeld & Arcavi, 1988). Schoenfeld and Arcavi (1988) found that the understanding of variable could be highly dependent on contexts and that the multiple connotations, meanings, and uses of the term variable make the understanding of variable even more complicated. Various definitions of variable have been provided by people from different disciplines, i.e. "symbol, placeholder, pronoun, parameter, argument, pointer, name, identifier, empty space, void, reference, instance." (Schoenfeld & Arcavi, 1988, p. 421). It was then concluded that the wide-range, multiple uses of the term variable makes it hard for students to understand. From the educational perspectives, nonetheless, it is unnecessary and even pointless to argue on a true definition of variable, but to focus on the significant use of variable as a generalization tool in forming relations and a notational model of the reality.

Students' understanding of variable has been extensively studied (e.g., Akgün & Özdemir, 2006; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Küchemann, 1978). Despite the vital role of definitions in mathematics learning, students have been found to carry with them multifarious interpretations, a phenomenon Wagner (1983) remarked as "Literal symbols (are) easy to use but hard to understand." (p. 474) Students have been generally found to treat variables, in markedly descending frequencies, as objects, some specific unknowns, and generalized numbers (Küchemann, 1978). And the misconception of variable has been proven to bear on students' mathematical performance (Knuth et al., 2005).

However, relatively few studies have been conducted on students' perception of groups of variables (variables consisting of other variables), especially when they occur in the context of other mathematical entities, such as equations and graphs, as in this study. We believe that classroom instructions have largely focused on single-letter variables and less on composite functions as variables, as in "Solve for the values of x and y, given that $2x + y = 5$ and $x – y = 13$", rather than "Solve for the value of $x + 2y$, given that $2x + y = 5$ and $x – y = 13$". The latter task self-evidently requires some extent of perceptual flexibility in viewing $x + 2y$ as a variable, and with some careful inspection, it is not difficult to obtain the answer simply by subtracting 13 from 5 to arrive at -8.

We consider gain in such perceptual flexibility critical in students' development of algebraic thinking and reasoning. From the Gestaltist educational perspective, classroom instruction should provide greater emphasis on students' perception of mathematical elements in addition to conceptual understanding. Students should learn to view a variable expression as a collective whole gestalt. It could be helpful to understand how students actually perceive a variable expression. When such expression as (2x – 5)2 enters their visual field, does it mean a process "to square the expression (2x – 5)" which will naturally trigger the need to expand the expression, or a product "the square of some value"? There is significance with such perceptual flexibility in switching between process and product in dealing with an algebraic expression. According to Gray and Tall (1994), it is the amalgam of both product and process perspectives that is associated with effective (mathematics) problem solvers. The two different perspectives are likely to influence a problem solver's solution choice, as shown in Figure 1 (Data from Low, 2016).

Taking on an a *posteriori* standpoint, we argue that the participants' past learning experience has not delved into such levels of detail to understand the possible mental reconfiguration of mathematical objects from the Gestaltist perspective. Students should be given the learning opportunity to perceive the various gestalts of variables. They should learn that if x is a variable, so do 5x, $x^2 + 6$, 2x + 5, etc. Otherwise, they would arguably not be able to generalize the concept of variable. This idea is significant if students are to be efficient problem solvers adept at producing elegant solutions. For instance, only if there is the ability to perceive (2x + 5) as a collective whole variable, will a student be able to solve such an equation as $3(2x + 5)^2 + 5(2x + 5) - 12 = 0$ efficiently and elegantly without first expanding $3(2x + 5)^2$ and 5(2x + 5).

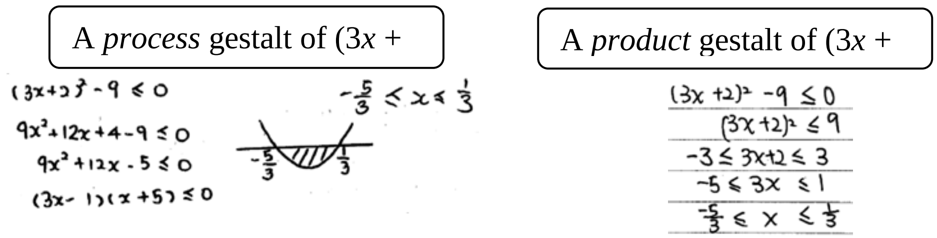


Figure 1. The effects of a *process* versus *product* gestalts of $(3x + 2)^2$ on solution choice.

