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Exploring Mathematical Knowledge for Teaching Teachers: Supporting Prospective Elementary Teachers' Relearning of Mathematics

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Abstract: *The growing number of studies on mathematics teacher educator knowledge have consistently argued that mathematics teacher educators require specialized knowledge in their work with prospective teachers (beyond the knowledge needed for teaching students), what researchers refer to as mathematical knowledge for teaching teachers. Drawing from existing research and aspects of our own work as mathematics teacher educators, we offer our own conceptualization of mathematical knowledge for teaching teachers and illustrate ways in which we as mathematics teacher educators use our own knowledge in teaching mathematics content to prospective teachers. We are particularly concerned with the knowledge mathematics teacher educators use to support prospective teachers' relearning of mathematics, which involves prospective teachers ultimately reconstructing their previously developed knowledge of mathematics. We will illustrate ways in which we use various aspects of mathematical knowledge for teaching teachers to support prospective teachers' relearning of mathematics through the lens of three different tasks of teaching. We conclude with a discussion of the implications of our analysis for informing the growing knowledge base for mathematics teacher educators.*

Keywords: teacher educator knowledge, preservice teacher education, mathematical knowledge for teaching teachers, relearning

Introduction

While there is an established research base on the knowledge K-12 teachers use in their work with students (Da Ponte & Chapman, 2006; Sullivan & Wood, 2008), research on the knowledge mathematics teacher educators (MTE) use in their work with elementary teachers or prospective teachers (PTs) is still in its infancy (Jaworski & Huang, 2014). Developing a research base on MTE knowledge is critically important as MTEs play a central role in the mathematical preparation of teachers. The growing number of studies on MTE knowledge have consistently argued that MTEs require specialized knowledge in their work with PTs (above and beyond the knowledge needed for teaching students), what researchers refer to as mathematical knowledge for teaching teachers (MKTT; e.g., Zopf, 2010).

Building on existing models of MKTT, our aim in this article is to contribute to the growing research base on MKTT in order to elaborate on, or "unpack," aspects of MKTT and consider how such knowledge is different from the knowledge required of K-12 teachers of mathematics. We are particularly concerned with the knowledge MTEs use to support PTs' *relearning* of mathematics, which involves PTs ultimately reconstructing their previously developed knowledge of mathematics (Zazkis, 2011). Drawing from existing research and aspects of our own work as MTEs, we offer our own conceptualization of MKTT and illustrate ways in which we as MTEs use our own knowledge in teaching mathematics content to PTs who are studying to obtain certification to teach students of ages 2-14, who we will refer to as elementary PTs. We conclude with a discussion of the implications of our analysis for informing the growing knowledge base for MTEs.

Theoretical Framework

Mathematics Content Courses for Prospective Elementary Teachers

Recent research on the nature of the mathematical knowledge teachers need to be effective has informed our understanding of what mathematics should be taught in content courses for PTs. Specifically, Ball, Thames and Phelps (2008) have reconceptualized mathematics content knowledge, arguing that teachers need to know mathematics in ways required exclusively for teaching, or mathematical knowledge for teaching (MKT), which includes both common content knowledge (e.g., knowing how to solve multi-digit subtraction problems) and specialized content knowledge (e.g., knowing how to connect representations to underlying mathematical ideas). The current purview of university-based mathematics content courses for PTs is to develop PTs' knowledge of mathematics in ways needed for teaching, including the development of both common and specialized content knowledge. In fact, in the U.S., recent reports released from the Conference Board of the Mathematical Sciences (2012) and the Council for the Accreditation of Educator Preparation (2013) recommend that mathematics teacher preparation coursework include a focus on the development of MKT as defined by Ball and her colleagues (2008). Yet, despite such recommendations, research on mathematics courses for PTs in the U.S. illustrates that such courses often focus solely on the learning of common mathematical content with limited attention given to the development of specialized content knowledge (Hart & Swars, 2009; Hart, Oesterle, & Swars, 2013).

Furthermore, researchers have shown that it is often challenging for PTs to develop their knowledge of mathematics in ways needed for teaching. For example, researchers have found that while PTs have shown proficiency in performing mathematical procedures, they often lack deeper conceptual understanding (e.g., Thanheiser et al., 2014). Given such challenges, developing knowledge of mathematics in ways needed for teaching requires PTs to *relearn* mathematics (Zazkis, 2011). Zazkis (2011) proposes the idea of *relearning* to describe how PTs learn mathematics by reassessing familiar concepts, reconsidering previously held ideas, and ultimately reconstructing their knowledge. For PTs, now as adult learners, relearning mathematics (in ways needed for teaching) involves revisiting procedurally understood content and revising their understanding in more conceptually-oriented ways. Expanding Zazkis' (2011) notion of relearning, we argue that relearning for PTs also involves learning new content for the first time and enhancing their understanding of previously learned content in ways needed exclusively for teaching. Thus, the work for MTEs in teaching mathematics content courses for PTs is supporting PTs' relearning of mathematics in ways needed for effectively teaching mathematics. Yet, little is known about the nature of the knowledge that MTEs need or use in their work supporting PTs' relearning.

Conceptualizing Mathematical Knowledge for Teaching Teachers

There is general consensus within the teacher education community that MKTT is the knowledge MTEs use in their work with PTs and that the domains of MKTT include not only knowledge that teachers must have, but also elaborations of that knowledge that extend beyond that which teachers must know. However, the precise nature of the different domains of MKTT has been discussed by researchers with varying degrees of specificity (e.g., Goodwin & Kosnik, 2013; Zaslavsky & Leikin, 2004). Much of the existing research conceptualizes domains of MTE knowledge as connected to and extending from domains of teacher knowledge, such as knowledge of how to support PTs in developing specialized content knowledge. Chauvot (2009), for example, conceptualized MTE knowledge as consisting of a number of different knowledge domains based on a self-study of her work as an MTE. Specifically, she constructed a knowledge map for MTEs consisting of three main categories—subject matter content knowledge, pedagogical content knowledge, and curricular knowledge (extending the categories put forth by Shulman's (1987) work on teacher knowledge). She also extends several of Ball and colleagues' (2008) domains of MKT to make them more specific to MTEs, including knowledge of how to develop PTs' specialized content knowledge and pedagogical content knowledge and knowledge of how to engage PTs with content in ways that are connected to teaching students. Notably, Chauvot's knowledge map (2009) also includes considerations of the context (Grossman, 1990) in which MTEs work, including school standards for mathematics and curriculum programs, which has implications for what MTEs have to know.

Perks and Prestage (2008) conceptualize domains of MTE knowledge as including ways of supporting PTs' content learning. They describe the domains of MTE knowledge as the connections among learner knowledge (i.e., knowledge developed from being teachers), practical wisdom (i.e., activities used during instruction), and professional traditions (i.e., ways of working with PTs, research on teaching and learning). Most notably, in their conceptualization, Perks and Prestage (2008) particularly highlight interactions among ways of working with PTs and activities for developing knowledge needed for teaching as part of the knowledge required by MTEs. Zaslavsky and Leikin (2004) similarly conceptualize MKTT as including ways of supporting PTs' content learning. Specifically, they include knowledge of ways of managing PTs' learning, as well as knowledge of the content that PTs must learn. Zaslavsky and Leikin go further and argue that MKTT also includes knowledge of how to be sensitive to PTs and how to assess PTs' mathematical understanding.

