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What Content PreK-8 Prospective Teachers Should Know and Why: Topic Recommendations for Content Courses

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Abstract: *There is a long history of mathematics educators trying to answer the questions of what mathematics is essential to learn and why, including the mathematical education of prospective teachers. Determining the answer to these questions and conveying that answer convincingly are not always easy tasks for new mathematics teacher educators (MTEs). As an MTE's career progresses (or in some cases as novice MTEs), they must also decide how much content is possible to cover in the context of one, two, or three semesters. That is, how deep is deep enough, and how much breadth is feasible? And how does an MTE make these decisions? To address these questions, we provide potential answers from three perspectives: policy, research, and textbook analysis. We begin with a brief summary of three policy documents: The Mathematical Education of Teachers II (MET-II) (CBMS, 2012), Standards for Preparing Teachers of Mathematics (SPTM) (AMTE, 2017), and the Statistical Education of Teachers (SET) (ASA, 2016). Drawing on these policy documents, a textbook analysis, the Common Core State Standards for Mathematics (CCSSM) (National Governors Association, 2010), and relevant research, we provide insights into how MTEs might choose topics for content courses, with the aim of aligning their teaching objectives with answers to why more math matters.*

Keywords: elementary teacher education, content course topics recommendations, content course topics for PreK-8 prospective teachers, Common Core State Standards for mathematics.

Introduction

Background

The history of mathematics education is long and varied, beginning with the age-old questions of what mathematics is essential and why. In the current era of standards-based

education much of the attention has turned to the preparation of prospective teachers (PTs) and those who teach them (Jackson et al, 2020; Castro Superfine et al, 2020). In this paper, our focus is on supporting students of mathematics aged 2 - 14 years by supporting those who teach PreK-8 prospective teachers (PTs), i.e., mathematics teacher educators (MTEs). As Hauk, Jackson, & Tsay (2017) and Jackson, Hauk, Tsay, & Ramirez (2020) point out, it is important for MTEs to stay current with what mathematical content knowledge is needed to teach PTs in "ways that resonate with the kinds of classrooms those future teachers are expected to sustain," (pp. 4) including the depth and breadth of essential mathematical knowledge, connections between topics and skills are required of PTs. That is, *how deep is deep enough? How much breadth is feasible, and what connections should be emphasized?*

In a national survey of programs offering mathematics courses for PTs, Masingilia, Olanoff, and Kwaka (2012) reported a general lack of support and/or professional development for instructors of these courses. They also concluded that the majority of respondents were not meeting recommendations from *The Mathematical Education of Teachers* (*MET*) (Conference Board of the Mathematical Sciences (CBMS), 2001) for at least 9 college level credit hours of mathematics (p. 8) taught by content experts. Specifically, the number of mathematics courses required for PTs still varied from 0 courses to 3 or more, with 54.4% of respondents requiring two courses. Although most respondents seemed to agree on the major mathematics topics covered by their courses, we believe that with such a wide range of required number of credit hours, it is possible the specific content covered within each topic varies just as much. Therefore, we have chosen to provide support to MTEs by synthesizing what content is recommended by major policy documents that needs to be included in these courses for PTs, including *MET*, textbook reviews, and the Common Core State Standards for Mathematics (CCSSM) (National

Governors Association, 2010), what Castro Superfine et al. refer to as MTE Knowledge of MKT (Castro Superfine et al., 2020) in their description of the mathematical knowledge for teaching teachers (MKTT).

Organization

This paper is organized into four sections. We begin with an explanation of our methodology. Second, we provide a brief summary of the history and intent of several policy documents published by five national organizations, the National Council of Teachers of Mathematics (NCTM), the Mathematics Association of America (MAA), the American Mathematical Society (AMS), the American Statistical Association (ASA) and the Association of Mathematics Teacher Educators (AMTE). Third, we discuss four textbooks used as resources related to these topics. Last, we provide a summary of content recommendations, in the form of a list of tables, based on the information from the previous sections.

Methodology

We approached this project as a synthesis of key policy documents, textbooks, and research to help us make recommendations for content courses for elementary (K-8) teachers. To create a list of topics, we first examined the recommendations from *The Mathematical Education of Teachers II* (*MET-II*) (CBMS, 2012), *Standards for Preparing Teachers of Mathematics* (*SPTM*) (AMTE, 2017), and the *Statistical Education of Teachers* (*SET*) (ASA, 2016) and listed the topics recommended in these documents. In analyzing the policy documents, we noticed that *MET-II* built upon recommendations from *MET*, and *SPTM* built upon the content recommendations from *MET-II*. Therefore, we began our list of content from *MET-II*. Then we analyzed four textbooks for PTs content courses to create a list of common topics covered by

these textbooks. After that, we went through the topics and identified the ones, which coincided with topics recommended by the three policy documents. Topics that were not explicitly supported by the policy documents and/or recommended for prospective *secondary* teachers were removed from Table 1-13 and listed after the table as a paragraph. Next, we matched the topics with corresponding K-8 grade levels in CCSSM where they were addressed (again separating high school topics). We examined policy documents and textbooks to produce a set of subtopics to highlight some essential understandings for each topic. Furthermore, if a subtopic was not mentioned by policy documents, we provided a research citation in support of this topic. Topics are supported with page references in each policy document. The last column aligns each topic with the corresponding grade level under the Common Core. We provided an example of how Table 1-13 might be used before presenting them. The chart below illustrates our process.

