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Abstract: *This paper aims to introduce the notions of knowing and grasping of a mathematics concept. We choose the concept of complex numbers to illustrate how these notions can be used to describe the understanding of students for this particular topic. As students develop their knowledge, supportive conceptions and problematic conceptions may occur in different ways. A student may know how to perform an algorithm or how to use a particular concept, without 'grasping' the meaning of the idea in a manner which enables him to comprehend more sophisticated ideas in an extended context. Grasping a concept means the ability to see the different aspects of an underlying concept, manipulate it and use it in different ways for different purposes. On top of that, one should be able to speak of it as a meaningful entity in its own right. In this paper, we will report the data which were collected through questionnaires and follow-up interviews of two third year undergraduate mathematics students to illustrate the subtle distinctions between "knowing" and "grasping". The results reveal that both participants, M1 and M2, couldn't grasp the concept of complex numbers as a coherent whole due to the problematic conceptions which arose from real numbers. There might be other factors which contributed to this phenomenon such as the theoretically abstract nature of complex numbers and surrounding factors that affect learning. This study leads us to realise the importance of creating the necessary experience for learners to make sense of complex numbers so that learners can build from their existing knowledge in real numbers which may conflict with complex numbers. Too often, we have focused on a particular context, teaching the content that learners should know without helping them to grasp the essential ideas for further leaning.*

Keywords: knowing, grasping, complex numbers, problematic conceptions

Introduction

It took several centuries to convince mathematicians to accept the imaginary number, i, as a part of a number system. After the imaginary number was created, the complex number which

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contains both real and imaginary part was developed, thus, making the real number a subset of complex numbers. The rules and conceptions of number systems are usually applicable when used in a broader context, for example, the set of natural numbers, N is a subset of rational numbers, Q. In this respect, the conceptions of natural numbers still work in the rational numbers. However this is not the case when we move from real numbers, R, to complex numbers, C. There are some rules that work in real numbers but do not work in complex numbers and these rules may contradict with learners' past experiences. The work of Chin and Tall (2012), Chin (2013), and Chin (2014), which proposes the framework of supportive and problematic conceptions in making sense of trigonometry, showed that problematic conceptions impeded the learning of more advanced concepts. In this respect, supportive conceptions refer to those conceptions which work in an old context and continue to work in a new context. On the contrary, problematic conceptions refer to those conceptions which work in an old context but do not work in a new context.

A study of how Swedish students understand the concept of complex numbers was performed by Nordlander and Nordlander (2012). Their findings show that students had a variety of concept images in describing the concept of complex numbers such as: (1) complex numbers as a way to make some calculations possible through the expansion of the set of real numbers, some of them also specified that it was a way of solving negative square roots; (2) complex numbers as two-dimensional numbers consisting of a real part and an imaginary part; (3) some students associated the complex numbers with the identity i^2=1, or simply with the symbol i; and (4) some students gave more emotional or negative answers (for example : 'abstract', 'complicated', 'difficult', 'complex, which explains the name', 'no conception' and 'don't know'). Their study also revealed a variety of misconceptions regarding complex numbers.

Most of the researchers and teachers used the notion of misconception to indicate a situation where a student didn't give a correct response in a particular context. However, we think that there is a source for this misconception. As an illustration, a misconception in a context might be a correct response in another context. For instance, in the context of real numbers, the multiplication of two real numbers will always give you a positive number. In the context of complex numbers, this is not true because i^2=-1. This shows that the conception of two negatives make a positive is true in the context of real numbers but it is not true in the context complex numbers. In this case, the conception of two negatives make a positive can be regarded as a problematic conception in complex numbers. Learners face difficulties in building a coherent understanding on advanced concepts due to the problematic conceptions which arise from previous learning (Chin, 2013). At the end, some learners end up with knowing some facts for a particular context without grasping the meaning of the idea in a manner which enables them to comprehend more sophisticated ideas in an extended context.

The Framework of Knowing and Grasping

Most of the students may know how to carry out a mathematical procedure without knowing why. As an illustration, in the multiplication of 1/2×1/3, most of the students know the procedure to get this answer by multiplying the denominators together however many of them couldn't answer why this procedure works. Several studies such as Aksu (1997), Bulgar (2003) and Prediger (2008) have shown that students have limited conceptual knowledge in comparison with their procedural knowledge in the multiplication of fractions. Skemp (1976) used a similar notion as instrumental understanding and relational understanding to distinguish between those who know what to do and those who know why it works. Students need to know mathematical procedures in order to pass examinations. Take for instance, they may learn specific algorithms that need to be used in a specific situation so that they are successful in solving simple problems but they do not grasp the essential ideas that will enable them to solve more sophisticated problems and to understand more advanced ideas.

Knowing vs Grasping

In this paper, we will use the distinction between knowing a mathematical idea and grasping that idea so that it can be used in more sophisticated ways. We operate the notion of grasping as if it is analogous to being able to hold an object in one's mind, to be able to look at it from different angles and to be able to manipulate it, to use it in different ways (Chin, 2013; Tall, 2013). On top of that, to say one can grasp an idea means that one can think of it and speak of it as a meaningful entity in its own right. Take for instance, the concept of parallelogram in Euclidean geometry, we can have a relational view of a parallelogram as the various properties are seen to be related to each other. In this case, a parallelogram has two sets of congruent opposite sides, two sets of congruent opposite angles, the consecutive angles are supplementary, two diagonals that bisect each other, one set of opposite sides which are both congruent and parallel (see Fig. 1). These properties are all equivalent and any one can be used as the basic definition of a parallelogram then the others will follow. If we reflect deeply, we will realise that these properties are not just equivalent, in fact they are different facets of the same underlying concept: parallelogram.

An Illustration of Knowing and Grasping: The case of parallelogram

Knowing the various properties of a parallelogram means knowing the different aspects of a parallelogram. Grasping the concept of parallelogram means seeing all these properties equivalent, recognising the different aspects of the same concept: parallelogram. Thus grasping a concept means the ability to see the different aspects of an underlying concept, manipulate it and use it in different ways for different purposes. Additionally, one should be able to speak of it as a meaningful entity in its own right. Students might know an idea at one level and do not grasp the idea at a higher level. Take for instance, student teacher ST1 in Chin (2013) described $\sin x$ as the ratio of the opposite and hypotenuse in a right angled triangle at the first instance. When he was asked to explain why sin270°equals minus 1, he accepted this fact solely based on the sine curve. This indicates he knows $sin270 = -1$ based on sine graph only but he doesn't grasp it as an extension of circle trigonometry. In this case, circle trigonometry involves dynamic angles of any size and sign with trigonometric ratios expressed as signed numbers and the trigonometric functions represented as graphs.