Building on the aforementioned studies, we conceptualize MKTT as connected to and extending from domains of mathematical knowledge for teaching (Ball et al., 2008). Specifically, we conceptualize MKTT as comprised of the following domains: (a) MTE subject matter knowledge (MTE-SMK), and (b) MTE pedagogical content knowledge (MTE-PCK) as displayed in Figure 1. Following Ball et al. (2008), we assume that MTEs need to be knowledgeable of mathematical content (MTE-SMK) as well as ways of facilitating PTs' learning and relearning of content, which we conceptualize as MTE's pedagogical content knowledge (MTE-PCK). We include MKT (for classroom teachers) as a subdomain of MTE-SMK, which we refer to as MTE-Knowledge of MKT. We elaborate on these domains, and their related sub-domains, in the paragraphs that follow.

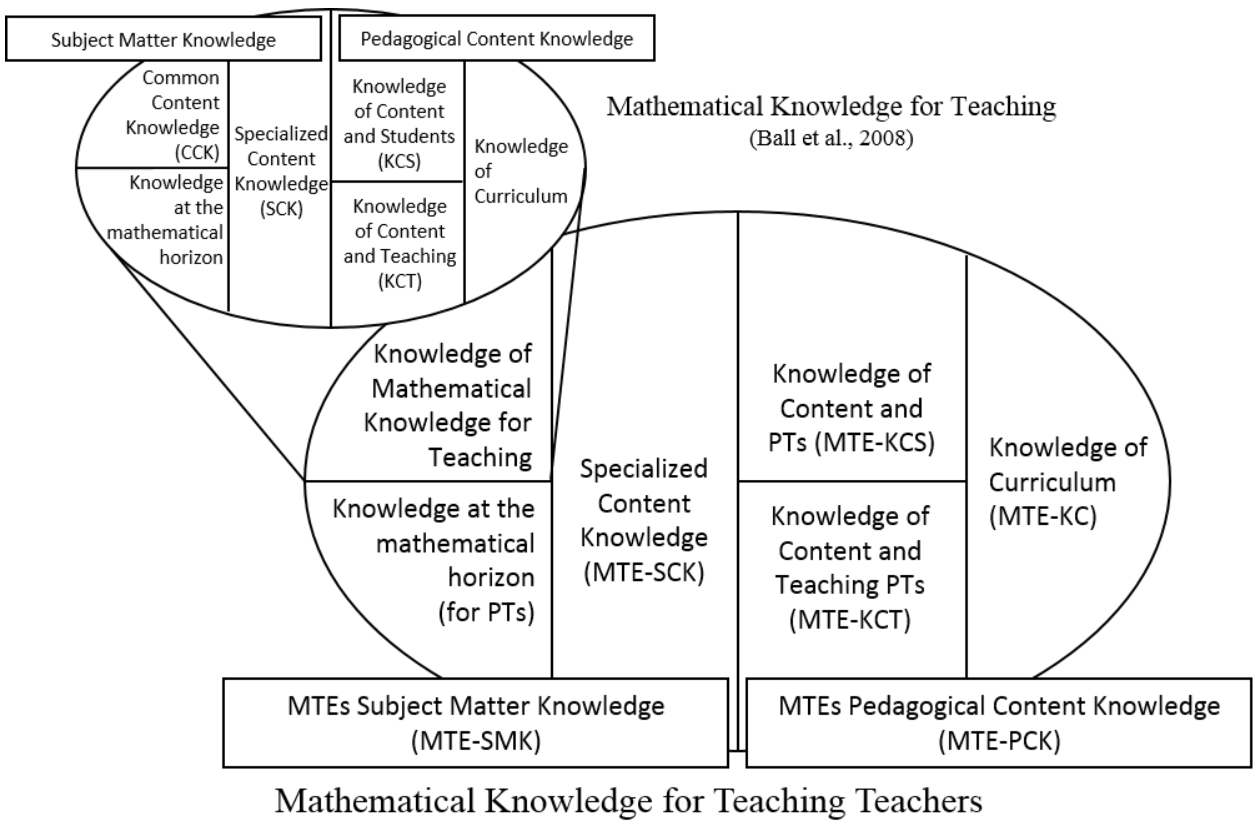


Figure 1. Potential conceptualization of mathematical knowledge for teaching teachers.

We conceptualize MTE-subject matter knowledge (MTE-SMK) as comprised of three sub-domains: MTE knowledge of MKT, MTE specialized content knowledge (mathematical content knowledge that is specific to teaching MKT to PTs), and knowledge at the mathematical horizon for PTs. Following researchers who have outlined new ways of understanding knowledge needed for teaching (e.g., Ball, et al., 2008; Shulman, 1987), we assert that PTs need to learn content in ways specific to teaching (i.e., mathematical knowledge for teaching as defined by Ball et al., 2008). Thus, MTEs need to be knowledgeable of the content PTs need to know (MKT), which we refer to as MTE-Knowledge of MKT. For example, PTs are required to understand multiple models of multiplication and division, including scaling, equal groups, and areas/arrays, as well as the representations, algorithms, and pedagogical considerations involved in teaching these models. Since MTEs are tasked with helping PTs develop this knowledge, MTEs must also possess knowledge of multiplication and division in such ways.

Furthermore, we assert that MTEs have specialized content knowledge that is unique to teaching MKT to PTs and includes understanding the mathematical purposes underlying the specialized content that PTs learn. This is what we are referring to as MTE specialized content knowledge (MTE-SCK). For instance, MTEs use this knowledge to modify tasks for PTs in ways that uncover PTs' procedural knowledge or misconceptions in order to facilitate relearning. MTEs also understand the mathematical ideas underlying different multiplication algorithms, for example, and use their MTE-SCK to articulate to PTs the mathematical purposes and consequences of using these algorithms with students. Finally, we assert that MTEs have knowledge at the mathematical horizon for PTs, which includes knowledge of the content PTs will encounter in their subsequent coursework as part of a teacher education program. Thus, MTE-SMK is comprised of MTE content knowledge (i.e., the entirety of MKT as defined by Ball et al., 2008), as well as other sub-domains of content knowledge that are unique to teaching mathematics content to PTs, including knowledge at the mathematical horizon for PTs and MTE-SCK that is unique to teaching MKT to PTs.

In addition to subject matter knowledge (MTE-SMK), we assert that MTEs also need pedagogical knowledge of how to facilitate PTs' learning and relearning of mathematics, which we conceptualize as MTE-pedagogical content knowledge (MTE-PCK). It is comprised of three sub-domains: knowledge of content and PTs (MTE-KCS), knowledge of content and teaching PTs (MTE-KCT), and knowledge of curriculum (MTE-KC). We hypothesize that MTE-PCK includes knowledge of content and students, where the students are now PTs (and thus, adult learners) and the content to be learned is now MKT, and knowledge of curriculum where curricular materials are those designed for PTs. It also involves knowledge of content and teaching, which translates to how MTEs facilitate PTs' learning of MKT and relate it to teaching mathematics to students (Zopf, 2010). This means that MTEs not only need to understand the specialized content knowledge needed for teaching students (SCK), they must also understand the intricacies of how such knowledge develops in adults (MTE-KCT). And, MTEs must understand how to use or develop curricular materials (MTE-KC) to create learning environments that provide opportunities for PTs to unpack and relearn mathematical content.