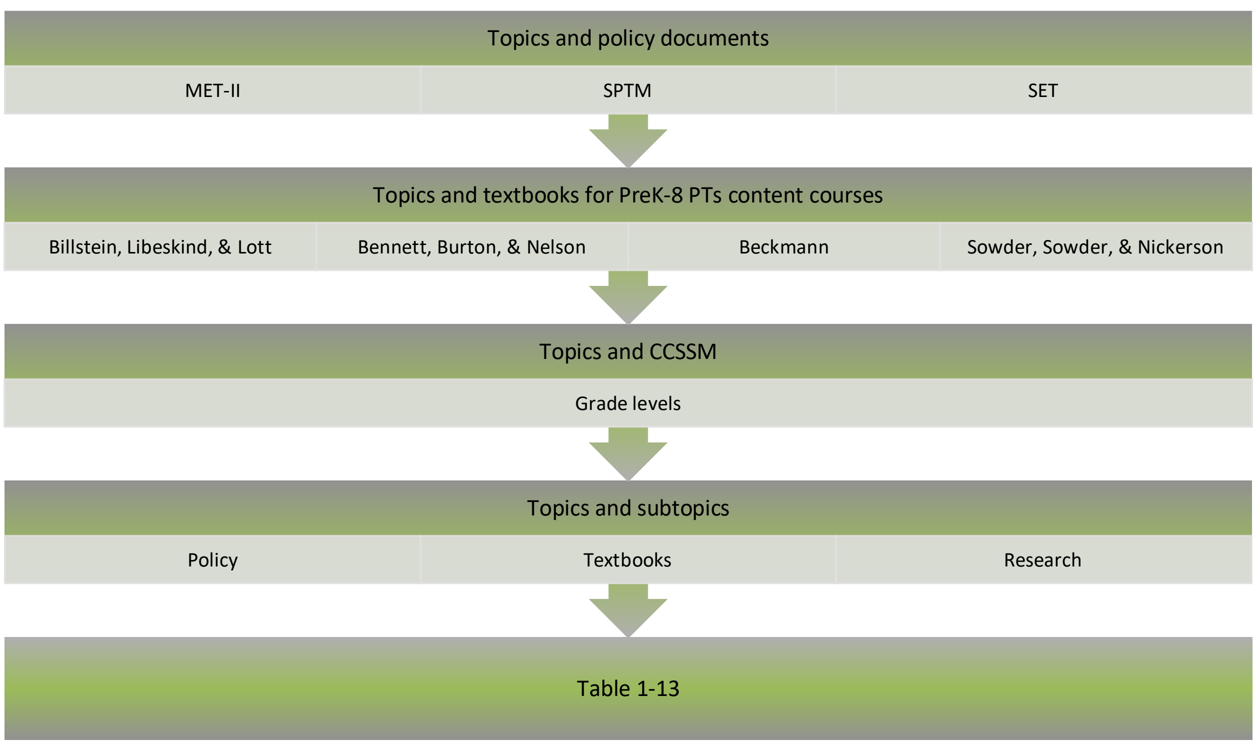

Figure 1. Methodology flow chart.

Policy

Our intention for this paper is to primarily provide support to novice MTEs who can freely make curricular decisions. For these MTEs, decisions for inclusion of certain topics in a mathematics course for PTs may not always be obvious. Granted that for some MTEs, decisions are made at the department level, or governed by the course syllabus, state guidelines or by a team of instructors, this paper can still help support them in their understanding of historical context, preparation and planning. We present a historical overview of how this discussion has developed over the years in the US, stemming from the earliest conversations of what mathematics students should learn. Our hope is that offering this historical context will help MTEs better understand why these topics have been recommended. For a more detailed history of school mathematics, please see NCTM (2003).