Nonetheless, it is still arguable that the functional interaction between perceptual flexibility and conceptual understanding could be far more complex than what has been understood or presumed as true. Is there a one-way or two-way causal relationship between the two? To what extent does perceptual flexibility support conceptual development? Does conceptual understanding influence perceptual flexibility? To what extent the two, separately or in combination, influence a problem solver's solution choice? These questions warrant further studies to be answerable.

Perception of a Line Crossing the Coordinate Axes

The findings of this study reveal that students are generally too absorbed in the theoretical elements of mathematics (i.e., superficial, apparent features), with little or no predilection to seek a new light in the interactive quality of the elements (e.g., the interactive effect of a line and the coordinate axes in giving rise to the gestalt of a right-angled triangle). Without such predilection for reconfiguring the perceptual field from a Gestaltist perspective, a problem solver is unlikely to gain insight into a wider solution space and reach out to alternative, some possibly more efficient, solutions. The findings starkly reveal that students generally attend to more explicit rather than implicit mathematical structures. We surmise that with such deficiency, it is less likely that they would relate to concepts associated with the implicit mathematical structures. This conjecture however requires further investigation for confirmation.

It is a common contention that students' inadequacy in problem solving is mostly an issue of conceptual understanding and knowledge application. We, however, would argue that it may not be absolutely so. What would students *see* or *identify* from a problem situation before they relate to their knowledge structures for *possible* solutions? This question seems to have been commonly neglected in mathematics teaching and learning. We contend that students may not always *perceive* mathematical elements or information the way educators or mathematicians do.

It is interesting to observe the seven responses indicating "line through axes". The terms "line" and "axes" are likely to imply the participants' success in perceiving the individual elements on the diagram. In Gestaltist terms, these seven participants have paid more attention to the "parts" rather than the interactive "whole" of the parts. This observation may show the participants' inability to flexibly switch between "parts" and "whole" gestalts. Based on the principles of perceptual learning (Kellman & Massey, 2013), we conjecture that the participants might have little or no past learning experiences focusing on flexible perception of mathematical elements.

A problem solution may entail such attributes and qualities as efficiency, elegance, intelligibility and certitude in addition to accuracy, which could be codetermined by task features, contextual influences and personal subjectivity (Verschaffel, Luwel, Torbeyns, & Dooren, 2009). When A-Level students were asked to find the shortest distance from the origin to a given line, i.e. $y = -\frac{2}{3}x + 4$, most successful solutions provided by the participants involved solving the equations of two perpendicular lines as shown in Figure 2 (Low, 2015). The findings of this study may well be an explanation. Most of the participants apparently were overly obsessed with explicit mathematical features closely related to the line (i.e., the equation of the line, gradient, *x*- and *y*-intercepts, etc.) to the extent that they did not *notice* a right-angled triangle, which may simply emerge from a mental reconfiguration of field elements, arising from the geometrical interaction between the line and the coordinate axes. We doubt that the A-Level students lack the conceptual understanding of basic trigonometry, Pythagoras theorem, similar triangles, and area of triangle, which they had learned long in their secondary studies. We argue from the Gestaltist perspective that such concepts would only be invoked for use in problem solving only if the existence of a right-angled triangle *visually* stands out to the problem solver (Figure 3, Data from Low, 2016). In other words, perceptual flexibility, namely a more dynamic and active mental reconfiguration of field elements or perceptual cues, could be a vital gateway to an alternative, possibly more efficient, solutions. From this perspective, it could be reasonably conjectured that few of the participants in this study would employ concepts of triangle for more efficient solutions such as those shown in Figure 3, if they were to find the shortest distance from the origin to the line $y = -\frac{2}{3}x + 4$. Polya's (1965) principle of economy states that "do not do with more what can be done with less." (p. 91) Certainly, the solutions in Figures 2 and 3 are equally accurate. However, the solutions in Figure 3 are relatively more procedurally and conceptually efficient, based on Polya's principle of economy and a purposeful interest to seek relatively more efficient and elegant solutions (Posamentier & Krulik, 1998). It should however be clarified that it is not the purpose of this study to comparatively rate the different solutions, but how perceptual outcomes may vary and possibly influence a solution choice.