For example, MTEs discussing quadrilateral hierarchy with PTs must engage deeply held preconceptions of the relevant definitions, such as "a square is a quadrilateral with four congruent sides." PTs may not only struggle with (re)conceptualizing a square as simultaneously a special type of rectangle and a special type of rhombus, but also with why this is true and the mathematical consequences of not considering a square as a rectangle. Thus, MTEs must engage in the work of uncovering and addressing such preconceptions, which means they must also be aware of the likely preconceptions PTs have. Thus, MTE-PCK not only includes knowledge of how to represent and explain mathematical concepts to PTs, but also knowledge of the conceptions and preconceptions that PTs bring to content courses (Shulman, 1987), and knowledge of how to leverage those conceptions and preconceptions in ways that will promote PTs' relearning of mathematics in ways needed for teaching.

Taken together, we conceptualize MKTT as comprised of knowledge of not only the content that PTs need to know, but also specialized knowledge of content that is unique to teaching PTs, and knowledge of how to facilitate PTs' relearning. Moreover, like Beswick and Chapman (2013), we consider MKTT to be not only an elaborated extension of teacher knowledge, but also to include domains of knowledge that are characteristically different from teacher knowledge. We argue that the ways in which MKTT is different from teacher knowledge stems from MTEs' work in supporting PTs' relearning of mathematics.

Mathematics Teacher Educators' Examples for Supporting Prospective Teachers' Relearning

Drawing from our own work as MTEs, in the following sections we illustrate ways in which we use various aspects of MKTT in our practice as instructors of content courses for elementary PTs. Specifically, we illustrate our work in supporting PTs' relearning of mathematics through the lens of three different tasks of teaching content to PTs, data from which come from multiple research projects which included one or more of the authors. The first example involves a task for PTs that models asking questions to probe students' thinking when it is unclear and provides examples of students' thinking that do not align with a traditional subtraction algorithm; such tasks support PTs' relearning of content by encouraging them to revise procedural understandings as they reflect on the conceptual underpinnings of an unconventional algorithm. The second example describes designing and facilitating tasks in such a way as to encourage PTs to move beyond their procedural understandings of fractions and develop conceptually based fraction comparison strategies, which also reflects Zazkis' (2011) notion of relearning. The third example describes revising tasks for PTs so that they are able to develop not only common content knowledge but also specialized content knowledge; although PTs may be learning new content, such tasks help MTEs ensure that PTs' understanding of that content is being enhanced in ways needed exclusively for teaching. Our aim is to contribute to the practices of MTEs who teach content to elementary PTs by sharing examples from our work in which we supported PTs' relearning and then reflecting on the domains of MKTT that we drew from in doing so.

Example I: Using MTE-SCK and MTE-PCK to Support Prospective Teachers in Relearning Subtraction

As noted above, MTE-PCK includes knowledge of the conceptions and preconceptions that PTs bring with them to content courses (MTE-KCS) and knowledge of how to leverage these conceptions and preconceptions in ways that will promote PTs' relearning of mathematics (MTE-KCT). Since PTs often feel confident in their understanding and ability to teach lower-level mathematical content (Thanheiser, 2018), an important aspect of the work of MTEs becomes creating opportunities for PTs to question the basis of their current knowledge and see an authentic need to restructure their understanding of seemingly "simple" content. The following example will highlight the roles of MTE-SMK and MTE-PCK in MTEs' development and implementation of tasks that have the potential to help PTs unpack their knowledge and facilitate the relearning of content at a deeper, more conceptual level.

In this example, PTs complete a task in which they review a written artifact of a young student's unconventional, yet natural thinking, analyze his understanding of the content based on this artifact, and then listen (and respond) to an audio recording of an interview conducted by one author with the student (see Welder, 2012) for an excerpt from the interview and reflection on the student's strategy). At the beginning of the task, PTs are shown a small piece of the student's written work, recreated for clarity in Figure 2. The PTs are informed that the work came from a first-grader, Dylan, who had not yet been exposed to any strategies for subtracting multi-digit numbers that would require the traditional regrouping algorithm. At the time of the interview, Dylan was not quite 7-years-old and his class was working on subtracting single-digit numbers from double-digit numbers less than 20 (e.g., 13 – 6) using counting back strategies (starting at 13 and counting backwards 6 numbers on their fingers) (Welder, 2012).

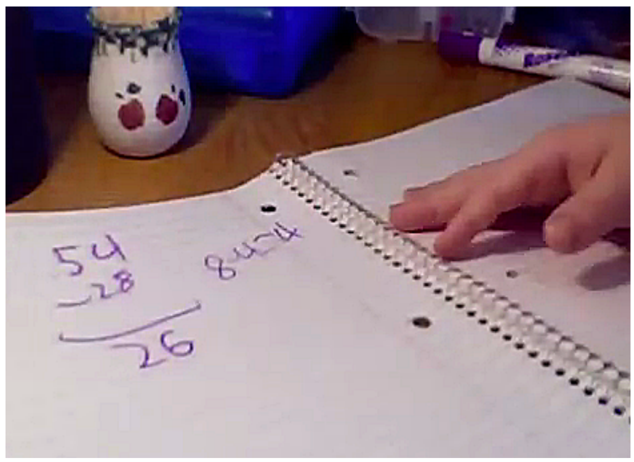


Figure 2. Written artifact of Dylan's work (showing 54 – 28 = 26; 8 – 4 = 4).

Since there is no clear evidence of Dylan applying any particular algorithm, most PTs find themselves unable to interpret his thinking and initially conclude that he had either (a) found the answer through the use of a calculator (or some other external means), or (b) erroneously subtracted the smallest digit from the largest in each place value column, regardless of their position in the problem (as suggested by the writing of "8 – 4 = 4", see Ashlock's (2010) Error Pattern S-W-1). Many PTs are able to recognize that applying such a procedure should have led Dylan to the incorrect answer of 34; but, unable to consolidate their understanding with his provided solution, they hypothesize that his arrival at the correct answer must have been independent of this calculation. “The only explanation that came to mind was that subtracting 4 from 8 (8 – 4) would be his initial thought to solving, but then corrected himself,” suggested one PT. Another speculated, “he must have figured out that his computations would make the answer incorrect, but maybe he forgot to scribble out his work (scratch work--not supposed to mean anything to the teacher). I think that he reached the answer by counting backward on his fingers from 54 down to 28.”

After analyzing Dylan’s written work, PTs are then prompted to listen to an audio recording of the author asking him a series of probing questions about his thinking and prompting him to record calculations he was performing mentally. Figure 3 shows how Dylan was thinking about the problem 62 – 49. By decomposing the ones digit of the subtrahend into two parts, using the ones digit of the minuend as one part (e.g., 9 can be decomposed into 2 and 7), he could first subtract the known part from the minuend (shown as 62 – 2 = 60) and then the remaining part (7) from this new total (shown as 60 – 7 = 53).

