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Experiencing Active Mathematics Learning: Meeting the Expectations for Teaching and Learning in Mathematics Classrooms

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Abstract: *Active learning mathematics classrooms incorporate meaningful activities that emphasize reasoning, thinking and active interaction with mathematics. Current mathematics standards and curricula recommend that Mathematics Teacher Educators (MTEs) use elements of active learning in their mathematics content courses specifically designed for Prospective Teachers (PTs) as they prepare PTs to learn and teach mathematics. However, it can be very difficult for PTs to shift their pedagogical dispositions towards instruction associated with active learning because they typically have not experienced mathematics taught in this way. This article focuses on two instructional practices for MTEs to use with PTs. First, selecting tasks that promote reasoning and problem solving. This includes practices to open tasks for multiple entry points or solution strategies, tasks that allow for analysis of examples and counterexamples, and tasks that evaluate multiple strategies. Second, facilitating meaningful mathematical discourse. This includes practices for whole class and small group discourse. When MTEs use active learning strategies with PTs in their mathematics content courses, PTs may begin to shift their beliefs and understandings about what it means to teach and learn mathematics. By highlighting how active learning experiences can enhance PTs' own understanding and their future students' mathematical understanding, MTEs will provide a valuable foundation for PTs to meet the expectations for teaching and learning in mathematics classrooms.*

Keywords: active learning; mathematics; problem solving; discourse; prospective teachers; Mathematics teacher educators

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Introduction

Teaching and learning mathematics in PreK-8 classrooms is more than just memorizing basic facts and procedures – it is about helping elementary students construct mathematics understandings and reason about the mathematics and strategies they are constructing (NCTM, 2014). An *active learning* classroom provides students opportunities to develop a deeper understanding of the mathematics through “meaningful learning activities that require students to think about what they are doing” (Prince, 2004, p. 223).

Mathematics curricula and standards published in the last few decades outline expectations to incorporate elements of active learning in PreK-8 classrooms as well as Prospective Teacher (PT) mathematics content and methods courses. Standards include the Common Core State (CCSSI, 2010) Standards for Mathematical Practices (CCSSMP), National Council of Teachers of Mathematics Principals to Action (NCTM, 2014), and the Association of Mathematics Teacher Educators (AMTE, 2017) Standards for Preparing Teachers of Mathematics (SPTM), which have instructional implications for Mathematics Teacher Educators (MTEs) as they prepare their PTs to learn and teach mathematics.

In teacher education programs, PTs are required to take mathematics content courses to develop a deeper understanding of the elementary mathematics they will teach and methods courses to help understand effective practices for teaching and learning mathematics. Sometimes, these courses can send mixed messages, which leads to an “incomplete and fragmented vision of how to enact effective mathematics learning environments for their students” (AMTE, 2017, p. 2). MTEs who are tasked with mathematics content courses, specifically designed for PTs, can support PTs’ continuing growth and development by modeling effective instructional practices, such as active learning, that PTs will be expected to use in their future teaching.

When MTEs use instructional practices that promote active learning, PTs "experience learning mathematics using methods that are consistent with the methods they should use as teachers" (AMTE. 2017, p. 31). These instructional practices often challenge PTs' preconceptions about how mathematics should be learned and taught. Persistent frustrations stem from PTs focusing primarily on "mathematics products" rather than the underlying instructional practices that promote the learning of those products (AMTE, 2017). It is very difficult for PTs to shift their pedagogical dispositions towards instruction associated with active learning because they typically have not experienced mathematics taught in this way. In fact, researchers have found that PTs will draw from their past learning experiences when designing their instructional experiences (Cady, Meier, & Lubinski, 2010). Similarly, we recognize it may be difficult for the MTE to experience resistance from their PTs. Thus, the following sections outline the standards-based expectations that MTEs can use to rationalize incorporating active learning in their course, as well as specific active learning instructional practices MTEs can use in their content or methods courses to prepare PTs to establish and teach in an active learning environment.

Creating Active Learning in Mathematics Content Courses

In active learning classrooms, students engage with the mathematics in ways that allow them to construct their own understanding of concepts and procedures. There are a variety of instructional practices that MTEs can use to engage PTs with the mathematics and promote an active learning environment. This article integrates standards focusing on two instructional practices: 1) selecting tasks that promote reasoning and problem solving; 2) facilitating meaningful mathematical discourse.