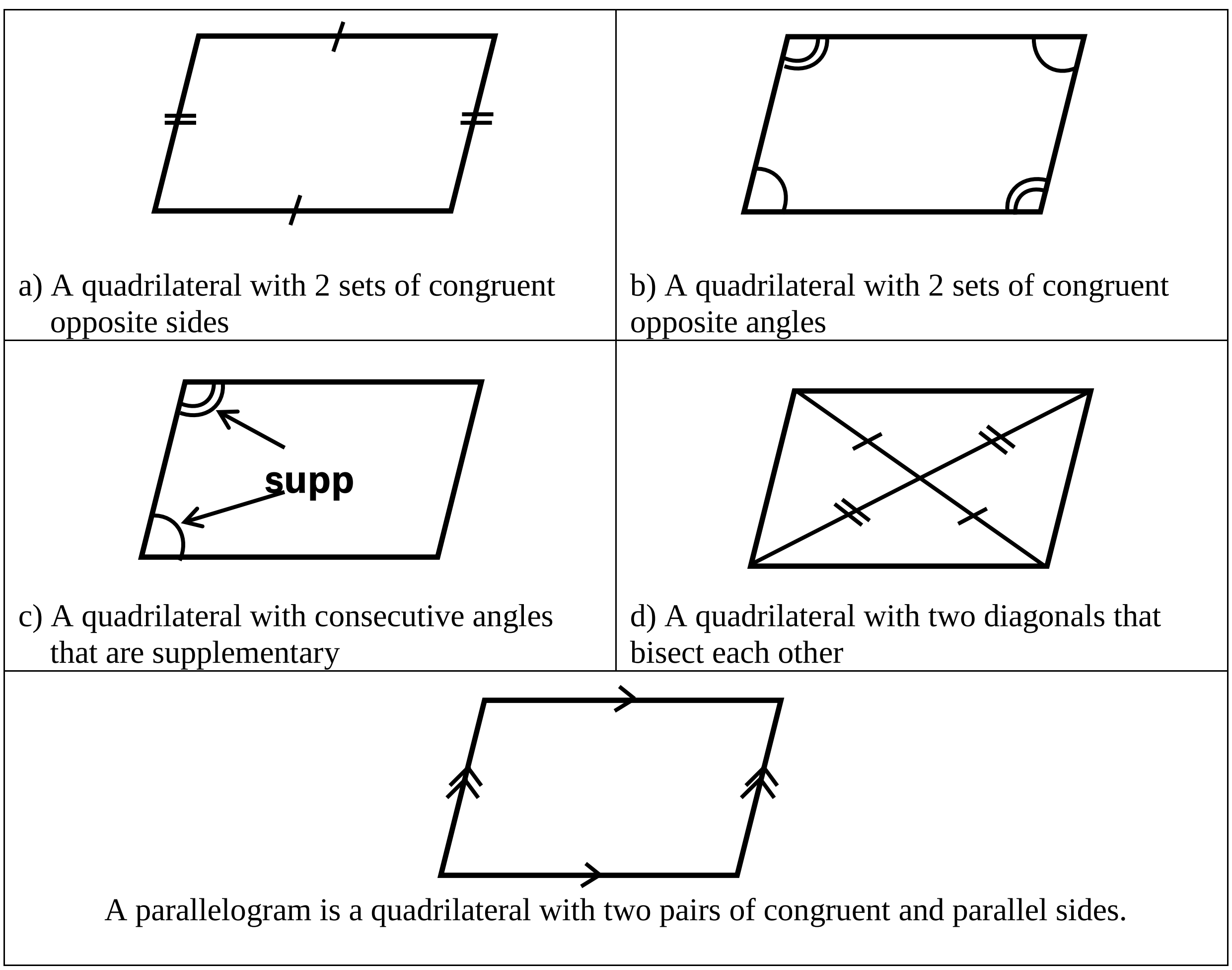

Figure 1. Properties of a parallelogram.

Theories Related to Mathematical Thinking

Sfard (1991) proposed the notions of operation conception and structural conception in describing mathematical concepts formation. In this case, structural conceptions mean "seeing a mathematical entity as an object means being capable of referring to it as if it was a real thing – a static structure, existing somewhere in space and time." (p.4). Meanwhile, operational conceptions mean interpreting a notion as a process rather than an actual entity. She illustrated the development of the concept of number as a cyclic process that involved operational phase and structural phase. She stressed that "when the symbol $5 + 2i$ is intrepreted as a name of legitimate object – as an element in a certain well-defined set – and not only (or even not at all) as a prescription for certain manipulations" (Sfard, 1991, p.20). In this case, grasping a mathematical concept also means having a solid structural conception of a particular concept in an individual's mind.

Bardini, Pierce and Vincent (2015) stated that a particular symbol can be examined from three aspects namely materiality, syntax and meaning. In this respect, "materiality" refers to the appearance of a symbol. "Syntax" means the position and conventions associated with a particular symbol. Take for instance the symbol $(\times)$ must have some symbols or expressions on both sides of it. "Meaning" refers to the meaning of a symbol in a context. As an illustration, the multiplication symbol $(\times)$ can be interpreted as repeated addition for 2×3 as it is equal to $3 + 3$ in the context of natural numbers. However in the context of fractions such as $\frac{1}{2} \times \frac{1}{3}$ it has to be interpreted as "of". Since a complex number can be represented by $a + bi$, where a and b are real numbers, and i is an imaginary number, thus the framework of knowing or grasping of the concept of complex numbers can be explored by looking at these three aspects.