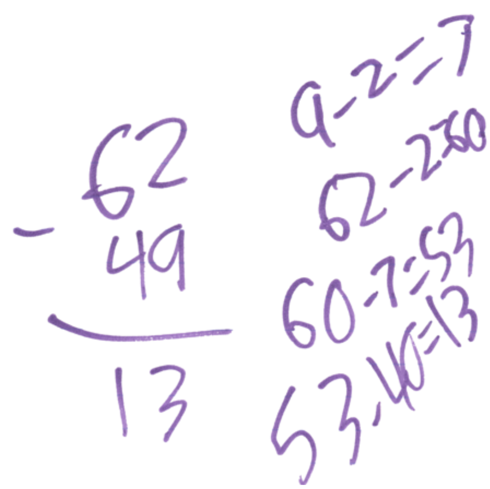


Figure 3. A written record of the mental computations Dylan performed in solving the problem 62 – 49 (showing 9 – 2 = 7; 62 – 2 = 60; 60 – 7 = 53; 53 – 40 = 13).

The interview is diagnostic in nature and models the type of investigative approach MTEs would want PTs to use in similar situations with their future students. By listening to the student explain his reasoning and correctly apply his unique strategy to additional problems, the majority of PTs are able to eventually realize that through strong conceptual understanding of numbers, place value, and operations, Dylan is able to reason through complex computations with surprising skill for such a young student. However, some PTs finish the interview without being able to make sense of Dylan's strategy or provide a conclusion about its validity and are thus left unsure about his understanding of the content.

PT responses suggested that this task was successful in helping them recognize the need, as future teachers, to unpack, reorganize, and at times relearn mathematics content to be better prepared to make sense of young students' thinking (part of their development of SCK). The task encouraged many PTs to reflect on their own learning of the content and consider the limitations of their current understanding: "I found this activity was a great challenge to my own thinking about the process of subtraction and especially challenging to my ideas about teaching it to young children." Another added, "I have only learned one way to subtract... I have never perceived math problems with the possibility of having patterns and multiple ways of solving. I am feeling challenged to expose myself more to new ways of doing math, so that I can best support my future students as they explore and try new problem-solving skills." Others specifically alluded to a need to *change* the way they think about this content or *acquire* additional knowledge to best meet the needs of their students: "I know that in a classroom environment I might need to bring in other resources to understand a child's methods of problem solving…" Another noted gaining insight into, "how each student won't learn the way I did and that I might have to change my problem solving to help them."

In this example, multiple domains of MKTT were utilized to design and implement a task to help PTs not only see an authentic need to relearn content, but to also help PTs connect that learning to teaching students. Recall that MKTT includes an understanding of the common dissonance between the depth of conceptual knowledge teachers need for teaching mathematics (MTE knowledge of MKT) and the typical procedural-based understanding that PTs bring with them to content courses (MTE-KCS). Specifically, MTEs assume that, as high-school graduates, incoming PTs have successfully mastered whole-number operations. Based on research with PTs, we also know that their knowledge will be primarily, if not exclusively, procedural (Thanheiser et al., 2014). Furthermore, MTEs' knowledge of how students learn mathematics helps us recognize the limitations of PTs' knowledge, as it pertains to helping students develop conceptual understanding. Thus, the motivation for developing a task around such a seemingly "simple" problem was based on an understanding of the need for MTEs to create learning opportunities that can support PTs in developing their SCK (MTE-KCT). Furthermore, knowledge of PTs, as adult learners, guides our understanding of the need for PTs to unpack their incoming common content knowledge, as a first step towards relearning content and developing SCK (MTE-KCS).

However, the specific way in which we approached the development of the task in this example was based on more than our knowledge of subject matter (MTE-SMK) – it evolved from our knowledge of PTs' preconceptions about their content knowledge (MTE-KCS). Most PTs find comfort and confidence in the familiarity of procedures they have learned to master and do not see a need to reconsider a problem that they can successfully solve (such as 54 – 28). PTs' preconceptions about what it means to *know* and *do* mathematics are initially tied to their abilities to perform algorithmic procedures, and most enter content courses believing that they will be successful teachers by showing students how to perform the same calculations (Thanheiser, 2018). Therefore, in addition to knowing that PTs with solely procedural knowledge would struggle to make sense of a student conceptualizing subtraction in an unfamiliar way (MTE-KCS), we also knew that PTs would need a genuine experience to convince them of the need to think differently (MTE-KCT). Listening to and trying to make sense of a young student as he explains his thinking provides an authentic opportunity for PTs to move beyond how they would *solve* the problem or even *teach* a student to solve the problem. We argue these data suggest that a critical shift in PTs' mindsets can take place when they are put in the role of the teacher and are able to see firsthand the limitations of their knowledge and the need to *relearn* content to better support the learning of their future students.

Lastly, this example also utilized the specialized content knowledge unique to helping PTs develop MKT (MTE-SCK). We see the application of this specialized mathematical knowledge in the number choices made in the various problems we asked Dylan to solve. For example, since the initial problem (54 – 28) involved an 8 in the ones column of the subtrahend, being broken into two equal parts of 4 and 4 (because of the 4 in the ones column of the minuend), we predicted that this would leave some ambiguity for PTs about the significance of the order in which these two parts were subtracted. To clarify the fact that 8 was not just broken in half but instead decomposed into two parts based on one portion being the size of the ones digit of the minuend (4), the next problem that was posed to Dylan was 62 – 49. Here the PTs could listen to Dylan discuss how the 9 was decomposed into 2 and 7 (unequal parts) and how he thought about first subtracting the portion of size 2 (62 – 2 = 60) to make a friendlier minuend, from which he subtracted the remaining portion of size 7 (60 – 7 = 53).

Furthermore, to test the strength of Dylan's conceptual understanding, the next problem we posed was the similar, but significantly different problem of 69 – 42. We wanted PTs to be able to see how Dylan would apply his thinking to an example that did not require "regrouping."

We knew that PTs would likely assume that he would view this in the more standard way of subtracting each column individually (9 – 2 = 7 and 6 – 4 = 2) to find the correct solution of 72. However, as we suspected, Dylan began this problem using the same thinking he had applied earlier by first stating that 2 – 2 = 0 and using this to rewrite the problem as 67 – 40. It was our mathematical knowledge that is specialized to helping PTs develop MKT (MTE-SCK) that drove the decision to pose this problem to Dylan instead of moving onto a new example that would require regrouping. We knew that seeing how Dylan applied his thinking to this situation and having to make sense of his first calculation (2 – 2 = 0) would provide an additional challenge for the PTs. It also provided a great opportunity for the PTs to discuss what problem they would have posed following the 62 – 49 problem and reflect on the purpose of our posing the problem we did (helping them to develop MKT).