Select Tasks that Promote PTs' Mathematical Reasoning and Problem Solving

The Standards for Preparing Teachers of Mathematics (SPTM C.2.2) recommends that MTEs select mathematics tasks for PTs that "promote reasoning and problem solving, provide multiple entry points, have high ceilings to offer challenges, and support varied solution strategies" (AMTE, 2017, p. 14). Tasks that promote reasoning and problem solving can engaging PTs in more complex mathematical thinking, which can help deepen their own understanding of the subject (Horn, 2007).

This recommendation aligns with Principals to Actions (NCTM, 2014) recommendations for PTs to implement these same types of tasks in PreK-8 classrooms. The Common Core Standards for Mathematical Practice (CCSS MP1) adds that when engaging in these types of tasks, PTs have opportunities to identify entry points, evaluate and modify their own strategies, and critique other approaches and strategies. Active learning classrooms typically use tasks with multiple entry points or solution strategies (i.e., open tasks) to promote reasoning, which increases opportunities for PTs to develop a deeper conceptual and procedural understanding of the mathematics (AMTE, 2017; Smith, Stein, & Arbaugh, 2004; Stein & Lane, 1996). Figure 1 provides an example of an open task.

> Solve the following Base-5 addition problems using two different addition strategies (traditional algorithm, partial vertical, chip abacus, partial horizontal, vertical left to right, etc).
>
> $33_5 + 42_5 =$ $\qquad\qquad$ $24_5 + 43_5 =$
>
> Compare and evaluate the two strategies. How might each strategy support students when solving addition problems in a Base-10 system? Cite evidence to support your thinking.

Figure 1. Example of an open mathematics task.

The open mathematics task in Figure 1 allows multiple entry points for PTs by offering a choice in the strategies they use. By requiring two different strategies, the problem also encourages varied solution strategies. PTs may choose to use a strategy they are comfortable with and also branch out to try something new. This task promotes reasoning by eliciting a comparison and evaluation of the strategies and supporting their thinking with specific evidence. PTs who make connections between their own strategies and possible student strategies are better prepared to evaluate their future students' work (AMTE, 2017).

Unfortunately, traditional mathematics textbooks generally offer mathematics tasks in which there is only one correct solution and one correct solution strategy (i.e., closed tasks). Small (2009) offers five strategies that MTEs could use to turn closed mathematics tasks into more open tasks to promote reasoning and problem solving: (1) turn around a question, (2) ask for similarities and differences, (3) allow for choice in numbers, (4) construct word problems from a number sentence, and (5) change questions to allow for multiple answers. Figure 2 provides an example for each of these strategies.

Example 1: Turn Around a Question:
(Closed Task) Solve: $2\frac{1}{2} - 1\frac{3}{4} =$
(Open Task) What are two mixed fractions with different denominators whose difference equals $\frac{3}{4}$?

Example 2: Ask for Similarities and Differences:
(Closed Task) List 3 properties of a square.
(Open Task) Compare the similarities and differences between squares and rectangles.

Example 3: Allow for Choice in Numbers:
(Closed task) Solve: 42 x 68
(Open Task) Roll a 10-sided die four times. Use these numbers to create and solve a 2-digit by 2-digit multiplication problem.

Example 4: Construct a Word Problem from a Number Sentence:
(Closed Task) Solve: $2x + 8 = 26$
(Open Task) Design a word problem that would be solved by constructing an equation in the form of $ax + b = c$, noting specific values for a, b, and c.

Figure 2. Changing closed tasks to open tasks.

In Figure 2, Example 1, the task was "turned around" by providing the answer and asking PTs to construct a question. This opens the question to have multiple entry points and varied solution strategies as PTs consider sets of fractions whose difference is ¾. Adding restrictions such as "different denominators" increases the difficulty or prompts PTs to consider other important mathematical concepts. Another example is to provide both the answer and strategy, and have PTs evaluate for accuracy or provide reasoning. This same principal could be applied to non-numerical problems such as identifying shapes; PTs could create a 20-Questions type problem rather than name shapes based on properties.