From as early as the 1892 Committee of Ten, the first recorded time a group of college and high school leaders met to set policy for what students should learn, the questions of what mathematics to teach and how to teach it have not changed. The committee was appointed by the National Education Association and represented several subject areas. They recommended the school structure we use today (i.e., eight years of elementary and four years of high school), with subject recommendations to help prepare students for success in college. What *has* changed is our knowledge and perspectives as a nation and profession. It is from this history that our current era of Common Core State Standards arose. In most recent history the National Council of Teachers of Mathematics (NCTM) (1989, 2000) published standards for school mathematics (i.e. grades K-12), providing a national vision for what and how students should learn. In response to this effort the American Mathematical Society (AMS) and the Mathematical Association of America (MAA) collaborated to formally begin the conversation around the preparation of K-12

teachers in *The Mathematical Education of Teachers* (*MET*) (Conference Board of the Mathematical Sciences (CBMS), 2001). Aligned with *Principles and Standards for School Mathematics* (NCTM, 2000), its focus was to emphasize the need for deeper and higher levels of mathematical knowledge among school teachers, giving general programmatic recommendations as well as specific recommendations for topics in courses for K-12 prospective teachers. It was the first-time mathematicians called for creating at least 9 credit hours of mathematics courses specifically for prospective K-12 teachers taught by content experts, stating "The mathematical knowledge needed by teachers at all levels is substantial, yet quite different from that required by students pursuing other mathematics-related professions." (CBMS, 2001, p. 7). *MET* encouraged MTEs to view PTs as novice learners of mathematics and further develop their mathematical ways of thinking, proposing to start from what teachers know and "help teachers make meaning for the mathematical objects under study (p.17)".

This document was later updated (CBMS, 2012) with the advent of CCSSM (NCACBP and CCSSO, 2010). While *MET* advocated developing future teachers' mathematical habits of mind in content courses, *The Mathematical Education of Teachers II* (*MET-II)* (CBMS, 2012) elaborated on these habits by listing specific topics to be covered in different areas, added to its recommendations for PTs knowledge in statistics, and made suggestions on how these topics should be addressed in mathematics content courses. In addition, *MET-II* put more emphasis on the importance of developing future teachers' mathematical ways of thinking by connecting its recommendations to the eight mathematical practices in CCSSM, stating instructors of content courses should "periodically remind teachers to review and reflect on the Standards for Mathematical Practice so that they become more familiar with [and acquire] the types of expertise described by these standards in the context of elementary mathematics (p. 24-25)"

(Max & Welder (2020) in this special issue addressed Standards for Mathematical Practices in content courses). *MET-II* also emphasized prospective teachers need to know more than what they will teach and be able to connect learning trajectories beyond teachers' current teaching. For example, PTs are recommended to understand the "connections among different bases" (base-10, base-2, base-5, etc.) (MET-II, p. 46; SPTM, p. 111) although this topic is not covered in CCSSM.

In 2015, the American Statistical Association (ASA) commissioned a report on the *Statistical Education of Teachers* (*SET*) (ASA, 2016). This report serves three purposes: (1) to describe differences in the statistical and mathematical education of teachers, (2) to give a view of the CCSSM from a statistical lens, and (3) to give recommendations for the statistical preparation and professional development of K-12 teachers. The report also includes recommendations on assessment of statistical knowledge and a brief history of PreK-12 statistical education. Among its recommendations, *SET* suggests having an entire course in statistical content or designing a special section of an introductory statistics course which covers both key statistical topics and pedagogical content knowledge. Like *MET-II*, *SET* suggests PTs need to experience the four parts of the statistical problem-solving process (formulate questions, collect data, analyze data and interpret results) in their content courses "at a level that goes beyond what is expected of elementary-school students (p. 14)" and understand connections between middle school statistics as well as elementary level mathematics. Future teachers should also be aware of connections between elementary statistics content and other subject areas. The focus of the statistical content course should not be practicing statistical formulas but emphasizing conceptual understanding. Other than learning statistical topics, future teachers

need to learn about common misconceptions held by elementary students and learn how to use various teaching strategies to address these misconceptions.

Most recently, the Association of Mathematics Teacher Educators (AMTE) published *Standards for Preparing Teachers of Mathematics* (*SPTM*) (AMTE, 2017), giving programmatic recommendations for the mathematical preparation of PreK-12 teachers. While *MET-II* specifically lists connections to the mathematical practices of the CCSSM, *SPTM* provides more discussion of pedagogy and connections to specific knowledge for teaching. For example, *MET-II* recommends PTs should understand the intricacies "of learning to count, including the distinction between counting as a list of numbers in order and counting to determine a number of objects" (p. 25). *SPTM* unpacks this sentence by discussing pedagogical concerns in the development of cardinality, quantity and their relation to counting in K-8 mathematics. *SPTM* also breaks PT education into early childhood (Pre-K-2) and upper elementary grades (3-5). Each grade band (i.e., Pre-K-2, 3-5, middle, and high school) is then divided into several sections, beginning with mathematical content and including aspects of pedagogical content knowledge such as dispositions, learning trajectories, tools, and assessment.