Example II: Using MTE-SMK and MTE-PCK to Support Prospective Teachers in Reconsidering Procedure-Based Knowledge

MTE knowledge becomes important in both designing and implementing high-level mathematical tasks that provide opportunities for relearning. This second example also challenges PTs' previously held procedural conceptions to provide opportunities for them to consider the conceptual foundations of the procedures that they use. In this example, we developed a task with the goal of helping PTs extend their knowledge of fractions as numbers, specifically that $\frac{a}{b}$ can be conceptualized as a measurement of a pieces of size $\frac{1}{b}$, and use this measurement definition to reconsider strategies for comparing and ordering fractions. Based on prior research (e.g., Olanoff, Lo, & Tobias, 2014), we knew that PTs often hold a procedural view of fractions, and are generally only able to compare fractions using a "common denominator approach." However, PTs often perform this procedure with little understanding of why what they are doing produces the correct answer, nor do they consider whether or not there may be a more appropriate strategy for comparing a particular set of fractions (Bartell, Webel, Bowen, & Dyson, 2013; Livy, 2011; Whitacre & Nickerson, 2011; Yang, Reys, & Reys, 2009). Our goal in this task was thus to encourage PTs to use their understanding of fractions as measurements to develop a variety of sense-making strategies for comparing fractions and to be able to explain why those strategies work.

In designing this task, we drew on our knowledge of fractions as measurements (as a certain number of pieces of a certain size) and how this understanding could support the development and justification of sense-making comparison strategies (MTE-SCK). Using our knowledge of the multiple strategies students are asked to use (according to the *Common Core State Standards of Mathematics* (National Governors Association Center for Best Practices and the Council of Chief State School Officer, 2010) and U.S. elementary curricula) (MTE knowledge of MKT), we identified five fraction comparison strategies that we believed should be part of PTs' MKT:

1. compare fractions that have the same size pieces (the familiar common denominator strategy) by looking at which has more of those pieces;
2. compare fractions that have the same number of pieces (common numerator strategy) by looking at which has larger pieces;
3. compare fractions that have both more and larger pieces with fractions that have fewer and smaller pieces (we called this the "greater number of larger pieces" strategy);
4. compare fractions to a benchmark value that is greater than one of the fractions and less than the other fraction; and
5. compare fractions that are both greater than or less than a benchmark fraction by comparing the distance that each fraction is from the benchmark.

For a full discussion of the development and implementation of this task, see: Feldman, Thanheiser, Welder, Tobias, Hillen, and Olanoff, (2016); Tobias, Olanoff, Hillen, Welder, Feldman, and Thanheiser, (2014); and Thanheiser, Olanoff, Hillen, Feldman, Tobias, and Welder (2016).

Once we had generated a list of comparison strategies, and a definition of fractions that we wanted PTs to explore and relearn, we drew upon our pedagogical content knowledge to design a task in such a way as to facilitate this relearning (MTE-PCK). Specifically, our knowledge of how PTs typically interact with procedurally based tasks (MTE-KCS) suggested that if we gave PTs a task that merely asked them to compare fractions, they would likely draw upon their prior knowledge and use the common denominator procedure without necessarily considering each fraction as a quantity. Thus, we needed to draw upon our knowledge of content and teaching to find ways to encourage PTs to develop additional ways of thinking about the relative magnitude of fractions and to use that understanding to construct alternative comparison strategies (MTE-KCT). See Figure 4 for the task instructions as they were presented to the PTs and Figure 5 for the fifteen comparison problems.

For each set of fractions below, circle the fraction that is greater (or if the fractions are equivalent, write "=" in between them), and provide a "sense-making" explanation for how you know. You may use pictures if that is helpful to you, but your explanation cannot rely solely on a picture.

- *Calculators may not be used.*
- *Feel free to work on these problems in any order that makes sense to you. If you find yourself struggling with any of the problems, skip them and revisit them later.*

Figure 4. Instructions for the fraction comparison task (Taskmasters, 2014).

One way we worked towards our goal of encouraging PTs to develop conceptually-based fraction comparison strategies was by not allowing the use of calculators, as we wanted to prevent PTs from converting fractions to decimals or percents and relying on their ability to compare the quantities in these alternative forms. Additionally, we deliberately designed problems in ways we believed would encourage the use of specific strategies besides common denominators.

Fractions to Compare											
#1	1/2	versus	17/31	#6	13/15	versus	17/19	#11	2/7	versus	3/8
#2	2/17	versus	2/19	#7	5/6	versus	6/5	#12	25/12	versus	31/15
#3	4/7	versus	9/14	#8	7/10	versus	8/9	#13	11/20	versus	19/36
#4	3/7	versus	6/11	#9	1/4	versus	25/99	#14	2/9	versus	3/8
#5	8/9	versus	12/13	#10	24/7	versus	34/15	#15	18/25	versus	16/27

Figure 5. The 15 sets of fractions compared in the fraction comparison task (Taskmasters, 2014).

For example, we asked PTs to compare 2/17 and 2/19. This problem could be efficiently solved by thinking about the size of the pieces in each fraction but requires more work to rewrite the fractions with common denominators. (Note that it is important to emphasize with PTs that comparing fractions requires that each fraction be from the same whole. A 1/17 piece is larger than a 1/19 piece, as long as they are referring to the same whole.) Another example is the problem comparing 15/17 and 19/18. By recognizing that one fraction is less than one and the other is greater than one, it is easy to use a 'benchmark between' strategy to determine that the second fraction is larger.

In facilitating the task, we drew upon our MTE-PCK to help PTs interact with the task. Using our MTE-KCS of PTs' prior knowledge of fractions (generally part-whole based) and fraction comparison (generally focused on finding common denominators), we began the task with a launch where we asked PTs to brainstorm everything that they knew about the fraction 7/8. We then asked them to find three fractions that were greater than and three fractions that were less than 7/8 that first had the same denominator and then the same numerator. The purpose of this launch was to help PTs think about both the number of pieces (the numerators) and the size of the pieces (the denominators.) This would help them in developing a better understanding of the measurement definition of fractions, which was one of the goals of the task. We gave the PTs all fifteen comparison problems and encouraged them to engage with the problems in any order, in order to provide multiple entry points (some of the later problems were potentially easier, so PTs who struggled with some of the earlier problems could skip them until later). As the PTs worked, the instructors walked around and encouraged PTs to think about what they knew about the fractions to be compared before attempting to use a procedure. We used our MTE-KCT to ask questions regarding the number and size of pieces, and we were able to get some PTs to naturally consider the number of pieces and size of pieces simultaneously in comparing 18/25 and 16/27. This problem would require extensive multiplicative computations without the use of a calculator, but drawing on the 'greater number of larger pieces' strategy (18/25 has more pieces of a larger size than 16/27) requires no such calculations.

Following PTs' work on the task, we used our MTE-KCT to facilitate a PT-led discussion of the different strategies they developed by asking PTs to write solutions to specific problems to present to the class. As a class, we named each strategy and discussed the problems on which they could be used, and those on which it made sense to use them. For example, while common numerators or denominators could be used to compare 18/25 and 16/27, this problem better lends itself to the greater number of larger pieces strategy, which requires much less work. The full task, in a format ready to use with PTs, is available online at https://mathtaskmasters.com. For a detailed guide for facilitating this task, the reader is encouraged to see the online Facilitation Guide (Taskmasters, 2014).