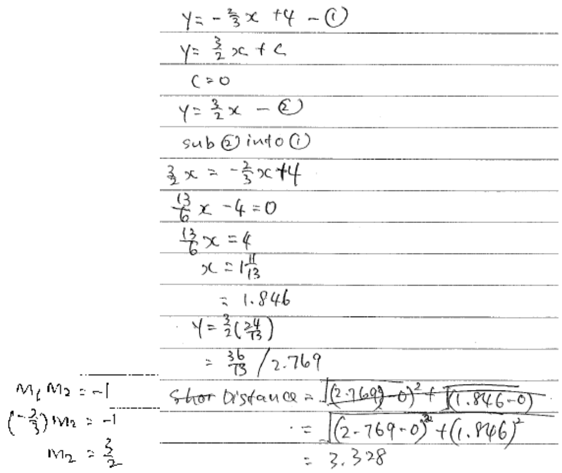


Figure 2. A typical solution: Solving the equations of two perpendicular lines in an attempt to evaluate the shortest distance from the origin to a line.

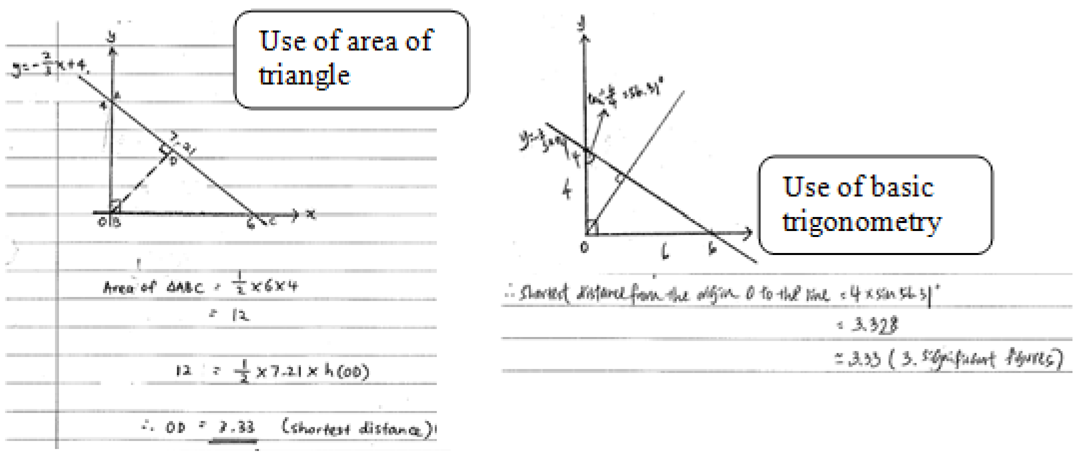


Figure 3. More efficient solutions capitalizing on the concepts of triangle.

Nonetheless, it is equally noteworthy that the solutions in Figures 2 and 3 are all viable, robust solutions which are equally accurate. From the student learning perspective, both the solutions in Figures 2 and 3 could be valuable learning tools. The solution in Figure 2 represents and manipulates the geometric relations between two perpendicular lines symbolically, while the solutions in Figure 3 exploit the geometric embodiment of the relations in attaining the same answer. Thus, the solutions entail different thinking processes, which are equally great learning opportunities for students in promoting mathematical thinking. Learning different ways of solving the same problem informs students that there is not always only one way for solving a problem and that solutions may vary in different aspects or attributes.