Based on Streefland (1985), an individual may shift from establishing a representation of a concept to using this representation for constructing and reconstructing the concept in new contexts, this can be described as shifting from a ‘model of’ to a ‘model for’. However, the issue might be, this personal ‘model for’ might be not appropriate in a new context due to the fact that the ‘model of’ is not constructed based on grasping the correct basic idea in previous contexts. Therefore we conjecture that the understanding of complex numbers is affected by personal conceptions that arise from previous contexts. The work of Tall and Chin (2012), Chin (2013) and Chin (2014) has highlighted the importance of humans’ conception in making sense of mathematics. They proposed the notions of supportive and problematic conceptions in making sense of mathematics. In this respect, one of the dominant ingredients that shapes humans’ conceptions is prior experiences but the issue is, what does it mean by making sense of mathematics. According to the respected USA National Council of Teachers of Mathematics (NCTM, 2009), “sense making may be considered as developing understanding of a situation, context or concept by connecting it with existing knowledge or previous experience” (p.4). This sparks the following questions such as how exactly this connection happens? What are the possible consequences of this connection? In this case, the notions of supportive conceptions and problematic conceptions can be regarded as a useful framework to understand this matter. A supportive conception refers to a conception that works in an old context and continues to work in a new context. On the contrary, a problematic conception refers to a conception that works in an old context but doesn’t work in a new context. This is not the whole story. Chin (2014) further elaborated that supportive conceptions may contain problematic aspects whereas problematic conceptions may contain supportive aspects. In short, supportive conceptions support generalisation whereas problematic conceptions impede generalisation. Based on Chin (2013), we notice that supportive conceptions enable learners to grasp more sophisticated ideas in an extended context of trigonometry. On the other hand, problematic conceptions may force learners to know mathematics ideas without grasping them. Previous study such as Norlander and Norlander (2011) has shown that students have various misconceptions on complex numbers however we regard these misconceptions as preconceptions that hinder students from grasping complex numbers. Thus in this study, we will try to provide plausible explanations for the root causes of the respondents' inability to know or grasp the concept of complex numbers based on the collected data. In analysing the data, we will seek to distinguish those instances in which the student grasps the fundamental ideas in making sense of a situation rather than just knowing how to cope with them in a routine manner.

Objective

In general, the objectives of this paper are to introduce the framework of knowing and grasping and to demonstrate how this framework can be used in explaining the sense making of participants on complex numbers. Supportive conceptions help learners to grasp sophisticated ideas easily whereas problematic conceptions impede learners from building a coherent understanding. Take for instance, the conception of addition makes bigger, this conception is correct when it involves natural numbers only. When we move to the context of positive fractions then the conception of addition makes bigger which arises from the context of natural numbers can be regarded as a supportive conception as this conception continues to work in this new context. This conception helps us to grasp the idea of addition. However, when we move to the context of negative integers then the conception of addition makes bigger will be regarded as a problematic conception because in this case addition makes smaller. As a consequence of this, learners need to reconstruct their knowledge structure of addition in order to cope with this problematic conception. For those who are able to sense the changes of context and grasp the underlying idea, they will proceed learning with confidence and flexibility. On the other hand, those who cannot cope with this problematic conception may turn to rote learning so that they can pass examination without understanding. The collected data are aimed to answer three specific research questions as outlined below:

1) To what extent do the undergraduate mathematics students grasp the concept of complex numbers?
2) What are the problematic conceptions in making sense of complex numbers?
3) How problematic conceptions hinder the participants from grasping the idea of complex numbers?

Methodology

We began the process of data collection by sending out invitations to a group of third year university undergraduate mathematics students in Malaysia. Four students contacted us in order to participate in our study on a voluntary basis. Then we arranged a specific time and venue with each of them for data collection. We have used four days to complete the data collection. Each day we collected the data from one participant only. At first, the participants were requested to complete a questionnaire about complex numbers then immediately after completing the questionnaire, one follow-up interview was conducted for each of them to enable them to explain and further elaborate their written responses. The data collection was performed individually with the four participants at four different time points. Each follow-up interview lasted for about 40 minutes. The interview protocols were formulated in a way so as to seek for the underlying reasons of why the participants have responded in such a way for the items in the questionnaire. The questionnaire comprises of 21 items about complex numbers in order to explore whether the participants grasp complex numbers as a coherent and meaningful object or not. These items involve asking the participants about the properties of complex numbers, the meaning of complex numbers, the operations of complex numbers, the relationship of complex numbers with real numbers and the different representations of complex numbers. In order to explore how participants deal with unfamiliar extended situations in complex numbers, we also formulated items that were purely virtual in this questionnaire so as to explore the basis of their sense making process. The content of the questionnaire was developed and checked by experts in mathematical thinking.

All the data were collected during the university semester in December 2016. The venue for data collection was the researchers' office. We audiotaped all the follow-up interviews. Four third year undergraduate mathematics students were involved in this study. All the participants have learnt complex numbers in the past. In this paper, we report the data of two students who are indicated as M1 and M2. M1 is a male student and M2 is a female student. We report the data of these two participants only because this data can show the spectrum of responses collected from this study. In reporting the data collected from the interviews, we will use R to represent the researcher. Both the researchers analysed the qualitative data separately then the researchers crosschecked the analyses in order to come to a consensus. This method can resolve all the interpretative dilemmas in the data analysis process. In order to be as objective as possible we were not only looking for the evidence which will support the framework of knowing and grasping but also those data which were contrary to the proposed framework. In this case, we employed the quasi judicial method of analysis as proposed by Bromley (1986, 1990). Rather than searching for data which supports the framework, this method also focuses on looking for evidence so as to remove as many of the suggested explanations as possible. Based on Bromley

(1986), a psychological case-study is sufficient to explain how and why an individual acts in a specific way in a particular context provided that it "contains enough empirical evidence, marshalled by a sufficiently cogent and comprehensive argument, to convince competent investigators that they understand something that previously puzzled them" (p.37). In order to analyse a case, we followed the following ten steps as suggested by Bromley (1986): 1. Express initial issues clearly; 2. State the background context for the case; 3. Propose first view explanations; 4. Examine first view explanations and search for additional evidence; 5. Look for enough evidence to remove as many of the suggested explanations as possible; 6. Examine evidence and sources of evidence to check for consistency and accuracy; 7. Perform a critical inquiry into the internal coherence, logic and external validity of the arguments in the favoured explanations; 8. Choose the 'most reasonable' explanation; 9. Formulate, if appropriate, what implications there are for action; 10. Write a coherent case-report (Bromley, 1986, p.26). The interpretation of data may be regarded as valid if it informs a coherent whole.

Results and Discussions

In this section, we will present the data collected through questionnaires and follow-up interviews of two participants, M1 and M2. The data of M1 and M2 were collected at two different time points. The written responses from the questionnaire are presented first then followed by the relevant follow-up interview excerpts so as to provide readers an understanding of why the participants have written such responses. Additionally, the follow-up interviews also aim to gain further responses in order to achieve a more specific or detailed understanding of the participants' knowledge or more detail on the written responses they gave in the questionnaire. The summaries for students M1 and M2 are presented at the end of each case. These summaries are written in a way to answer the proposed research questions based on the data of each case.

The Case of Student M1

1) Describe complex numbers in your own words.

Complex numbers is numbers that doesn't have orders. It is larger than real number because it consist of real number and imaginary number

M1: …complex numbers are not like integers. For example, after 0 is 1 then 2 then 3. We cannot say after this complex number is that complex number, because they have no order…a complex number consists of real numbers, that's why it is larger than real numbers.