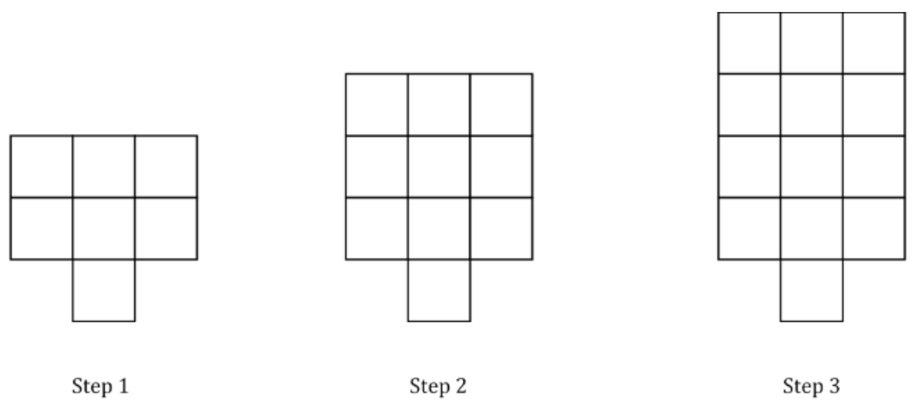
In studying PTs' work on the fraction comparison task, we determined that the task was helpful in eliciting many of the comparison strategies that we outlined above (see Thanheiser et al., 2016 for more information on the results of the task). We drew upon MTE-SMK in designing the task: we used our knowledge of the MKT we hoped PTs would develop through the task and our MTE-SCK in choosing and facilitating the development of the measurement definition of fractions. We also drew upon MTE-PCK in designing and implementing the task, as we considered the procedural understandings that PTs often utilize in comparing fractions (MTE-KCS) and attempted to challenge these understandings (MTE-KCT). Overall, data collected confirmed that we were able to design and implement a task that was beneficial in helping PTs relearn mathematical content related to fractions.

Example III: Using MTE-PCK to Revise a Task to Deepen Prospective Teachers' Exploration of Potentially New Mathematical Knowledge

In our work as MTEs, we design and redesign tasks in order to uncover and address PTs' preconceptions. This work is not only integral to facilitating PTs' relearning (or, indeed, new learning) of mathematical concepts, but also to our own ongoing development of MKTT. Specifically, this third example discusses how the process of revising a task helped us develop our own knowledge of content and PTs (MTE-KCS) and knowledge of content and teaching PTs (MTE-KCT). We chose to implement the Continuous Improvement framework for lesson revision (Berk & Hiebert, 2009) as a way of investigating PTs' knowledge about elementary mathematics concepts and developing and improving lessons that address their common content knowledge and SCK.

One task focused on growing visual patterns. This topic became an interest to us because we noticed that PTs have difficulty using a variable to represent the index when constructing an algebraic expression that describes a growing visual pattern, which reflects the findings of

Warren and Cooper (2008) in regards to students' thinking about visual patterns. Our learning goal for developing this task was that PTs would be able to use a variable to represent an index in a sequence of visual patterns and build an expression in terms of the variable to represent an arbitrary step in the sequence. After selecting a task from Boaler (2015) (see Figure 6) involving a linearly growing visual pattern and pre-assessing PTs' knowledge, we implemented this task in two sections of a content course.



In groups:

a. Draw the next three steps of the pattern.
b. Draw the 10th step of the pattern.
c. Express in words a general rule describing how to create any step in the pattern, given the step number.
d. Using a variable, write a general expression that tells how many tiles are in a step of the pattern, given the step number.

Figure 6. Original growing visual patterns task adapted from Boaler (2015).

During the discussions motivated by this task, we noticed that the first iteration of the task uncovered, but did not explicitly address, two different procedural understandings. First, PTs used their previous knowledge of the formula for an arithmetic sequence $(a_n + (n - 1) \times d)$ and fit the numbers derived from the growing pattern into that formula (see Figure 7). It should be noted that although these PTs recognized that they could describe the numerical pattern as an arithmetic sequence, most of them did not remember the formula but looked it up online before proceeding through the rest of the task.

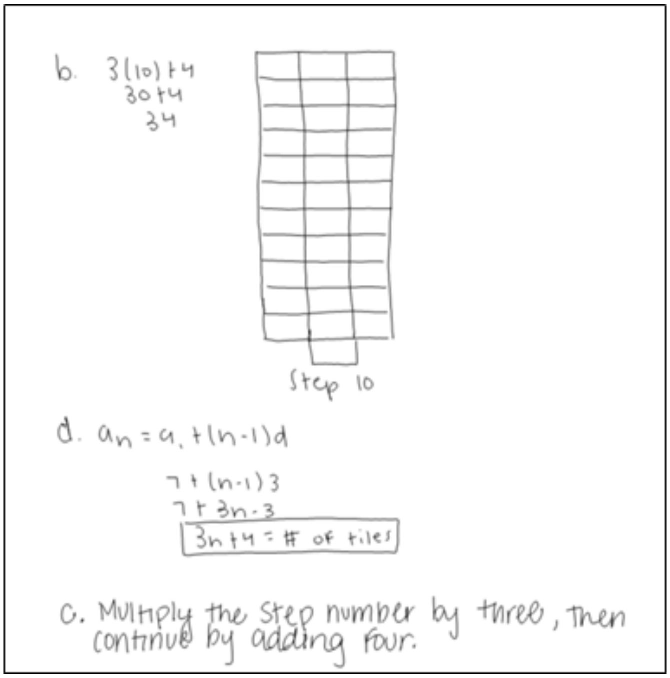


Figure 7. PTs fitting the numbers derived from the growing pattern into the formula for an arithmetic sequence.

Second, PTs approached the task by numerically building a linear expression from a table, disregarding the growing visual pattern once the numerical table was built (see Figure 8). Once PTs had remembered a procedure that would produce an answer, they immediately applied it to all of the other given visual patterns. Since all of the given patterns grew linearly, the original task only reinforced PTs' use of their procedural approaches (i.e., applying the formula for an arithmetic sequence or extracting the information about the pattern's growth into tabular form and finding a way to relate the numerical quantities).

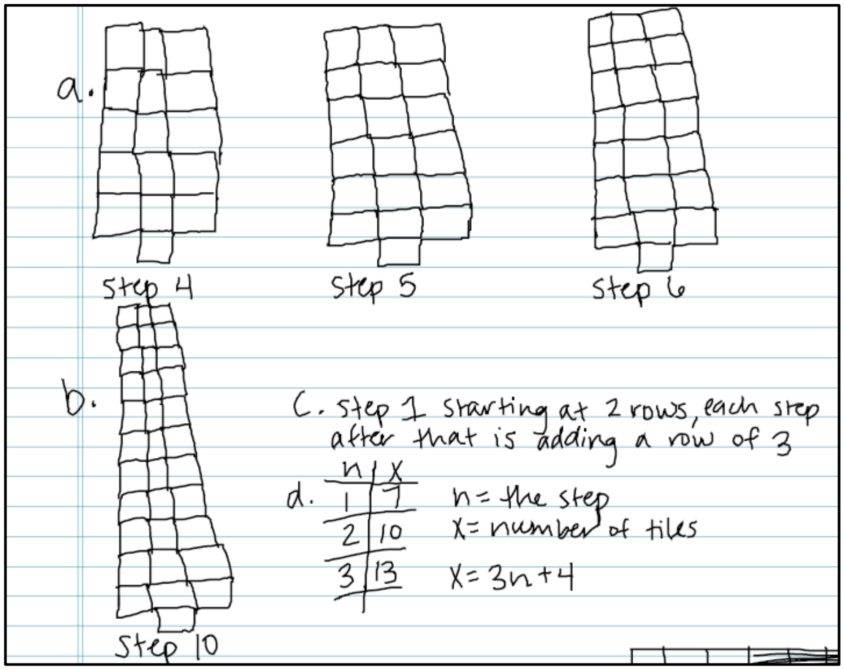


Figure 8. PTs in this group did not connect the development of their expression in part (d) with the visual attributes of the pattern expressed in part (c).