However, it is not immediately clear as to whether the knowledge of mathematical elements, i.e. the elements in association with a linear equation and the Cartesian system, have ironically hindered the participants from noticing a right-angled triangle in the second task. Should the line and the coordinate axes be simply reduced to three plain lines (Figure 4), will the appearance of a right-angled triangle become more salient to the participants? This seems to be yet another interesting area that requires further exploration before a theory could be justified.

Possible Ways to Nurture Perceptual Flexibility

This study strongly suggests that students' perceptual outcomes may vary significantly even with simple visual stimuli. As such, students' perceptual outcomes with mathematical elements should not be taken lightly. Instructional practice should seek to understand not only the conceptual understanding but the perceptual aspects of students in mathematics learning. We propose that students be exposed to both structural similarities and dissimilarities of mathematical elements underlying the same concepts. For instance, while learning to solve a quadratic equation, it could be more beneficial if students are shown not only the standard form of a quadratic equation, i.e. $3y^2 + 5y - 12 = 0$, but also such structurally-similar form as $3(2x + 5)^2 + 5(2x + 5) - 12 = 0$. Similarly, it may not be sufficient to expose students to the concept $\sqrt{x} \geq 0$, for $x \geq 0$ alone, but also to illustrate such other structurally-similar expressions as $\sqrt{2x+5} \geq 0$, for $2x + 5 \geq 0$. In other words, we are suggesting that course delivery should be sufficiently "closed" and "complete" to make space for generalization on the same concept and to enhance perceptual flexibility with algebraic expressions.

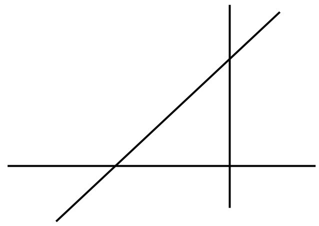


*Figure 4.*Three lines forming a right-angled triangle.

In addition, we argue that encouraging students' attempt at conceptually-varied solutions to multiple-solution tasks could be an effective way to help them nurture perceptual flexibility (Low, 2016). In order to be able to churn out as many as possible conceptually-varied solutions, a student needs to be actively varying or mentally reconfiguring his perceptual elements in order that he could establish connections with distinct concepts for varied solutions. For instance, when required to find the coordinates of a vertex D of a parallelogram ABCD knowing the coordinates of the other three vertices, a student may flexibly perceive D as the point of intersection of lines AD and CD, thus finding the coordinates of D by solving the equations of the two lines. If not, he may perceive the diagonals AC and BD as sharing a common mid-point, thus finding the coordinates of D by equating the mid-points of AC and BD. Or else, he may also find the coordinates of D by two equivalent vectors and . These examples sufficiently elucidate how perceptual flexibility could lead to conceptually-varied solutions.

Limitations

While the participants of this study had churned out a plethora of perceptual responses to the administered tasks, the open-ended nature of the task instruments might have appeared to be overly general to the participants. It is not impossible that findings could be rather sensitive to the ways the tasks are phrased. Should the tasks have been phrased differently to make the requirements more heavily-tipped, such as “State as many as possible variables in the linear equation.” and “State what the diagram geometrically show.”, the results could have differed significantly. This possibility makes the findings of this study specific to the tasks employed. There is definitely ample room for future studies to ascertain the extent to which findings would differ due to the different ways the tasks are phrased. Furthermore, collecting the data in the written form without interviewing the participants has certainly inhibited the effort to portray a true picture of the participants’ cognizing process and their schemas.

In addition, the participants could have been asked about their understanding of the notion of variables prior to being administered the two tasks. By so doing, it would be interesting to see how their understanding would relate to their responses.

Conclusion

This study looked into what pre-university students would perceive as variables in a linear equation and what would appear in their visual fields when they were shown a line crossing the coordinate axes. The findings indicate that students may perceive mathematical elements very differently, a phenomenon which should not be taken lightly in mathematics learning. The findings suggest that students are generally too obsessed with theoretical elements of mathematics to the extent that they fail to perceive or extract essential information from visual stimuli which may only arise from a conscious effort to actively reconfigure one’s perceptual elements in order to arrive at possible gestalts. Such perceptual flexibility is deemed to be critical in any problem solving situations so as to enable access to alternative solutions, some of which could be relatively more, if not highly, efficient.