In agreement with the CBMS and ASA, the AMTE recommends that content courses for future teachers should be highly specialized to teaching mathematics and cover broader content than what they will teach. Pre-service teachers need a solid background and flexibility in the processes and practices of mathematical thinking. They must be able to recognize these processes in students' thinking and know how to develop and encourage them. They should be able to analyze mathematical thinking and curriculum and use technological tools to engage students in learning mathematics. Specifically, AMTE believes effective mathematics teacher preparation programs should provide candidates with "opportunities to learn mathematics and

statistics that are purposefully focused on essential big ideas across content and processes that foster a coherent understanding of mathematics for teaching (p. 26)" and take the guidance for specific mathematics content set by *MET-II* (CBMS, 2012) and *SET* (ASA, 2016). Furthermore, AMTE supports the recommendation by *MET-II* that PTs have 9 credit hours of mathematics. It is these three policy statements *MET-II*, *SPTM*, and *SET*, that we will use to make content recommendations in Table 1-13.

Most of the topics in the first column of Table 1-13 are specifically listed as recommendations in *MET-II*. This is where we began constructing our list of topics. We added some topics, such as *estimation and mental computation*, because they were specifically mentioned in *SPTM*. We encourage the readers to see each policy document for its complete list of recommendations.

Mathematics Textbooks for Preparation of PTs

Various textbooks are available that have been specifically developed for preparation of prospective elementary mathematics teachers. Often, teacher education programs adopt a textbook depending on the goals of the content courses and the number of content courses offered.

Reys, Reys, and Chavez (2004) summarized that mathematics textbooks usually serve as the primary resource for the sequence of topics, the content that should be covered, and the activities for engaging learners. They identified five characteristics for a good textbook: coherent presentation of material, in-depth development of ideas, emphasis on sense making, engaging students, and generating motivation for learning. In the process of textbook selection, Reys et al. (2004) also suggested that some guiding questions may be considered to help to evaluate the efficacy of the textbook within the context of elementary teacher preparation, including:

1. What key mathematical ideas are to be addressed in the content course? How does the content of the textbook align with these ideas?
2. What types of activities does the textbook provide? Does it involve mathematical thinking and problem solving? Do these activities promote sense making rather than drill practices?

Through a systematic analysis of textbooks, Valverde at al. (2002) found that elementary textbooks in mathematics and science can vary substantially in content and the way they present and structure that content. Textbooks for PTs are no different. We wanted to offer in this paper a brief textbook analysis specifically related to the content (more specifically, mathematical topics) from four different textbooks. We selected textbooks that reflect these differences in order to include a substantial breadth of content available to MTEs.

Specifically, we analyzed the textbooks in terms of inventory of commonly identified topics covered across these four textbooks. We also used Trends in International Mathematics and Science Study (Valverde et al., 2002) and the two overarching questions (Reys, Reys, and Chavez, 2004). We found that the four textbooks varied in terms of organization, length, and depth of coverage of the topics. Using the method in TIMSS (Valverde et al., 2002), where a list of topics was created to cross-analyze the connections between standards and textbook content, we identified how our "textbook" topics coincided with the topics recommended by the three "policy documents" described in the previous section. This created a list of commonly identified topics.

From here, the content in the "textbooks" as well as "policy documents" was used to create *subtopics* (see second column of Table 1-13) to highlight essential understandings of each topic in Table 1-13 (e.g., the specific integer models listed under integer operations). The

subtopics generated from the "textbooks" were further supported by research citations. For example, the entry "patterns in counting" was further supported by Von Glasersfeld's (1982) concept of "subitizing", which entails assigning of number words to specific collections of items. Von Glasersfeld suggests that learning to count through pattern recognition provides children with a strong foundation for number sense development.

The four textbooks are: (a) *A Problem Solving Approach to Mathematics for Elementary School Teachers, 12th edition* (Billstein, Libeskind, & Lott, 2016), (b) *Mathematics for Elementary Teachers - A Conceptual Approach, 8th edition* (Bennett, Burton, & Nelson, 2010), (c) *Reconceptualizing Mathematics for Elementary Mathematics Teachers, 3rd edition* (Sowder, Sowder & Nickerson, 2017), and (d) *Mathematics for Future Elementary Teachers, 5th edition* (Beckmann, 2018). A brief description of each textbook is provided below.

- *A Problem Solving Approach to Mathematics for Elementary School Teachers* includes extensive mathematical topics for both elementary and middle school teachers, as well as activities that the authors claim to promote critical thinking and problem-solving skills. It is heavily concept- and skill-based, and goes into details of rigorous proofs for mathematical theorems and properties. It focuses on the development of strong content knowledge (both procedural and conceptual) for PTs with moderate connections to standards and classrooms. It is paired with online resources including lecture videos and MyMathLab assessment bank.
- *Mathematics for Elementary Teachers - A Conceptual Approach* focuses on the conceptual understanding of essential topics for elementary PTs. It features activities for future classrooms, colorful visual representations, problem-solving applications and technology connections for almost every topic, and interactive mathematics applets that

are available online. The authors claim that this book promotes sense-making of abstract topics. Laws and properties are demonstrated with concrete examples and models. Some advanced concepts are introduced without formal definitions (e.g. complementary events in probability).