Example 2 shows how a geometry task could be opened by comparing similarities and differences between the properties of two shapes. This prompts students to look at relationships between shapes and their properties rather than one shape in isolation. A numerical example of comparing similarities and differences is to provide students with different strategies for solving the same operations. For example, when solving the equation $9 - 5x = -2x$, you may have students compare three strategies for finding the solution, $x = 3$: 1) guess and check, 2) isolating x on the left side of the equation, and 3) isolating x on the right side of the equation. By comparing the different strategies, PTs can evaluate the efficiency of each strategy as well as identify potential misconceptions (e.g., variables must be isolated on the left side of the equation). Other comparison examples could include PTs comparing multiple numbers, shapes, graphs, or even story problem structures.

Example 3 uses dice to provide PTs random numbers when developing number sentences. MTEs can also provide PTs real world contexts to generate numbers, such as consulting a local restaurant menu for prices when adding decimals. In another example, students could compare the difference between two Olympic skier run times. Additionally, MTEs can replace numbers in word problems with the word "some" and have PTs discuss (un)reasonable solutions. Technology apps can also provide ways to customize or randomize numbers. For example, the *Map Maker Co-ordinates* app allows users to place icons on a map to identify coordinate locations in up to four quadrants.

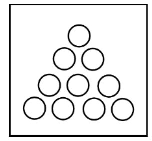
Example 4 requires PTs to create a context for specific mathematics content, which prompts PTs to consider the mathematics in relation to everyday experiences. When creating a story problem for an equation in the form of $ax + b = c$, PTs must consider the arithmetic relationships between each of the symbols and variables. For example, "a" represents multiple sets of "x," (e.g., 3x could be 3 boxes of apples at a fruit stand, with a "box" having an unknown quantity of single apples); "b" represents single items (e.g., 5 single apples), with "+" representing single items gained and "-" representing single items lost (e.g., -5 could be 5 apples were eaten by workers); and "c" represents the total number of single items (e.g., 25 apples remained at the stand). MTEs can modify the difficulty of the problem by changing the magnitude of each variable, the number of variables, or the operations used within the equation.

In addition to selecting tasks that promote reasoning and problem solving, MTEs' actions can also play a role in promoting or limiting opportunities for PTs to reason while they solve mathematics tasks. In the following two examples in Figure 3, consider how Mr. Wiley's and Mr. Actins' actions may or may not promote reasoning.

In Figure 3, each instructor affords PTs opportunities to solve challenging problems. However, Mr. Actins' approach of explicitly modeling solutions for PTs (e.g., by initially modeling how to flip the triangle) may have limited PTs' opportunities to reason through their own strategies. Additionally, when Mr. Actins hinted at using specific coins, he limited the group's strategy choices to only those coins. In contrast, Mr. Wiley shifts the responsibility for

Flip the Triangle Mathematics Task:

Can you turn this triangle upside down by moving only three of the coins?
Add another row making the total number of counters 15. How many moves do you expect to flip this triangle?



Mr. Wiley's posted the problem on the board and had his PTs work in groups of 2-3 to solve the problem. One group was struggling to find a solution. Mr. Wiley asked the group to describe what they had tried so far and how it had affected the number of moves required to flip the triangle. He then moved on to other groups while the group talked together about their strategies. After 15 minutes, Mr. Wiley pulled the class together and asked the PTs to share their strategies.

Mr. Actins placed a triangle with three coins under the document camera. He showed PTs how he could flip the triangle by moving the top coin down and moving the bottom two coins up. He then added another row and asked PTs to identify ways they could flip this new triangle before posting the full problem. One group was struggling to find a solution. Mr. Actins pointed out the three coins on the edges and asked the group to find a way to move only those three coins. After 15 minutes, Mr. Actins shared two common strategies.

Figure 3. Teacher actions that promote or hinder reasoning.

sense making to the PTs. Mr. Wiley's scaffolds (e.g., reflecting on the details of the problem, working in small groups) provide enough information for PTs to know what they need to do without prescribing their strategy development (Store & College, 2015). Additionally, Mr. Wiley had his PTs describe their own strategies, which (1) provided Mr. Wiley opportunities to use PTs' reasoning when extending their thinking to more difficult problems and (2) provided PTs opportunities to explain their thinking and reasoning. PTs need opportunities to grapple with challenging tasks and construct their own ideas to make sense of the mathematics.