(Response 1. M1, Item 1)

Apparently at this particular instance, M1 noticed a conflicting property of complex numbers with real numbers which was about the nonexistence of order for complex numbers (see Response 1). He further illustrated the notion of order in real numbers by giving specific examples. The existence of order in real numbers may be considered as a problematic conception in complex numbers because complex numbers are not ordered. Obviously M1 was aware of this problematic conception. At this particular instance, we were unable to claim that he knew the

2) Which of the following is a complex number?

(i) 2 **(ii) 0** **(iii) -1** **(iv) $2 + 4i$**

Answer : All of them is complex number. i, ii and iii is complex number that only have real number but iv is complex number that have both real and imaginary number

M1: Complex numbers can be either only real numbers, only imaginary numbers or the combination of these two.

(Response 2. M1, Item 2)

basic idea of complex numbers because he only described one of the properties of complex numbers but he didn't describe what was a complex number. We probed this further in the follow-up interview by asking him to comment on his responses for the second item of the questionnaire. He also said that complex numbers were larger than real numbers and we speculated that he must be thinking of real numbers as a subset of complex numbers. We probed this issue further in item 4.

M1 considered all the given numbers were complex numbers and he further elaborated by saying that "Complex numbers can be either only real numbers, only imaginary numbers or the combination of these two (see Response 2)." This showed that he knew the basic meaning of a complex number which enabled him to identify the given complex numbers.

3) Given a complex number $z = -2 - i$, do you think z is a negative complex number? Explain your response.

Answer : Yes ~~no~~. If we look at the real number of Z, which is −2, ~~it might be said~~ it is negative number ~~but we also must consider the~~ the imaginary number also in negative which is −1. So, it is basically negative complex number since both real and imaginary part have ~~in~~ negative sign.

R: *For this question, you gave an answer "yes". What makes you think that z is a negative complex number?*

M1: *A complex number has two parts. z has a negative for the real part and also negative for the imaginary part. So if both of the parts are negative, we can consider it as a negative complex number. But if only one part has a negative, either real or imaginary, we cannot consider it as a negative complex number.*

R: *In that case, if the z is –2 + i, can it be considered as a positive complex number?*

M1: *No, it's not either negative or positive. When it has a positive and a negative, we cannot determine whether it is negative or positive.*

R: *If we have two parts positive? For example 2 + i?*

M1: *It is a positive complex number.*

(Response 3. M1, Item 3)

Based on his written response for Item 3 (see Response 3), we can see that he scribbled out his initial spontaneous response then he changed to an opposite answer. He seemed likely influenced by the conceptions of positive numbers and negative numbers in real numbers. We all know that in real numbers, a negative sign in front of a positive real number will be identified as a negative number such as -3 whereas without a negative sign then it will be interpreted as a positive number such as 3. Zero is neither positive nor negative. Based on this conception that arises from real numbers, M1 must have formulated a mental model of negative numbers. He transferred this model of negative numbers to the context of complex numbers which led him to give the above responses. The constructed model of negative numbers in the context of real numbers was not based on the basic meaning of negative numbers. In a simplest sense, a negative number is a real number that is less than zero. This shows that negative numbers are placed on an ordered field because they are less than zero. If M1 was building the concept of negative numbers based on this basic meaning of negative numbers then he might have sensed the cognitive conflict inside his mind because at an earlier instance he did mention that complex numbers were not ordered. Additionally, the basic meaning of negative numbers also states that a negative number is a real number.

If M1 was thinking of negative numbers on a number line then he should be able to see that all the negative numbers are on the left side of zero whereas positive numbers are on the right side of zero. We speculate that he doesn't use the representation of negative numbers on a number line because if he does that, then he must have sensed that it is problematic in determining negative numbers on a complex plane. Take for instance, if he conceives $z=-2-i$ as a point on a complex plane then he should be able to sense that his way of categorising negative complex numbers is problematic as complex numbers are on a complex plane but not on a number line. We can determine negative numbers easily on a real number line with zero in the middle of the line and all the negative numbers are located on the left side of the zero. But this is a different story for complex numbers as they are on a complex plane. Obviously, he was building the model of negative numbers based on the materiality of negative numbers without focussing on the meaning of negative numbers then he applied this model to the context of complex numbers. This indicates that he was building his own incorrect notion of negative complex numbers based on a problematic conception that he wasn't aware of. The conception of negative numbers is a problematic conception because it works in real numbers but it does not work in complex numbers. M1 didn't grasp the meaning of negative numbers that enables him to make sense of this extended context.

4) Explain the relationship between the set of real numbers and the set of complex numbers.

The set of real number is smaller than the set of complex numbers ~~the~~ since real number is a part of complex number. Just like integer is a part of real number.

M1: …A real number is a part of the complex number so we have integers in real numbers, and real numbers in complex numbers (he drew a figure, see Fig. 2).

(Response 4. M1, Item 4)

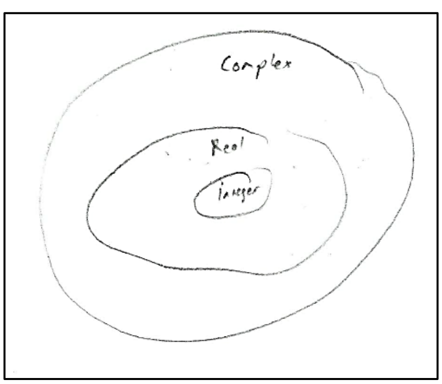

Figure 2. M1, Item 4.

According to Response 4, M1 knew the set of real numbers was a subset of complex numbers and he drew a correct diagram (see Fig.2) as shown above to represent this relationship. Hadamard (1949) stated that "I need [an image] in order to have a simulation view of all elements…to hold them together, to make a whole of them…; to achieve synthesis…and give the concept its physiognomy" (p.77). This mental picture drew by M1 allowed him to observe complex numbers from different contexts such as real numbers and integers. He was accessing his structural conception of complex number through this picture. This resonates with Sfard (1991) who stated that mental images are compact and integrative and support structural conception.

5) Do you think that all the properties in Real Numbers will also be the properties of Complex Numbers? Please explain your response.

Answer : No, the real number can be explain by ~~straight line~~ which is -1 comes before 0 ~~but~~ in complex number, we cannot say that 2 + i is greater than 2.

Yes

M1: Because real numbers are in complex numbers. So all the properties in real numbers will also be the properties of complex numbers.