- *Reconceptualizing Mathematics for Elementary Mathematics Teachers* was written *by* teachers *for* teachers. It provides activities that help teachers model the kind of student-centered instruction that would be expected of PTs in their future work with students. Each topic in the book is introduced through explorations, argumentation, and proof. The authors claim that the book seeks to "deepen (students') knowledge and instill confidence in their ability to understand and respond to student learning in their own classrooms". Each chapter contains learner-focused lessons and activities to not only cover the content but also to include discussion questions so that students learn individually and collaboratively, while also critically reflect on elementary students' mathematical thinking and learning.
- *Mathematics for Future Elementary Teachers* uses an inquiry-based approach, connecting foundations of teaching elementary mathematics to mathematical algorithms, models and reasoning. The text comes in two sections: content and worksheet activities. It uses a student-centered approach focused on in-class collaboration and activities to encourage student engagement in mathematical conversations and discovery. The text is paired with online resources such as videos of student thinking and individualized practice and assessment.

Content Recommendations

As explained in the previous sections, the following Table 1-13 are the result of analyzing policy documents on prospective teacher education, mathematics textbooks specifically designed for teaching content to PreK-8 PTs, and the CCSSM. Please note that this inventory is not exhaustive, and may not represent a complete list of mathematical knowledge and skills necessary for mathematics preparation of prospective elementary teachers. Recommendations for programs with a single semester of mathematics are discussed at the end of this section.

Table 1-13 contain four columns: topic, subtopic, policy/research source, and CCSSM grade. To help distinguish topics within different grade levels, we separated K-5 topics and Grade 6-8 topics into two groups (Grade K-5 contains Table 1-7; Grade 6-8 contains Table 8-13), especially given that some programs divide their teacher certifications into two categories: K-5 and Grade 6-8. Overall, the topics in both groups are broken down into seven content areas similar to the *MET-II* document: 1) Counting and Cardinality; 2) Operations and Algebraic Thinking; 3) Number and Operations and Number Theory; 4) Number and Operations - Rational Numbers and Representations; 5) Measurement; 6) Geometry; 7) and Data Analysis & Probability.

As an advisory note to the reader for using Table 1-13, we suggest the "top down" approach. Specifically, before selecting a set of topics to be included in the content courses, it might be helpful to first check the state mathematics requirements for K-8 teacher certification. For example, the topic of *line of symmetry in geometric shapes* was listed as a middle school topic in the *Principles and Standards for School Mathematics* document (NCTM, 2000). Currently, *SPTM* recommends *symmetries and transformations* as an essential topic for

elementary school teachers (p.47) as well as middle school curriculum (p.98), while *MET-II* recommends these topics primarily for middle and high school teachers (pp. 44; also p. 63). Although these publications are only five years apart, the concrete differences directly illustrate the changes driven by the demands of socio-political, economic, and socio-cultural teacher education contexts, and (consequently) the varied expectations within the mathematics education communities. We also suggest starting within the first column of Table 1-13 (for selecting the topics required by the state), and then including additional topics based on the Common Core State Standards, and the recommendations by the national and policy documents and research. Lastly, when designing lessons around specific topics, the *subtopics* column in Table 1-13 may be particularly helpful given that it provides references for essential understandings of these topics PTs may not have for concepts they think they “know” (Johnson et al, 2020).

Content Recommendations for Grade K-5

Table 1

Counting and Cardinality

Topics	**Subtopics**	**Policy/Research Source**	**CCSSM grade(s)**
Counting under different number-based systems: base-10, base-2, base-5, etc. place value	Cognitive development of counting	SPTM, p.49	Kindergarten, Grade 1, 2, 3, 4, 5
	Place value	MET-II, p. 27	
	Common counting errors	MET-II, p. 26	
	Patterns in counting	Von Glasersfeld (1982)	
	Connections among different bases	MET-II, p. 46, SPTM, p. 111	
Models for adding, subtracting, multiplying, and dividing whole numbers	Base blocks, area models, rectangular arrays	MET-II, p. 34, 26 SPTM, p. 76	Grade 1, 2, 3, 4, 5
	Understand multiple meanings for multiplication and division	SPTM, p. 76	
Algorithms for whole number operations	Understand algorithms rely on decomposing numbers, place value and algebraic properties, order of operations	MET-II, p. 26, 27 SPTM, p. 51	Grade 1, 2, 3, 4, 5