MTEs can provide opportunities for PTs to explain their thinking and reasoning by asking *why?*. Although this phrase may seem simple, it prompts PTs to reflect on their thinking and reasoning in order to justify their response. However, PTs may not always be able to articulate their reasoning, so to help PTs further develop skills in constructing viable arguments, MTEs can ask specific questions to guide PTs as they analyze a situation. For example, rather than just asking *why*, the MTE may ask, *"Why did you choose that number?"; "How did you* *know that strategy would work?"*; or, "C*an you draw a picture/diagram to illustrate what you did*. These questions provide PTs with a specific way to represent and organize their thinking. Carpenter, Franke, and Levi (2003) explain that PTs who "justify their own mathematical ideas, reason through their own and others' mathematical explanations, and provide a rationale for their answers develop a deep understanding" (p. 6) of the mathematics.

MTEs can also encourage PTs to construct viable arguments by providing opportunities to use examples and counterexamples. A more robust conceptual understanding is obtained from both understanding the boundaries of what that concept is as well as what that concept is not (Komatsu, 2010). In mathematics, examples and counterexamples can be used to develop conceptual understanding of principles and procedures. PTs can also use these examples and counterexamples to justify their arguments. Figure 4 illustrates one activity MTEs can use to support PTs' understanding of examples and counterexamples.

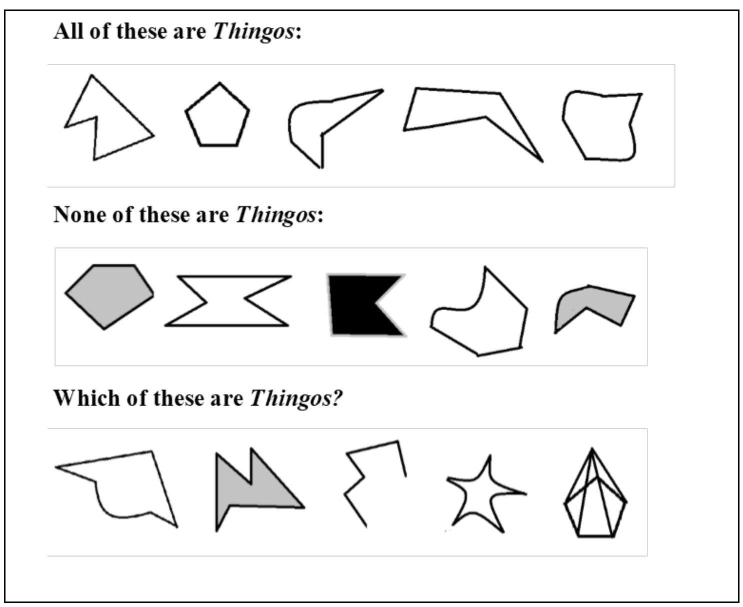


Figure 4. Thingos example and counterexamples.

PTs presented with the first set of Thingos (examples) in Figure 4 reason about the properties of Thingos as they compare the similar attributes between each shape. PTs may come up with ideas about the number of sides and the shape of the sides. PTs presented with the second set of non-Thingos (counterexamples) now compare the examples with the counterexamples. Through this comparison, PTs refine their understanding as they coordinate similar and dissimilar attributes within and between the two sets. In the third set (evaluate new shapes), PTs may be able to easily identify the first shape is a Thingo and the second shape is not; However, the last three shapes may raise additional questions (e.g., *Are Thingos closed or open shapes? Do Thingos need at least one straight side?*). This illustrates the idea that in mathematics, it is nearly impossible to provide every counterexample.

At each stage of the Thingos task, MTEs may want to have PTs compare and contrast what makes a Thingo versus what does NOT make a Thingo. By comparing aspects of each set of shapes, PTs are provided opportunities to reflect on the properties of a Thingo. This task provides PTs with opportunities to construct and refine a definition for a Thingo, which affords them opportunities to determine how their own students could construct definitions in mathematics classrooms. As PTs share their ideas, other PTs "try to use clear definitions in discussion with others and in their own reasoning." (CCSSI, 2010, MP6).