Based on our classroom observations and the post-assessment responses, we concluded that the design of the task successfully guided most PTs toward developing an algebraic expression for a given linearly growing visual pattern. However, many PTs found ways to proceduralize this task in ways we did not expect, based on their previous knowledge of linear relationships and arithmetic sequences, resulting in knowledge that reflected, at its deepest, the common content knowledge that Warren and Cooper (2008) documented with third grade students.

PTs' tendency towards procedure prevented them from developing SCK related to variables that we aim to develop in the course. As written, the task provides opportunities for PTs to fully exercise their mathematical authority by explaining their reasoning, assessing their expressions, and justifying their conjectures (Prasad & Barron, In-Press). However, during our class observations, we noticed that the first iteration of the task did not encourage PTs to justify their expressions by connecting the visual representation of the pattern to each term of their expression. We determined that this understanding was an integral part of their MKT for this concept for a variety of reasons: (1) to create or assess tasks about visual patterns for students, teachers need to understand the visual growth of patterns, (2) teachers need opportunities to create, assess, and justify their own algebraic conjectures to support their students in doing the same, and (3) to support students' learning of algebraic reasoning, teachers must be able to attach meaning to variables and terms in expressions arising from contextual problems.

This directly affected the *development* of our own MTE-PCK; we now better understood the nature of the prior knowledge and preconceptions that PTs brought into this course (MTE-KCS), as well as the task's ability to encourage the development of PTs' MKT (MTE-KCT). In order to deepen PTs' knowledge beyond common content knowledge and into SCK, we concluded that the word "represent" in the learning goal meant not only that PTs should understand that the variable represented the index of any particular term of the pattern, but also that the index is represented visually in the pattern; and, an algebraic expression can be built by identifying the parts of the pattern that grow with the index and the parts that remain constant. Therefore, we revised the task to encourage PTs to reflect on the relationship between the algebraic expression and the visual representation of the pattern. Being able to extend and (eventually) create a growing visual pattern may not be a necessary skill for young students, but it is for their future teachers.

We used the data we collected along with our observations in the classroom to redesign the task to address both the newly uncovered preconceptions and our more complete understanding of the learning goal. To help PTs establish the connection between the algebraic expression that describes the pattern and its visual representation, we redesigned the task to create four subtasks (borrowing some components from Beckmann, 2013). Part I asks PTs to attend to the visual attributes of the terms in a pattern by asking them to identify which of four given visual patterns (that follow the same linear relationship, $3n + 5$) fit a given verbal description of a pattern's growth. Part II presents a linearly growing pattern, which is visually growing with respect to the index, and asks PTs to extend the pattern with drawings, describe the visual aspects of the growth in general, and write a general expression for the growth algebraically using variables. Part III repeats Part II, but PTs are now asked to investigate a pattern that grows quadratically to challenge them to move beyond their reliance on previous conceptions of linear relationships. Lastly, Part IV asks PTs to draw a visual pattern following a given linear expression ($2n + 3$). The fully revised task can be found in the Appendix.

The analysis of PTs' work and subsequent redesign of the task is not only integral to the development of the curriculum for these courses, but also as a window into the MKTT used by the MTEs involved in this project. Tasks aimed at teaching mathematics to students can focus on developing their mathematical knowledge; tasks for PTs must not only develop their mathematical knowledge, but also uncover and address their previous conceptions. In order to develop PTs' MKT, tasks for PTs, even when based on tasks for students, must facilitate *relearning*, instead of just learning. Thus, the purpose of designing tasks for PTs differs fundamentally from the purpose of designing tasks for students. As these processes have different purposes, it stands to reason that they require different knowledge bases. Moreover, the shift towards closer analysis of the visual attributes of the pattern and problem-posing in the revised task points to an awareness on the part of the MTEs of the ways in which PTs will need to operationalize this knowledge in their teaching careers.

Implications for Our Work as Mathematics Teacher Educators

Our aim in this paper is to contribute to the growing research base on MTEs' knowledge, by elaborating on domains of MKTT through the lens of our practice as MTEs, and to support MTEs who teach (or are preparing to teach) content courses for K-8 PTs. We are particularly concerned with the knowledge MTEs use to support PTs' *relearning*. Relearning not only involves learning new content in ways needed for teaching, but also revisiting and revising procedurally understood content in more conceptually-oriented ways.

As illustrated through our examples, the purposes of designing tasks for PTs differ from the purposes of designing tasks for students, thus illustrating differences between the knowledge needed to teach mathematics to students (MKT) and the knowledge needed to teach MKT to PTs (MKTT). Specifically, the first example highlights a task used to support PTs' relearning of basic mathematical content by having them engage in a student's unconventional thinking. The second example describes how MTEs designed and implemented a task that facilitated PTs' relearning of fractions by encouraging them to develop sense-making strategies to use in place of familiar procedural strategies. And lastly, the third example focuses on revising tasks for PTs in ways that will help them develop SCK, which unexpectedly helped MTEs develop their own knowledge of content and PTs (MTE-KCS). Although these examples are not particularly novel, together they illustrate that the work of MTEs involves carefully considering mathematical tasks and pedagogical practices that support PTs' learning and relearning. Therefore, MTEs' work requires specialized knowledge that takes into account reshaping the knowledge of adult learners, which includes, but is fundamentally different from, the knowledge required to teach students.

While these examples contribute to the small, yet growing research base on MTE knowledge, there is a need for more research that pays particular attention to ways in which MKTT is different from MKT and how MTEs develop such knowledge. We see teachers' use of their MKT to support students' learning of mathematics as analogous to MTEs' use of their MKTT to support PTs' relearning of mathematics, as displayed in Figure 9 below. Thus, we posit that the research on teachers' development of MKT may not only provide insight into ways in which MKT is different from MKTT, but also inform ways that researchers might conceptualize a research base for MTE knowledge and the ways in which this knowledge develops over time.

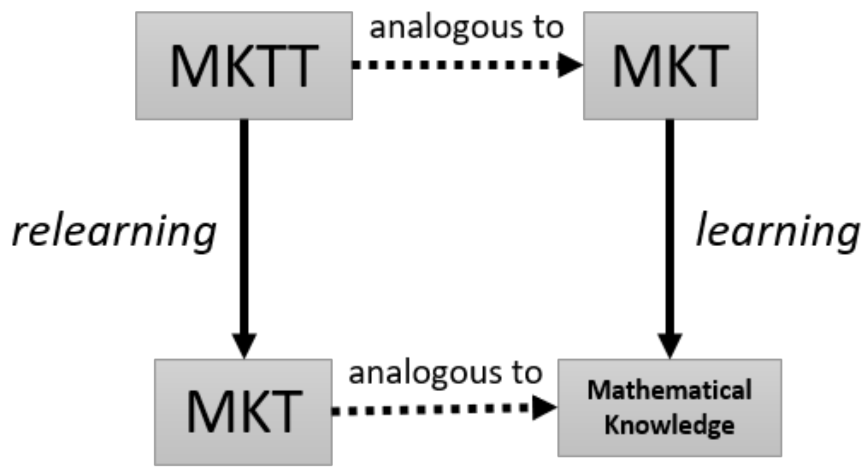


Figure 9. Parallels between mathematical knowledge for teaching teachers (MKTT) and mathematical knowledge for teaching (MKT).