Table 2

Operations and Algebraic Thinking

Topic	**Subtopics**	**Policy/Research Support**	**CCSSM grade(s)**
Properties for integer operations	Use properties of operations to explain operations with integers	MET-II, p. 41 SPTM, p. 97	Grade 1, 3, 6
	Recognize foundations of algebra in elementary math	MET-II, p. 26 SPTM, p. 51	
Arithmetic and geometric sequences	Understand arithmetic sequences as special cases of linear relationships	MET-II, p. 42 SPTM, p. 98	Grade 3, 4, 5
	Deduce *n*th term and sum of *n* terms	Russell, S.J., Schifter, D., & Bastable, V. (2011)	
Expressions, equations and their applications	Solve word problems by using the meaning of operations; Rewrite statements into algebraic expressions	SPTM, p. 80,	Grade 1, 2, 3, 4, 5, 6, 7, High School
	Understand equal sign as "same amount as"	MET-II, p. 26 SPTM, p. 51	
	Connect proportional relationship, arithmetic sequence & linear relationship	MET-II, p. 42 SPTM, p. 98	

Table 3

Number and Operations and Number Theory

Topic	**Subtopics**	**Policy/Research Support**	**CCSSM grade(s)**
Prime and composite numbers	Prime numbers Fundamental Theorem of Arithmetic	MET-II, p. 47 SPTM, p. 111	Grade 4
Mental computation and estimation for whole numbers	Breaking up and bridging Trading off Using compatible numbers Real life applications	Otto, Caldwell, Lubinski, & Hancock (2011)	Grade 1, 2
	Use properties of operations to perform mental calculations	MET-II, p. 28 SPTM, p. 76	
Ordering decimals	Compare and order decimals	SPTM , p. 77	Grade 4, 5, 6
	Connect decimals to fractions	SPTM, p. 77,	
	Understand decimals as an extension of the base 10 system Use base ten blocks to model decimals	MET-II, p. 27 SPTM, p. 78	
Decimal operations	Relate to fraction operations Expanded fraction model	Sowder, J., & Markovits, Z. (1991)	Grade 5
	Extend the whole number operations to decimals through place value	MET-II, p. 2, 27, 34, 79, SPTM, p. 77	
Rounding & mental computation with decimals and percent	Rounding based on real life situations; Breaking/bridging, using/making compatible numbers Balancing	Otto, Caldwell, Lubinski, & Hancock (2011)	Grade 3, 4, 5
	Mental arithmetic and estimations with decimals	SPTM, p. 77	
	Mental arithmetic with percentage	MET-II, p. 29	

Table 4

Number and Operations - Rational Numbers and Representations

Topic	Subtopics	Policy/Research Support	CCSSM grade(s)
Rational numbers and its representations	Multiple interpretations of fractions Fraction circle, number-line, area, and set models	MET-II, p. 28, 79 SPTM, p. 77	Grade 3, 5
Equivalent fractions	Any fraction has infinitely many equivalent fractions Simplifying fractions	MET-II, p. 28, 79 SPTM, p. 77	Grade 3, 4
Ordering rational numbers	Using different models to compare rational numbers	MET-II, p. 28, 79 SPTM, p. 77	Grade 4
Operations of rational numbers	Understand algorithms for fraction operations using various models	MET-II, p. 28, 41, 79 SPTM, p. 97	Grade 4, 5, 6, 7
	Understand connection between fractions and division	MET-II, p. 28	
Estimation and mental computation with rational numbers	Estimate and compute mentally based on understanding the magnitudes	SPTM, p. 77	Grade 5
Proportions and scale drawings	Fraction as ratio	MET-II, p. 28 SPTM, p. 78	Grade 5
	Reason about proportional relationships in scaling shapes up and down.	MET-II, p. 30	
Conversion between repeating decimals and fractions	Terminating and repeating decimals are rational numbers (p/q) Convert between fractions and repeating decimals	MET-II, p. 41 SPTM, p. 97	Grade 4

Table 5

Measurement

Topic	Subtopics	Policy/Research Support	CCSSM grade(s)
Understand measurement process	Principles of measurement	MET-II, p. 29	Grade 1, 2, 3, 4, 5
	Process of iterations Role of units Connecting linear measurements with measures of Area and volume	MET-II, p. 29 SPTM, p. 78	
	Select appropriate tools Attention to precision	SPTM, p. 78, 80	
Unit conversion and unit analysis	Among and between English and Metric Systems Estimate measurement with unit for daily objects Convert unit ratios	Clements (1999)	Grade 4, 5, 6
Perimeter and circumference	Given perimeter determine side lengths	MET-II, p. 29	Grade 3
	Deduce formula for circumference through measurement	SPTM, pp. 78	Grade 7
Area of different polygons and circle	Understand area	MET-II, pp. 29 SPTM, p. 98	Grade 3, 4, 6, 7
	Explain area formulas for regular and irregular shapes	MET-II, pp. 44	
Volumes of three dimensional shapes	Understand what volume is Give rationales for volume of rectangular prisms.	MET-II, p. 29 SPTM, p.78	Grade 3, 6, 7, 8, High School