Although the Thingo task is a contrived set, working with this novel shape allows PTs the opportunity to evaluate how they define mathematical representations in traditional mathematics tasks as well as how they can help students develop a conceptual understanding of those representations. When PTs share their thinking and compare this with examples and counterexamples, they develop precise language. PTs bring with them a variety of experiences with mathematics. This range of experience offers PTs engagement with a variety of examples and counterexamples that their MTE may not have considered before enactment of a task.

In addition to attending to the precision of their language, MTEs can provide PTs with opportunities to "analyze elementary students' reasoning that may be atypical or different from their own" (AMTE, 2017, p. 10). Figure 5 provides one example of this type of activity.

Four students were presented with the following task:
Curly used a shovel to dig his own swimming pool. He also planned to build a rectangular concrete deck around the pool that would be 6 feet wide at all points. The pool is rectangular and measures 14 feet by 40 feet. What is the area of the deck?

Rajesh, May, Simon, and Wang each solved this problem using a different method:

Rajesh:
52 ft × 6 ft = 312 sq ft
312 sq ft × 2 = 624 sq ft
14 ft × 6 ft = 84 sq ft
84 sq ft × 2 = 168 sq ft
Total = 624 sq ft + 168 sq ft = 792 sq ft

May:
40 ft × 6 ft × 2 = 480 sq ft
14 ft × 6 ft × 2 = 168 sq ft
6 ft × 6 ft × 4 = 144 sq ft
Total = 797 sq ft

Simon:
52 ft + 26 ft + 52 ft + 26 ft = 156 ft
156 x 6 ft = 936 square feet.

Wang:
40 ft × 14 ft = 560 sq ft
1352 sq ft – 560 sq ft = 792 sq ft

Compare and evaluate the reasoning behind these students' solutions and strategies. Cite evidence to support your thinking.

Figure 5. Evaluating student strategies.

The four strategies in Figure 5 provide PTs with a variety of student strategies and misconceptions they may encounter in their future classrooms. Rajesh's strategy breaks up the deck into two sets of equivalent deck components and then adds the parts together for a correct answer of 792 sq ft. May's strategy provides an alternative way to break up the deck. However, the intentional incorrect total (797 sq ft) provides opportunities for PTs to evaluate how a wrong answer does not always indicate an incorrect strategy. Simon's strategy multiplies the perimeter by the width of the deck, bringing forward a misconception between perimeter and area that doubles the area of each corner of the deck. Wang's strategy may be unique in that the area of the pool is subtracted from the combined area of pool and deck to arrive at the area of the deck. Depending on their prior mathematical experiences, PTs may gravitate to one or more strategies and find it difficult to understand the logic behind certain strategies.

When implementing this task in content courses, MTEs should encourage PTs to draw a diagram of the problem and consider how the numbers in each strategy relate to the diagram. These considerations can help PTs begin to make sense of the meanings of each computation in the context of the problem and build their own strategic competence (National Research Council, 2001). Tasks that require PTs to analyze strategies that do not align with traditional computation algorithms can support PTs relearning of content and the "conceptual underpinning of an unconventional algorithm" (Castro Superfine, Prasad, Welder, Olanoff, & Eubanks-Turner, 2020, p. 375). Additional benefits of having PTs evaluate students' strategies in mathematics content courses is that PTs may be more open to considering alternative strategies introduced during subsequent lessons, as well as better prepared to analyze and unpack their future students' mathematical strategies (AMTE, 2017; Feldman, Wickstrom, Ghosh Hajra, & Grupta, 2020).

Facilitate Meaningful Mathematical Discourse Among PTs

Mathematical discourse is a discussion between two or more people that focuses on specific mathematics properties or procedures and provides PTs with opportunities to share and clarify mathematical ideas (Brodie, 2011; Hufferd-Ackles, Fuson, & Sherin, 2004; Wouters et al., 2013). Teacher preparation standards recommend that MTEs provide PTs with opportunities to cooperatively discuss and explore mathematics content (SPTM P.2.2, AMTE, 2017). Principles to Actions explains that discourse helps PTs "build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments" (NCTM, 2014, p. 10). Learning is a social process. PTs can use their individual experiences and content knowledge to help support each other as they learn mathematical content and pedagogy (Clements & Battista, 2009). The co-construction of learning through discourse promotes multiple strategies and deeper understanding of the mathematics (Hwang & Hu, 2013; Mueller, 2009; Wachira, Pourdavood, & Skitzki, 2013). Mathematical discourse can take place as a whole class or in a small group/partner format. Below, we describe discourse in each format.