Such a knowledge base could inform the design of professional learning opportunities for MTEs and those training to become MTEs. As the majority of content courses for elementary PTs are taught in mathematics departments by mathematics faculty with little or no experience teaching mathematics to young students (Masingila, Olanoff, & Kwaka, 2012), researchers are concerned with finding ways to better support the preparation and development of this group of MTEs (Greenberg & Walsh, 2008). MTEs would benefit from research-informed training offered through graduate programs, conferences, and/or professional development workshops to further develop all domains of their MKTT, as well as a repository of materials and resources designed for MTEs.

Our intention in presenting the examples above is to make visible the ways in which MTEs use domains of MKTT to support PT learning and relearning in content courses. With more MTEs reflecting on their own knowledge and experience as teacher educators, researchers and practitioners can help to build a knowledge base and design practices necessary for developing effective and knowledgeable MTEs who are able to design and teach effective courses for PTs.

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Appendix: Revised Growing Visual Patterns Task

Activity: Growing Patterns

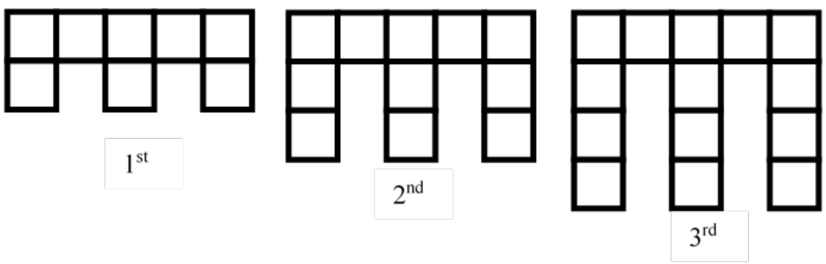
Part I

Which of the following patterns can be described by **both** the given expression (where *n* represents the step number) and the written description? Explain your reasoning.

Expression: $3n + 5$

Verbal Description: A column of five squares to the left; immediately to the right of that, *n* columns of three squares each, centered on the first column.

A



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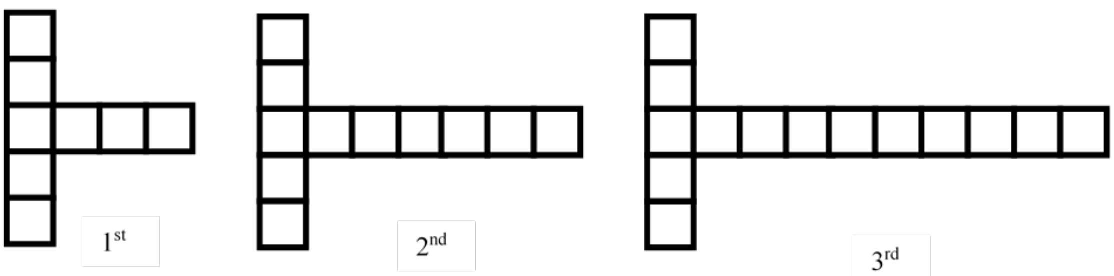

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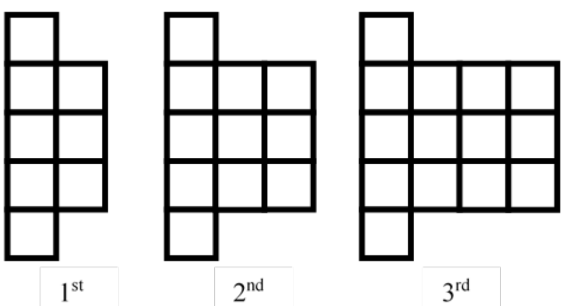

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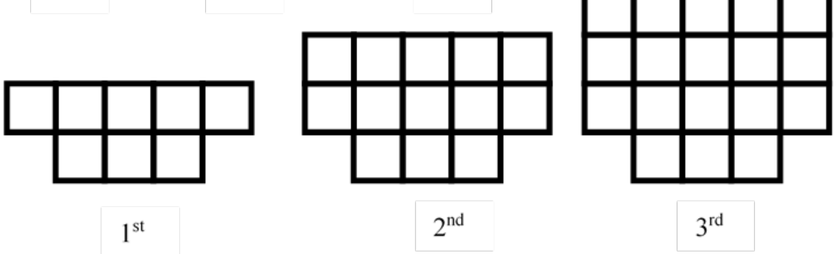

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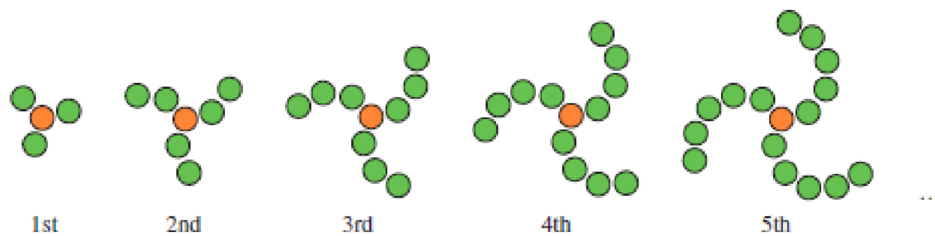
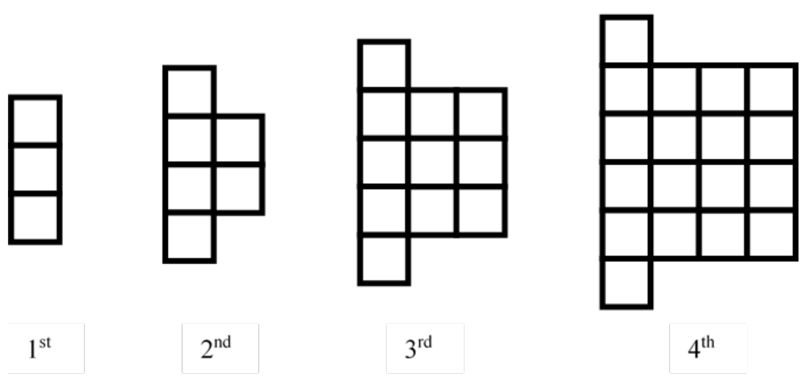


For the growing pattern shown above:

a. Draw the next three steps of the pattern.
b. Draw the 15th step of the pattern.
c. Express in words a general rule describing how to create any step in the pattern, given the step number.
d. Using a variable, write a general expression that tells how many tiles are in a step of the pattern, given the step number.

Part III



For the growing pattern shown above:

a. Draw the next three steps of the pattern.
b. Draw the 15th step of the pattern.
c. Express in words a general rule describing how to create any step in the pattern, given the step number.
d. Using a variable, write a general expression that tells how many tiles are in a step of the pattern, given the step number.

Part IV

Draw a growing visual pattern that grows following the given expression: $2n + 3$