Table 6

Geometry

Topic	**Subtopics**	**Policy/Research Support**	**CCSSM grade(s)**
Basic notations	Definitions and common misconceptions of geometric shapes	MET-II, p. 30 SPTM, p. 52	Kindergarten Grade 4, 7
Classification of polygons	Inclusive and exclusive definitions of polygons Polygon hierarchy in relation with properties	MET-II, p. 30 SPTM, pp. 52, 78	Kindergarten Grade 1, 2, 3, 4, 5
Symmetries	Line, rotational, and point symmetry	MET-II, pp. 47, 58 SPTM, p.78 (line)	Grade 4 (line symmetry only)
	Classification of polygons by symmetries	Knuchel, C. (2004)	
*Parallel and perpendicular lines	Angle properties with transversal Informal proofs Lines in two-dimensional figures	MET-II, p.44	Grade 4, 8
Three-dimensional geometry	Definition of prisms, pyramids, cylinders, and cones, real life examples	MET-II, pp.29-30	Kindergarten, Grade 1

Table 7

Data Analysis and Probability

Topic	**Subtopics**	**Policy/Research Support**	**CCSSM grade(s)**
Displaying data	Representing and summarizing data graphically Selecting appropriate displays	MET-II, p.29 SPTM, p.52 SET, pp.14-15	Grade 1, 2, 3, 4, 5, 6, 8, High School

Content Recommendations for Grade 6-8

Table 8

Operations and Algebraic Thinking

Topic	**Subtopics**	**Policy/Research Support**	**CCSSM grade(s)**
Integers and models for integer operations	Chip, Charged-Field, Number-Line, and Pattern Models	Pierson, J., Lamb, L., Phillip, R, Schappelle, B., Whitacare, I., & Lewis, M. (2012)	Grade 6
	Explain why rules for integer operations make sense	MET-II, p. 42	
Irrational numbers & roots	Connect square root symbol to its meaning	SPTM, p. 97	Grade 8, High School
	Understand why irrational numbers are needed & how the number system expands from rational to real numbers	MET-II, p. 41	
Function and its representations	Meaning of functions	MET-II p.80	Grade 6, 8, High School
	Table and ordered pairs Function machines Arrow diagrams and graphs Equations and sequences	SPTM, p.97-98 MET-II p.43	
	Different types of functions & how they model real-world relationships	MET-II p.47, 81	

Table 9

Number and Operations and Number Theory

Topic	**Subtopics**	**Policy/Research Support**	**CCSSM grade(s)**
Divisibility rules	Divisibility rules; Applications of divisibility rules	MET-II, p. 47 SPTM, p. 111	Grade 6
Factors and multiples	Greatest common divisor and least common multiple; Euclidean Algorithm; Using Colored Rods, Factor trees; Intersection of Sets	MET-II, p. 47 SPTM, 97, p. 111	Grade 6
Polynomials	Polynomial algebra; The division algorithm	MET-II, p. 47	Grade 8 (linear) High School

Table 10

Number and Operations - Rational Numbers and Representations

Topic	**Subtopics**	**Policy/Research Support**	**CCSSM grade(s)**
Proportions and scale drawings	Use unit rates to solve problems Multiple approaches to proportion problems	MET-II, p. 41, SPTM, p. 96	Grade 6, 7
	Multiple ways to representations proportional relationship	MET-II, p. 41	

Table 11

Measurement

Topic	**Subtopics**	**Policy/Research Support**	**CCSSM grade(s)**
Area of different polygons and circle	Understand area	MET-II, pp. 29 SPTM, p. 98	Grade 3, 4, 6, 7
	Explain area formulas for regular and irregular shapes	MET-II, pp. 44	
Volumes of three dimensional shapes	Understand what volume is Give rationales for volume of rectangular prisms.	MET-II, p. 29 SPTM, p.78	Grade 3, 6, 7, 8, High School
	Use volumes of prisms and cylinders to deduce formulas for pyramids and cones	Clements (2009) Clements & Sarama (2014)	Grade 8

Table 12

Geometry

Topic	**Subtopics**	**Policy/Research Support**	**CCSSM grade(s)**
Surface areas of three dimensional shapes	Hands-on models to develop and calculate surface areas of prisms, cylinder, pyramid, cone	MET-II, pp.47, 49 SPTM, p.111	Grade 6, 7
Pythagorean theorem	Explain why theorem is valid Apply converse of theorem	MET-II, p.44 SPTM, p.98	Grade 8, High School
	Applications	SPTM, p.98	
Parallel and perpendicular lines	Angle properties with transversal Informal proofs Lines in two-dimensional figures	MET-II, p.44	Grade 4, 8
Sum of interior/exterior angles of polygons	Informal proof of interior angles sum for triangles Deduce the sum of interior/exterior angles sum for polygons	MET-II, p.44	Grade 8
Straightedge and compass construction	Parallel lines Perpendicular lines Angle bisector Inscribe/Circumscribe a circle	MET-II, p.67	Grade 7, High school
Transformations	Translations Rotations Reflections Glide reflections Dilations	MET-II p. 47 SPTM, p.98	Grade 8, High school
Networks	Graph theory Real-world applications	MET-II, pp.48, 66	