Whole Class Mathematics Discourse

Whole class mathematics discourse during a lesson or activity can be a great way to socially and collectively solidify goals and concepts set by the MTE (Bruce & Flynn, 2011; Hallman-Thrasher, Rhodes, & Schultz, 2020; Iiskala, Vauras, Lehtinen, & Salonen, 2011). MTEs can purposely select who will share during the whole class discourse by first monitoring the strategies that PTs used to complete the task and asking questions regarding their strategies. This provides the MTE with an awareness of the state of the class as well as insights into what the PTs do or do not understand (Smith & Stein, 2011). Next, the MTE can select specific PTs to share their strategies that address specific lesson goals or contrasting points of view. Purposefully selecting the order and content shared allows MTEs to "manage the content that will be discussed and how it will be discussed" (Smith & Stein, 2011).

PTs' written responses to mathematics tasks are a great source for both examples and counterexamples. Although MTEs or PTs may not have issues sharing accurate or correct examples of different strategies, they may hesitate to share inaccurate or incorrect counterexamples. In our experience, this hesitancy stems from two common misconceptions: 1) PTs may have used a faulty strategy and sharing it may confuse others in class, and 2) PTs may be embarrassed if their error is pointed out (Bray, 2011). Therefore, in Figure 6, we share two common classroom situations involving PT errors and the ways in which MTEs may respond. In both situations, the MTEs strive to create classroom environments where errors are framed as a source of learning rather than an indication of failure, and where all solutions (correct or incorrect) are honored and respected (Dweck, 2010; Leinhardt & Steele, 2005). In these types of environments, PTs may feel "safe" to share a solution strategy regardless of correctness, or to bring forward an example they know has an error as a learning opportunity

Situation 1: Response to an Unexpected Error
Dr. Laura, an MTE, asked her PTs to share strategies to identify different measures of center in a data set. As Micah is sharing her strategy, she suddenly stops and realized she has made an error. Dr. Laura asks Micah to continue explaining her strategy and points out an important conceptual understanding brought forward by the strategy. She then asks Micah to reflect on factors that may have contributed to her error.

Situation 2: Purposefully Have a PT Share an Error
As Mr. Trent walks around the room, he notices that many of his PTs are making the same error and wants to bring the misconception forward. To do this, he wants to be sensitive to the fact that many PTs may feel self-conscious if they knew they made an error. So, Mr. Trent decided to provide the class an anonymous example of the misconception and have PTs identify the error.

Figure 6. Examples of MTE responses to PTs' errors.

As illustrated in Figure 6, MTEs will encounter situations where PTs share a solution they thought was correct only to discover it was not correct. How MTEs respond to these types of situations may affect PTs' attitudes towards future errors that their own students encounter in the classroom. Dr. Laura's response is powerful for two reasons: (1) the MTE has created a safe environment for errors to be discussed openly; and (2) the PT focuses her attention on the *strategy* rather than the accuracy of the problem or the mathematics abilities related to the *PT.* In contrast, had Dr. Laura simply fixed the error and moved to another student in an attempt to save Micah from embarrassment, she might have sent an unintentional message that the solution is more important than the strategy and that Micah's thinking is flawed because her solution was incorrect. Instead, MTE may ask the PT, *Why do you think there is an error with that strategy?,* which allows the PT an opportunity to evaluate his or her argument and change his or her thinking. This can also open up opportunities to use this error as a counterexample as PTs identify points of divergence within an ineffective strategy to construct more effective strategies (Kazemi & Stipek, 2009). Further, this is an effective way to reduce PTs' mathematics anxiety (Karunakaran, 2020).