(Response 5. M1, Item 5)

Initially M1 wrote something to express that not all properties of real numbers will be the properties of complex numbers by focusing on the nonexistence of order in complex numbers (see Response 5). This was a problematic conception as real numbers were ordered whereas complex numbers were not ordered and he did realise this problematic conception. However he scribbled out his answer because he must be using his mental image (see Fig. 2) to arrive at his final conclusion. Sfard (1991) mentioned that when a meaningful entity exists in an individual's mind then he/she should be able to speak of its properties. Thus this item aims to explore whether a participant conceives complex numbers as a meaningful entity or not by asking one of the properties of complex numbers in relation to real numbers. His conception as represented by his mental image of complex numbers (see Fig. 2) was supportive because he can see that all the elements of real numbers were contained in the set of complex numbers. Thus he felt that it was logic to conclude that all the properties in real numbers will also be the properties of complex numbers. He was focussing on his mental image (i.e. his drawing) and suppressed the problematic conception such as the nonexistence of order in complex numbers. This showed that he didn't grasp complex numbers as a new meaningful entity with new properties that might differ with the properties of real numbers.

6) Do you think that $2 + 3i < 4 + 3i$? Explain your response.

Answer : No because ~~it~~ Complex number cannot be explain in straight line.

M1: For example real numbers have order. But for complex numbers, they are random and no order. So we cannot tell which one is bigger because they have no order.

(Response 6. M1, Item 6)

Based on Response 6, we can notice that he had a conception that all the complex numbers cannot be explained or located on a straight line thus they are not ordered. Clearly, this was a conception which arose from the context of real numbers as all real numbers are located on a number line. This can be regarded as a problematic conception in the context of complex numbers because complex numbers were not ordered. In this case, he used his conception of order based on a number line to make sense of a new extended context. This conception of order in real numbers became the conception for order in complex numbers. This showed that he has built the idea of order correctly in real numbers and used it to make sense a more advanced situation. It would be interesting if we could follow up with M1 to ask further questions about whether he conceives 2> 0 is correct. In his earlier response, he mentioned that 2 and 0 were complex numbers and at this moment he was saying that complex numbers were not ordered.

Thus we can see the hidden problematic aspect in this conception.

7) $\frac{2}{i} = \frac{\sqrt{4}}{\sqrt{-1}} = \sqrt{\frac{4}{-1}} = \sqrt{-4} = \sqrt{-1}\sqrt{4} = 2i$

Do you think that the working shown above is correct? Explain your response.

Answer : no.

$$\frac{\sqrt{4}}{\sqrt{-1}} \times \frac{(\sqrt{-1})}{(\sqrt{-1})}$$

$$\frac{\sqrt{4}\, i}{-1}$$

$$= -2i$$

R: *Based on your answer, which part of the working shown here is wrong?*

M1: *We cannot combine* $\sqrt{\frac{4}{-1}}$ *like this.*

R: *Why we cannot combine* $\frac{\sqrt{4}}{\sqrt{-1}}$ *to become* $\sqrt{\frac{4}{-1}}$ *?*

M1: *This is a bit difficult to explain. The answer is different when we combine and do not combine them. For* $\frac{\sqrt{4}}{\sqrt{-1}}$*, we multiply both numerator and denominator with* $\sqrt{-1}$*. Then we will get* $\sqrt{4}i$ *over -1, eventually we get -2i.*

R: *Why must we multiply with* $\sqrt{-1}$ *but not group them together under one big surd?*

M1: *If not mistaken, it is not the properties of complex number. For example,* $\frac{\sqrt{2+i}}{\sqrt{2+i}}$*. If we follow the properties of real numbers, we can just cancel out these two to get 1. But in the case of complex numbers, we need to find its conjugate then multiply both numerator and denominator with* $\sqrt{2-i}$*.*

(Response7. M1, Item 7)

Response 7 indicates that he knew that $\frac{\sqrt{4}}{\sqrt{-1}}$ and $\sqrt{\frac{4}{1}}$ were different in terms of answer and he needed to multiply both the numerator and denominator with $\sqrt{-1}$ based on the properties of complex numbers. He accepted this fact by building on his previous experience working with complex numbers therefore he can further elaborate it by giving other example which shows the difference in procedure of working with complex numbers and real numbers. M1 realised this particular problematic conception which arose from real numbers and he could convince himself that he needs to work differently in this kind of situation otherwise it will yield different answers. However he couldn't explain why $\frac{\sqrt{4}}{\sqrt{-1}} \neq \sqrt{\frac{4}{-1}}$ because M1 didn't grasp $\sqrt{-1}$ as a single entity. If he grasps $\sqrt{-1}$ as a single mental object then he should be able to reason that $\sqrt{-1}$ is not the same as taking square root of a number which involves an operation to a number. He was trying to justify why $\frac{\sqrt{4}}{\sqrt{-1}} \neq \sqrt{\frac{4}{-1}}$ by thinking about the division process in real numbers and complex numbers. He didn't grasp the basic idea of division in complex numbers. In fact, if he grasps the basic idea of multiplication in complex numbers then he could easily make sense of the complex division as an opposite of complex multiplication which up to certain extent can help him to reason this situation. In this case, the basic ideas of complex multiplication are scaling and rotation. In real numbers, the operation of division can be embodied by using a number line whereas in complex numbers, the operation of division can be embodied through an Argand diagram. Obviously, M1 realised the problematic conception of division arose from real numbers but he didn't grasp the meaning of the idea which enables him to comprehend extended ideas.

Summary of Student M1

The summary of Student M1 is written in a way to address the proposed research questions in this paper so that the readers can see how the data are used to answer these research questions.

1) To what extent do the undergraduate mathematics students grasp the concept of complex numbers?

M1 is a male undergraduate third year mathematics student. He knows the basic meaning of complex numbers but he doesn't grasp it as a coherent meaningful entity. This can be seen from his incoherent responses for item 5 and item 6 above. For item 5, initially he expressed that not all properties of real numbers will be the properties of complex numbers by focusing on the nonexistence of order in complex numbers. After that, he changed his response and agreed that all the properties of real numbers will also be the properties of complex numbers. Obviously he wasn't sure whether the nonexistence of order in complex numbers can be considered as a property or not thus he suppressed this conception and focused on his mental image. This indicates the idea of complex numbers is not a legitimate object in his mind. According to Sfard (1991), one should be able to speak of the properties of a meaningful entity if this entity exists in his/her mind. M1 formulates his own notions of negative and positive complex numbers and this indicates that there should be some sorts of order in complex numbers but then in another item he responded that $2 + 3i < 4 + 3i$ was not true because they cannot be explained on a straight line. Does that mean he can arrange his notions of negative complex numbers and positive complex numbers on a straight line? Again this shows that he couldn't conceive coherently on the properties of complex numbers. Item 7 shows that he doesn't grasp $\sqrt{-1}$ as a single mental entity. All these show that he doesn't grasp the concept of complex numbers.

2) What are the problematic conceptions in making sense of complex numbers?