Table 13

Data Analysis and Probability

Topic	**Subtopics**	**Policy/Research Support**	**CCSSM grade(s)**
Designing studies	Collecting, organizing, and reasoning about data	MET-II, p.29 SPTM, p.52 SET, pp.14-15	Grade 6, 7, High School
Measures of central tendency	Mean, median, mode Selecting appropriate measures	MET-II, p.29 SPTM, p.52, 99 SET, pp.14-16	Grade 6, 7, High School
Variation	Deviations Summarize, describe, and compare distributions in terms of shape, center, and spread	MET-II, p.44 SPTM, p.99 SET, pp.15-16	Grade 7, 8
Bivariate data	Represent and interpret Compare two data sets Making inferences	MET-II, p.44 SPTM, p.99 SET, p.21	Grade 7, 8
Definition of probability	Experimental and theoretical Using Simulations Simple probabilities	MET-II, p.44 SPTM, p.99 SET, pp.16, 21	Grade 7

Please note that high school mathematics topics include concepts such as the distance formula between two points, circle's equation, specific rules of probability (e.g., $P(A \cup B)$, $P(A \cap B)$, and $P(A^c)=1-P(A)$), conditional probability, expected value, and combinations and permutations. These topics were actually included in the textbooks that we reviewed, and would provide natural extensions of K-8 mathematics topics. Thus, if desired, these topics may also be included in the content courses for prospective elementary teachers; they are however considered (by *MET-II*, *SPTM*, & *SET*) as more critical for *secondary* teachers.

If course offerings are limited to a single semester of mathematics, we suggest following recommendations from the literature on Mathematical Knowledge for Teaching (MKT) (Ball, Thames, & Phelps, 2008) and common difficulties among PTs. Research on MKT argues that in

order to offer quality teaching to elementary students, PTs need a deep understanding of elementary topics that goes beyond computational and procedural knowledge. They need to be able to make connections between topics, representations, and strategies. They must be able to break down ideas, choose appropriate models and relate the development of mathematical topics to pedagogy. In a review of studies from 1990 - 2014, Auslander et al. (2019) found improving PTs content knowledge is "linked to … experiences that involved drawing and explaining their own representations, understanding how algorithms work, solving problems that allow for multiple solution strategies, inventing their own solution strategies …, and sharing and analyzing these strategies with peers" (p.32), and problem-based learning contributed to a "deeper and more flexible understanding of … mathematics" (p. 35). In a review of literature from 1978 – 2012, Browning et al. (2014) found that PTs understanding of elementary mathematics is limited to using algorithms, while experiencing difficulties in justifying *why* the algorithms work. In addition, many PTs can experience anxiety in the mathematics classroom (Hart et al., 2016). Focusing on the development of number sense and counting gives PTs the confidence to develop sophisticated mathematical reasoning and habits of mind (Cuoco, Goldenberg, & Mark, 1996). Therefore, if an MTE finds they are limited to offering a single semester of mathematics for PTs, we suggest following recommendations outlined in the above-mentioned studies. That is, a PT's mathematical education for teaching should *begin* with whole number concepts and operations (including models for whole numbers and the development of the four operations), an in-depth study of fractions, modeling fractions in various problem situations, modeling operations involving fractions, and the development of and models for algorithms for operations involving fractions, integers, decimals (including studying the structure of decimals, place value, and the density of decimals), geometry and measurement, ratio as measurement, patterns in algebra, and

interpreting and connecting algebraic symbolic representations. We also suggest a focus on problem-based experiences that emphasize collaboration, communication, explanation, and joyful learning in order to allay any anxiety around mathematics and build confidence in PTs as future teachers (Kuennen & Beam, 2020; Hallman-Thrasher, Rhodes, & Shultz, 2020).

Conclusion

Policy and research in the field of mathematics education have outlined core principles for developing essential teacher knowledge for teaching mathematics in elementary grades. In this paper, we offered an inventory of topics as a dynamic contribution to the field for continued exploration and expansion of the mathematics topics taught in the elementary (mathematics) content courses. We believe this inventory will help inform the work of MTEs in the field and provide a beginning platform for policy and research documents to continue informing our professional community in the efforts of meeting the demands of scientific, socio-political, economic, and socio-cultural teacher education contexts.

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