It is far too easy to attribute students' failure in solving a mathematical task to procedural or conceptual understanding. We however argue that it is equally important to first explore what the students would perceive from a problem situation which we believe would play a vital, if not more important, role in the learning of mathematical problem solving. Instructional efforts should emphasize not only procedural and conceptual knowledge but also the understanding of students' perceptual effects on choices of solution to a task.

The findings of this study show that majority of the participants gave responses relating to more explicit rather than implicit mathematical elements. It was starkly amazing that only one participant noticed a right-angled triangle being formed by the line with the two perpendicular coordinate axes. We conjecture that without noticing a right-angled triangle in the diagram, a participant would less likely relate to any concepts associated with a right-angled triangle, i.e. Pythagoras Theorem, the concepts of similar triangles and Basic Trigonometry, which could provide more efficient solutions to some mathematical tasks. However, although it is reasonably arguable that a solution choice is likely to be, at least partially if not entirely, dependent on a problem solver's perceptual flexibility (i.e., flexible mental reconfiguration of visual stimuli), such premise requires further studies to gain more solid grounds.

Nonetheless, this study is only at a preliminary stage. While this study involved anonymous participants, extended exploration for further understanding and clarification has been thwarted. Further studies should investigate the extent to which perceptual flexibility would influence a problem solver's solution choice, as well as the way classroom instruction would influence students' subjective interpretations of mathematical elements. In addition, it is still largely uncertain about the strength of perceptual reconfiguration of field elements in making connections with existing knowledge structures. It would be equally interesting to study if the participants would be more likely to notice a right-angled triangle in a non-mathematical context, i.e. with simply three straight lines compared to a line crossing two coordinate axes. We conjecture that it would. If that is really the case, knowledge of mathematical elements may influence a person's visual field characteristics which in turn may hinder, rather than facilitate, access to known concepts in a problem solving situation. All these require further explorations for further understandings.

We hope this study would draw attention to the need for instructional strategies which would support learners' flexibility in forming visual fields that presumably may facilitate access to their known concepts for both effective and efficient problem solving. We also hope this study would stimulate more interest in greater understanding of perceptual flexibility- a much neglected role in mathematics learning.

References

Akgün, L., & Özdemir, M. E. (2006). Students' understanding of the variable as general number and unknown: A case study. *The Teaching of Mathematics*, (16), 45-51.

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational studies in mathematics*, *52*(3), 215-241.

Booth, R. D., & Thomas, M. O. (1999). Visualization in mathematics learning: Arithmetic problem-solving and student difficulties. *The Journal of Mathematical Behavior*, *18*(2), 169-190.

Carden, J., & Cline, T. (2015). Problem solving in mathematics: the significance of visualisation and related working memory. *Educational Psychology in Practice*, *31*(3), 235-246.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *The Journal of Mathematical Behavior*, *15*(4), 375-402.

Eisenberg, T., & Dreyfus, T. (1991). On the reluctance to visualize in mathematics. In W. Zimmermann & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics* (pp. 25-38). Washington, DC: Mathematical Association of America.

Gibson, E. J. (1969). *Principles of perceptual learning and development*. East Norwalk, CT: Appleton-Century-Crofts.

Gibson, J. J., & Gibson, E. J. (1955). Perceptual learning: Differentiation or enrichment? *Psychological Review*, *62*(1), 32-41.

Goldstone, R. L., Landy, D. H., & Son, J. Y. (2010). The education of perception. *Topics in Cognitive Science*, *2*(2), 265-284.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A" proceptual" view of simple arithmetic. *Journal for research in Mathematics Education*, 116-140.

Healy, L., & Hoyles, C. (1999). Visual and symbolic reasoning in mathematics: making connections with computers?. *Mathematical Thinking and Learning*, *1*(1), 59-84.