M1 has one problematic conception in making sense of complex numbers that he doesn't aware of. The conception of negative numbers only valid for real numbers but not for complex numbers. This can be regarded as a problematic conception. The issue is he doesn't aware of this problematic conception. He formulates his own model of negative complex numbers by focussing on the materiality of negative real numbers. For him, any number with a negative sign in front of it is regarded as a negative number then he extents this to the context of complex numbers. He does aware of one problematic conception such as the nonexistence of order in complex numbers. When thinking of the operation on complex numbers, he is aware of the problematic conception of $\frac{\sqrt{4}}{\sqrt{-1}} \neq \sqrt{\frac{4}{-1}}$. In real numbers, we can always combine the radicals of numerator and denominator. Obviously this is a problematic conception which arises from real numbers.

3) How problematic conceptions hinder the participants from grasping the idea of complex numbers?

Based on the reported data, we can notice how the problematic conceptions have impeded M1 from grasping the idea of complex numbers as a coherent whole. For instance, M1 knew that complex numbers don't have order but at the same time he had an incorrect mental model of negative complex numbers that in fact contradicted with his knowing. When thinking of the operation of complex numbers, he couldn't make sense of why $\frac{\sqrt{4}}{\sqrt{-1}} \neq \sqrt{\frac{4}{-1}}$ because in real numbers context we can combine the radical of the numerator and the denominator. He only knew that if the radicals of denominator and numerator are combined in the complex numbers then another answer will yield.

The Case of Student M2

1) Describe complex numbers in your own words.

Considering $z = x + iy$.
where x is the real number part
and y is the imaginary number part.

M2: The y part is imaginary because of it is iy. y itself is actually a normal number, a constant, but this is i, because this i we usually assume as $\sqrt{-1}$ *where in the real number part it is actually do not exist.*

(Response 8. M2, Item 1)

According to Response 8, we can see that M2 conceived a normal number as a real number. Up to certain extent, we can speculate that she had a sense of discomfort with the existence of i because i was not considered as a normal number.

Response 9 shows that M2 didn't grasp complex numbers as a meaningful entity. She considered a number as a complex number when there was a specific purpose such as in the addition of complex numbers. This showed she had a strong operational view on complex numbers. Under a normal situation without a specific purpose, she wouldn't categorise a number as a complex number. This showed that she wasn't very comfortable with complex numbers and she might have a sense of doubt regarding the existence of complex numbers. M2 sensed the complexity of operating complex numbers thus she didn't conceive a real number as a complex number as it didn't consist of imaginary part. This doesn't mean she didn't know about the basic meaning of complex numbers. From Response 9, we can see that she formulated her own way to differentiate real numbers and complex numbers but at the same time she did realise that 2, 0 and -1 can be categorised as complex numbers. She possessed a strong operational conception about complex numbers and she was able to illustrate a particular instance which she would need to conceive a real number as a complex number in order complete an operation. For M2, thinking a real number as a complex number was harder in comparison of thinking of it as a real number. Apparently, she conceived those numbers given differently under different situations. It seemed like she had a conception that working in a complex number context was more difficult. We speculate that her experience of working with complex numbers led her to have this conception.

2) Which of the following is a complex number?

(i) 2 **(ii) 0** **(iii) -1** **(iv) $2 + 4i$**

Answer : iv

R: *Why only 2+4i considered as a complex number? What about the other three?*

M2: *Consider as real numbers.*

R: *Why they are not complex numbers?*

M2: *The absence of imaginary term. Actually a real number can be a complex number, but a complex number cannot be a real number. You want to say 2, 0, -1 are complex numbers…not wrong, but I will not consider them as complex numbers because they don't have the imaginary term although it seems easier to compute without the imaginary term.*

R: *But just now you said they can be considered as complex numbers, right? So under what conditions we can consider 2, 0 or -1 as complex numbers?*

M2: *Because behind is all zero i. So, this means the i still exists but just that we don't see it.*

R: *Ok, so in that case, all of these are complex numbers, am I right?*

M2: *Ya. But of course if you want to include them then you just think of them as real numbers, easier la, why want to make your life so hard.*

R: *So you still believe that 2, 0 and -1 are not complex numbers?*

M2: *Ya, not complex number unless you want to sum it up with another complex number. For example 2 plus 1+3i, so you have to consider 2 as a complex number that has zero imaginary term. But by itself, I think it as a real a number.*

(Response 9. M2, Item 2)

3) Given a complex number $z = -2 - i$, do you think z is a negative complex number? Explain your response.

Answer : No.

As there is a presence of i where $i = \sqrt{-1}$ in which it is actually undefined.

$\therefore$ z is not considered a complex number

R: *Why you said that z is not a negative complex number?*

M2: *Even the term is negative, doesn't mean that z is negative. Because we don't know the actual value of i.*

R: *We are familiar with i^2 equals to -1 right? So if i, can we just write it as $\sqrt{-1}$?*

M2: *Yes, i itself is $\sqrt{-1}$ but I wouldn't think it as a negative complex number. I will still look at complex numbers as complex numbers without direction.*

(Response 10. M2, Item 3)

Response 10 indicates that she couldn't see an actual value of i in the system of real numbers even though she knew that $i = \sqrt{-1}$. This is critical because a negative number must be a real number which is less than zero. If she could locate $\sqrt{-1}$ on a number line then she probably would be able to say that whether $z = -2 - i$ is a negative complex number or not as we all know that the numbers on the left hand side of the zero on the number line will be negative numbers. Indeed this was a structural view on negative numbers. Additionally she also possessed a operational view by saying that positive can be represented as movement to the right and negative can be represented as movement to the left therefore in this respect we can see that it involves direction when thinking about positive numbers and negative numbers. At last, she stated that complex numbers don't have direction and this means that she couldn't accept the idea of negative complex numbers. She made sense of this situation by using her conception of negative numbers that was constructed in the context of real numbers. She grasped the basic idea of negative numbers that enable her to formulate a correct conception of negative numbers. M2 tried to apply this conception of negative numbers to the context of complex numbers and she sensed the problematic aspects of this conception. As a consequence of this, she knew that was a problematic conception which arose from the context of real numbers.

M2 didn't conceive $z = -2 - i$ as a negative complex number because she has grasped the basic idea of negative real numbers in a manner which enables her to make sense of an extended context. Obviously, the conception of negative numbers was a problematic conception because it cannot be generalised from real numbers to complex numbers. More importantly, we can see that all the real numbers on the left hand side of the zero on the number line will be negative numbers. For the case of complex numbers, all the complex numbers will be located on an Argand diagram or a complex plane which is obviously different from the case of real numbers. This shows that grasping the underlying idea of a concept in a previous context can enable a learner to successfully make sense of an unfamiliar extended context.

4) Explain the relationship between the set of real numbers and the set of complex numbers.

Answer :

Set of real numbers is subset of comple numbers.

Response 11 shows that M2 knew the relationship between real numbers and complex numbers.

5) Do you think that all the properties in Real Numbers will also be the properties of Complex Numbers? Please explain your response.

Answer : ~~No.~~

~~Properties of imaginary part differ slightly.~~

Yes.

M2: Since the set of real numbers is a subset of complex numbers, so definitely all the properties in real numbers will also be the properties of complex numbers."

(Response 12. M2, Item 5)

M2 has given a similar response with M1 (see Response 12). Initially, in her written response, she stated that the properties of an imaginary part make the properties of complex numbers differ slightly to the properties of real numbers (see Response 12) then she scribbled out her answer and gave an opposing answer. She was focussing on her conception about the relationship between real numbers and complex numbers which led her to another opposing conclusion. We speculate that she might have a similar mental image as M1 (see Fig. 2). When she said "set of real numbers is a subset of complex numbers", this also implies that all the elements of real numbers will be the elements of complex numbers therefore intuitively the properties of real numbers should also be the properties of complex numbers since all the real numbers are contained in the set of complex numbers. Building on this intuitive conception which is visually convincing, therefore she came out with her conclusion. At this particular instance, she had two intuitive conceptions which will lead her to two different judgements. However, the intuitive conception based on visual that conceived real numbers as a subset of complex numbers was more convincing because she could see the relationship of real numbers and complex numbers in her mind. We claimed that this was an intuitive conception as it didn't operate at an analytical level. She didn't grasp complex numbers as a new meaningful mental entity that existed in its own right with new properties. Due to this, it was not surprising for us that she came out with this conclusion. Realising the fact that complex number is a new entity in its own right even though it is an extension of real numbers therefore the properties of real numbers do not necessary become the properties of complex numbers. This shows how a person's existing knowledge can become a barrier for understanding of a more advanced idea due to problematic intuitive conceptions which are not realised by an individual.

6) Do you think that $2 + 3i < 4 + 3i$? Explain your response.

Answer : Yes

As both imaginary part of the equation is equals to 3i we can omit it and observe only the real part.

R: *Why you think that this is correct?*

M2: *Assuming that these 3is have the same value, so we don't look at them anymore. Then 4 is bigger than 2, that's why I said* $2 + 3i < 4 + 3i$ *is correct.*

R: *What if they have different imaginary parts? For example 3i and 4i, can we find out which is bigger or which is smaller?*

M2: *No, we cannot because we don't know the actual value of the i.*

R: *If we have different imaginary parts but the same real parts, can we compare? For example* $2 + 3i < 2 + 4i$*?*

M2: *No we cannot compare as I told you we don't know what is i.*

(Response 13. M2, Item 6)

Based on Response 13, she was conceiving $3i$ analogous to an unknown in real numbers or a pronumeral. In this case, both sides of the given inequality have the same pronumeral therefore she didn't bother about $3i$ as she could cancel out the effect of it. This showed that she interpreted the meaning of $3i$ wrongly. Then she only focused on the real parts of the two complex numbers and she noticed that 2 was smaller than 4 hence she arrived at an incorrect conclusion of saying that $2 + 3i < 4 + 3i$ was correct. Her conception of 3*i* as an entity with a value was a problematic conception and we speculate that this conception was originated from her working experience in the real numbers context. This showed the difficulty in thinking a complex number as a single meaningful mental entity especially in the form of a+b*i* as it involved an operator between the real part and the imaginary part thus this left rooms for the participant to compare the real part of a complex number to the real part of another complex number when the imaginary parts of the two complex numbers were the same. M2 didn't aware of the difference in meaning between $3i$ and a pronumeral. If M2 conceives 2+3*i* and 4+3*i* as two different points on a complex plane then she might be able to notice that $2 + 3i < 4 + 3i$ is problematic because there is no way for her to order these points. This shows the divergence of grasping the complex number as a single mental entity and knowing a+b*i* as a complex number. Clearly M2 knew that a+b*i* was a complex number however she didn't grasp it as a single meaningful entity in its own right.

7) $\mathbf{\frac{2}{i} = \frac{\sqrt{4}}{\sqrt{-1}} = \sqrt{\frac{4}{-1}} = \sqrt{-4} = \sqrt{-1}\sqrt{4} = 2\boldsymbol{i}}$

Do you think that the working shown above is correct? Explain your response.

Answer : No

$\frac{2}{i} = 2 \cdot \frac{1}{i} \neq 2i$

R: Can you tell me which part of the operation here is wrong?

M2: None of the operation is wrong, if you want to talk in the sense of real number, this one is correct. Somehow with the presence of $\sqrt{-1}$ *like…not correct.*

R: Alright. Can you explain what you have written here?

M2: I multiply 1 over i. Because it is like 2 times 1 over i, this one not necessary equal to 2i.

R: So, does everything made sense to you?

M2: No no no. If you want to consider -1 as 1, then is correct. But because of the -1, because when comes to surd negative right, like…very weird, because we always say that $\sqrt{-1}$ *does not exist.*

(Response 14. M2, Item 7)

Response 14 shows that M2 sensed there was nothing wrong with the given mathematical working but at the same time she did sense there existed a problem due to the presence of $\sqrt{-1}$ by noticing the initial form and the final form of the given mathematical expressions. In this case, she compared the initial form (i.e. $\frac{2}{i}$) to the final form (i.e. $2i$) and responded that they were not the same based on the materiality of $\frac{2}{i}$ and $2i$. Therefore this made her felt not confident with the given working because the operation had transformed the initial form into a very different final form. Obviously she didn't grasp $\sqrt{-1}$ as a single entity which made her unable to explain explicitly why $\frac{\sqrt{4}}{\sqrt{-1}}$ cannot be transformed to $\sqrt{\frac{4}{-1}}$. If M2 grasps $\sqrt{-1}$ as a single meaningful mental object then she should be able to reason that $\sqrt{-1}$ is a single entity and this is not the same as taking square root of a number which involves an operation to a number. M2 did say that "…we always say that $\sqrt{-1}$ does not exist." and this indicated that M2 cannot accept $\sqrt{-1}$ as a single meaningful mental entity. This clearly showed that she didn't grasp the concept of complex numbers. This sparks an interesting question of why she couldn't accept $\sqrt{-1}$ as a single meaningful entity. Clearly, the problematic conception of cannot square root a negative number has impeded her from grasping $\sqrt{-1}$ as a single meaningful entity. This problematic conception was arisen from working in the context of real numbers.