Hiebert, J. (Ed.)(1986). *Conceptual and procedural knowledge: The case of mathematics* (pp.113−132). New Jersey: Lawrence Erlbaum Associates, Inc.

Hiebert, J., & Handa, Y. (2004). *A modest proposal for reconceptualizing the activity of learning mathematical procedures*. Presented at the annual meeting of the American Educational Research Association, San Diego.

Hiebert, J., & Lefevre, P. (1986). Conceptual and Procedural Knowledge in Mathematics: An Introductory Analysis. In J. Hiebert (Ed.), Conceptual and Procedural Knowledge: The Case of Mathematics (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kellman, P. J., & Garrigan, P. (2009). Perceptual learning and human expertise. *Physics of life reviews*, *6*(2), 53-84.

Kellman, P. J., & Massey, C. M. (2013). Perceptual learning, cognition, and expertise. *Psychology of Learning and Motivation, 58*, 117–165.

Kellman, P. J., Massey, C., Roth, Z., Burke, T., Zucker, J., Saw, A., Aguero, K. E., & Wise, J. A. (2008). Perceptual learning and the technology of expertise: Studies in fraction learning and algebra. *Pragmatics & Cognition*, *16*(2), 356-405.

Kellman, P. J., Massey, C. M., & Son, J. Y. (2010). Perceptual learning modules in mathematics: Enhancing students' pattern recognition, structure extraction, and fluency. *Topics in Cognitive Science*, *2*(2), 285-305.

Knuth, E. J., Alibali, M. W., McNeil, N. M., Weinberg, A., & Stephens, A. C. (2005). Middle school students' understanding of core algebraic concepts: Equivalence & variable. *ZDM*, *37*(1), 68-76.

Küchemann, D. (1978). Children's understanding of numerical variables. *Mathematics in school*, *7*(4), 23-26.

Lauritzen, P. (2012). *Conceptual and procedural knowledge of mathematical functions*. University of Eastern Finland.

Leikin, R. (2007, February). Habits of mind associated with advanced mathematical thinking and solution spaces of mathematical tasks. In *Proceedings of the Fifth Conference of the European Society for Research in Mathematics Education: Early childhood mathematics* (pp. 2330-2339).

Libera, C. D., & Chelazzi, L. (2006). Visual selective attention and the effects of monetary rewards. *Psychological Science*, *17*(3), 222-227.

Low, C. S. (2015, March). *A steep battle between flexible knowledge and for-the-test mentality*. Paper presented at the 1st International Conference on Language, Education, Humanities and Innovation. http://icsai.org/procarch/1iclehi/index.html.

Low, C. S. (2016). *Exploring A-Level Students' Mathematical Flexibility and Adaptivity Through Continual Exposure to Multiple-Solution Tasks and Strategies*. Unpublished PhD thesis. Universiti Sains Malaysia.

Low, C. S., & Chew, C. M. (2013, November). *Flexibility and Adaptivity in Mathematical Problem Solving: Some Observations*. Paper presented at the 5th International Conference on Science and Mathematics Education: Empowering The Future Generation Through Science and Mathematics Education, RECSAM, Penang, Malaysia.

Murray, D. J. (1995). *Gestalt psychology and the gestalt revolution*. New York: Harvester Wheatsheaf.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

National Research Council (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Olson, M. H., & Hergenhahn, B. R. (2009). *An introduction to theories of learning* (8th Ed.). New Jersey: Pearson Prentice Hall.

Perls, F. S., Hefferline, R. F., & Goodman, P. (1976). *Gestalt therapy: Excitement and growth in the human personality*. England: Penguin Books Ltd.

Polya, G. (1965). *Mathematical discovery: On understanding, learning, and teaching problem solving (Volume II)*. New York: John Wiley & Sons, Inc.

Pomerantz, J. R., & Portillo, M. C. (2011). Grouping and emergent features in vision: toward a theory of basic Gestalts. *Journal of Experimental Psychology: Human Perception and Performance*, *37*(5), 1331-1349.

Posamentier, A. S., & Krulik, S. (1998). *Problem-solving strategies for efficient and elegant solutions: A resource for the mathematics teacher*. California: Corwin Press, Inc.

Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics. *Handbook of Research on the Psychology of Mathematics Education*, 205-235.

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other?. *Journal of Educational Psychology*, *91*(1), 175-189.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: an iterative process. *Journal of Educational Psychology*, *93*(2), 346–362.

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology*, *99*(3), 561–574.

Rittle-Johnson, B., & Koedinger, K. (2009). Iterating between lessons on concepts and procedures can improve mathematics knowledge. *British Journal of Educational Psychology*, *79*, 483–500.

Rivera, F. (2011). *Toward a visually-oriented school mathematics curriculum: Research, theory, practice, and issues* (Vol. 49). Springer Science & Business Media.

Rosnick, P. (1981). Some misconceptions concerning the concept of variable. *The Mathematics Teacher*, 418-450.

Schoenfeld, A. H., & Arcavi, A. (1988). On the meaning of variable. *The mathematics teacher*, 420-427.

Selden, J., & Selden, A. (2005). Consciousness in enacting procedural knowledge. *Mathematics Education (JRME), 405*.

Sills, C., Fish, S., & Lapworth, Phil (1996). *Gestalt counseling*. UK: Speechmark Publishing Ltd.

Simons, P. M. (1988). Gestalt and functional dependence. In B. Smith (Ed.), *Foundations of Gestalt theory* (pp. 158-190). Munich, Germany: Philosophia Verlag.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics teaching*, *77*(1), 20-26.

Star, J. R. (2013, April). On the relationship between knowing and doing in procedural learning. In *Fourth international conference of the learning sciences* (pp. 80-86).

Star, J. R., Caronongan, P., Foegen, A. M., Furgeson, J., Keating, B., Larson, M. R., Lyskawa, J., McCallum, W. G., Porath, J., & Zbiek, R. M. (2015). Teaching strategies for improving algebra knowledge in middle and high school students.

Steenpaß, A., & Steinbring, H. (2014). Young students' subjective interpretations of mathematical diagrams: elements of the theoretical construct "frame-based interpreting competence". *ZDM*, *46*(1), 3-14.

Steinbring, H. (2005). *The construction of new mathematical knowledge in classroom interaction: An epistemological perspective* (Vol. 38). Springer Science & Business Media.

Summerfield, C., & Egner, T. (2009). Expectation (and attention) in visual cognition. *Trends in cognitive sciences*, *13*(9), 403-409.

Tall, D. (2006). A theory of mathematical growth through embodiment, symbolism and proof. *Annales de didactique et de sciences cognitive*, *11*, 195-215.

Verschaffel, L., Luwel, K., Torbeyns, J., & Dooren, W. V. (2009). Conceptualizing, investigating, and enhancing adaptive expertise in elementary mathematics education. *European Journal of Psychology of Education, XXIV*(3), 335–359.

Von Glasersfeld, E. (1987). Learning as a constructive activity. *Problems of representation in the teaching and learning of mathematics*, 3-17.

Wagemans, J., Feldman, J., Gepshtein, S., Kimchi, R., Pomerantz, J. R., van der Helm, P. A., & van Leeuwen, C. (2012). A century of Gestalt psychology in visual perception: II. Conceptual and theoretical foundations. *Psychological Bulletin, 138*(6), 1218–1252.

Wagner, S. (1983). What are these things called variables?. *The mathematics teacher*, *76*(7), 474-479.

Wagner, S., & Parker, S. (1993). Advancing algebra. *Research ideas for the classroom: High school mathematics*, 119-139.

Wertheimer, M. (1938). Laws of organization in perceptual forms. In W. Ellis (Ed.), *A source book of Gestalt psychology* (pp. 71–88). New York: Harcourt Brace. Cited in Gredler, M. E. (1997). *Learning and instruction*: *Theory into practice* (3rd Ed.). Upper Saddle River: Prentice-Hall, Inc.

Zuya, H. E. (2017). Prospective teachers' conceptual and procedural knowledge in mathematics: The case of algebra. *American Journal of Educational Research,5*(3), 310-315.