Summary of Student M2

The summary of Student M2 is written in a way to address the proposed research questions in this paper so that the readers can see how the data are used to answer these research questions.

1) To what extent do the undergraduate mathematics students grasp the concept of complex numbers?

M2 is a female undergraduate third year mathematics student. She doesn't grasp the concept of complex numbers as a meaningful coherent entity but she knows the basic meaning of complex numbers. This can be seen when she possessed different meanings for complex numbers for different situations. She didn't think complex numbers as a meaningful and coherent entity when thinking of the properties of complex numbers. For instance, she agreed that all the properties of the real numbers will be the properties of complex numbers and responded that this was because the set of real numbers is a subset of complex numbers. This showed that she didn't conceive complex numbers as a new entity with its new properties but she was thinking of complex numbers are an extension of real numbers thus she arrived at such conclusion. In responding to $2 + 3i < 4 + 3i$, M2 was thinking in the context of real numbers and conceived $3i$

as a pronumeral. Again this showed that she didn't see neither $2+3i$ nor $4+3i$ as a single meaningful entity thus she thought that she could ignore $3i$ by canceling off them on both sides of the inequality. This shows that she doesn't aware of the change of meanings that arise from the change of contexts. In thinking of $\frac{\sqrt{4}}{\sqrt{-1}}$ and $\sqrt{\frac{4}{-1}}$, she couldn't justify why the radicals of the numerator and the denominator cannot be combined into one. This showed that she didn't conceive $\sqrt{-1}$ as a single meaningful entity but she was thinking like an operation to be performed on a number. All the above instances show that M2 didn't grasp the concept of complex numbers as a meaning entity in its own right.

2) What are the problematic conceptions in making sense of complex numbers?

M2 was aware of the problematic conception of negative numbers which arose in the context of real numbers. She knew that the notion of negative numbers was not applicable to the context of complex numbers and she could make sense of it. She had a problematic conception of thinking that she can compare the order of two complex numbers. Apparently she was not aware of this problematic conception that arose from real numbers. She noticed the problematic conception in operation when thinking of $\frac{\sqrt{4}}{\sqrt{-1}}$ and $\sqrt{\frac{4}{-1}}$, but she couldn't make sense of it.

3) How problematic conceptions hinder the participants from grasping the idea of complex numbers?

Based on the reported data, we can see that M2 was impeded from grasping the idea of complex numbers due to problematic conceptions. Take for instance, she couldn't see $2+3i$ and $4+3i$ as two single entities due to her problematic conception in working with pronumerals. On top of that, her experience of working with radicals in real numbers contributed to her inability to conceive $\sqrt{-1}$ as a meaningful entity. In real numbers, square root of a number means an operation to be performed and at the same time we cannot square root negative numbers. Clearly this problematic conception makes her unable to grasp the idea of complex numbers as a single meaningful object.

Conclusion

This paper shows that the framework of knowing and grasping can be a potentially useful framework to describe the nature of understanding of students. Based on the reported data, both the participants didn't grasp i as a meaningful and coherent entity with new properties. They know the basic meaning of complex numbers but they don't grasp the concept of complex numbers. This resonates with Sfard (1991) which suggests that fully structural versions of the most basic concepts such as number and functions are hard to achieve. Norlander and Norlander (2011) suggests that many students have misconceptions in complex numbers due to the abstract nature of this concept. Thus it is not surprising that both the participants could not grasp this concept. This study has extended the work of Norlander and Norlander (2011) by exploring the possible root causes of misconceptions in complex numbers. In this case, one of root causes is problematic conceptions. The participants possess problematic conceptions which arise from the context of real numbers and sometimes they are not aware of these problematic conceptions. When they are not aware of these problematic conceptions then this will lead to either incorrect judgement or impede the sense making of complex numbers. Both the participants focus on certain aspects of complex numbers to make sense of a given situation. The way they make sense of those aspects is heavily affected by their conceptions that arise from the context of real numbers. It is important to help learners to grasp the concept of complex numbers as a meaningful and coherent entity so that they can make sense of unfamiliar extended situation with confidence. More future studies should be done on how to help learners to grasp the concept of complex numbers.

References

Aksu, M. (1997). Student performance in dealing with fractions. *Journal of Educational Research, 90*(6), 375-380.

Bardini, C., Pierce, R., & Vincent, J. (2015). Contemplating symbolic literacy in first year mathematics students. In M. Marshman, V. Geiger & A. Bennison (Eds.) *Mathematics Education in the Margins. Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia* pp 77-84. Adelaide, Australia: MERGA.

Bromley, D. B. (1986). *The Case-study Method in Psychology and Related Disciplines*. Chicheter: John Wiley & Sons.

Bromley, D. B. (1990). Academic contributions to psychological counselling: 1. A philosophy of science for the study of individual cases. *Counselling Psychology Quarterly, 3*(3), 299-308.

Bulgar, S. (2003). Children's sense-making of division of fractions. *Journal of Mathematical Behavior, 22*, 319-334.

Chin, K. E. (2013). *Making sense of mathematics: supportive and problematic conceptions with special reference to trigonometry.* (Unpublished Doctoral thesis). University of Warwick, England. Retrieved from http://wrap.warwick.ac.uk/58403/

Chin, K. E. (2014). Supportive and problematic aspects in mathematical thinking over the longer term. In Oesterle, S., Nicol, C., Liljedahl, P., & Allan, D. (Eds.), *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics*

Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education, 6, 41. Vancouver, Canada: PME.

Chin, K. E., & Tall, D. O. (2012). Making sense of mathematics through perception, operation and reason: The case of trigonometric functions. In Tai-Yih Tso (Eds.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, 4, 264. Taipei, Taiwan: PME.

Hadamard, J. S. (1949). *The Psychology of Invention in the Mathematics Field*. New Jersey, NJ: Princeton University Press.

National Council of Teachers of Mathematics (2009). *Focus in High School Mathematics: Reasoning and Sense Making.* Reston, VA: NCTM.

Nordlander, M., & Nordlander, E. (2012). On the concept image of complex numbers. *International Journal of Mathematical Education in Science and Technology,* 1-15.

Prediger, S. (2008). The relevance of didactic categories for analysing obstacles in conceptual change: Revisiting the case of multiplication of fractions. *Learning and instruction, 18*(1), 3-17.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1-36.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, *77*, 1-7.

Streefland, L. (1985). Wiskunde als activiteit en de realiteit als bron [Mathematics as an activity. and the reality as a source]. *Tijdschrift voor Nederlands Wiskundeonderwijs (Nieuwe Wiskrant)*, *5*(1), 60-67.

Tall, D. O. (2013). *How humans learn to think mathematically*. New York, NY: Cambridge University Press.