MTEs can also purposefully bring forward errors such as those in Figure 6, Situation 2 when selecting strategies to share during the whole class discussion (Leinhardt & Steele, 2005). When multiple PTs make similar errors, MTEs should address the underlying reasoning associated with the misconception. Otherwise, PTs may continue to use a flawed strategy (Leinhardt & Steele, 2005). By bringing the error forward anonymously, Mr. Trent was able to address the error without PTs feeling like they are being individually targeted. This can allow PTs to focus on the reasoning (accurate and inaccurate) associated with the task. Other strategies might include the MTE solving the problem using the error and then asking PTs to evaluate his solution, or the MTE writing the error on the board and having PTs volunteer why the problem might be solved using this strategy. Either of these responses provides opportunities for PTs to critique the reasoning of others and evaluate their own reasoning if they made the same error, all without revealing whether they committed the same error or not. Once underlying misconceptions are exposed, other PTs or the MTE can address how to overcome these misconceptions. Providing PTs opportunities to analyze errors allows them the opportunity to change their mindset from errors indicating failure, towards a mindset where errors lead to success (Boaler, 2015; Dweck, 2015).

Once MTEs have selected the strategies they would like shared during the discussion, MTEs can facilitate classroom discussions by using *talk moves* (Chapin & Anderson, 2013). Talk moves are questions that help facilitate discussion among PTs, such as: *How did you figure it out? Who can repeat Lucy's strategy in their own words? Do you agree or disagree and why? Who can add to Julio's thinking?* Chapin, O'Connor, and Anderson (2009) specifically name and define five talk moves: revoicing, repeating, reasoning, adding on, and wait time, which provide both MTEs and PTs a simple framework and question stems for facilitating mathematical discourse. Revoicing is a talk move that involves students in clarifying their contributions or in simply providing an opportunity for students to hear the idea again, for instance, *So, you are saying that as adults we easily use place value ideas, but for young children, those ideas might not be fully developed?* In using a Revoicing talk move, the MTE repeats a comment a PT makes back to the PT. Repeating is a question such as *Who can repeat Lucy's strategy in their own words?* and is used to provide a reiteration of a contribution, clarify an idea, and/or let students know that others are listening and their ideas are being considered. In using a Repeating talk move, the MTE asks PTs to repeat each other's statements in their own words. Reasoning moves such as *Do you agree or disagree and why?* are to encourage students to apply their thinking to their peers' ideas and make reasoning explicit. In order to encourage PTs to do this, an MTE uses Reasoning talk moves to ask PTs to justify their thinking around a statement. Adding On is a talk move for encouraging further development of an idea by asking questions such as, *Who can add to Julio's thinking?* Finally, Wait Time is a talk move in which the MTE encourages more participation by letting students know they do not need to answer quickly, rather the community provides wait time to allow everyone to be thoughtful. This can often be until the MTE sees that everyone has had time to arrive at a solution (via a gesture) or a standard 10-second to 30-second wait time, depending on the activity.

Figure 7 provides an example of an MTE modeling talk moves for the first time during a semester.

PTs had been asked to solve 52+25 with mental math, and the MTE explained she would be using talk moves to facilitate a discussion around a key idea.
MTE: How did you solve for 52+25?
Elise: I know 5 plus 2 is 7 so I just added both of those to get 77.
MTE: Can you repeat what Elise said in your own words? (*Repeating*)
Marta: Elise said she added the 5 and 2 to get 7 and that combination is in both places.
MTE: So, you are saying that 5 plus 2 shows up in two places. (*Revoicing*) Would someone like to add to what Marta is saying about Elise's strategy? (*Adding On*)
Kaylee: When Elise and Marta are talking about two places, they are talking about the tens place and the ones place.
MTE: The tens and ones places…do you agree or disagree and why? (*Reasoning*)
Elin: I agree that Marta is talking about the tens and ones places when she's saying two places, because its 50 and 20 and 2 and 5. That's what we are really adding. But I don't know if Elise was doing that.
Elise: Honestly, I just saw the 5s and 2s and I did it so fast in my head. I wasn't really thinking about tens and ones. I mean I know place value, but I didn't stop and think 'I'm adding ones and I'm adding tens.'
Elin: Yeah, and when you say it that way, I think this problem is really simple for us, but for a second grader, they might need to think about place value or might not know it mentally as quickly.
MTE: So, you are saying that as adults we easily use place value ideas, but for young children, those ideas might not be fully developed? (Elin nods) Let's talk about that, turn and talk to the people at your table about what we've discussed so far (*Small Group Talk Format*).
(After small group discussions, the MTE continues to facilitate the whole-class discussion.)
Lidia: After talking with my group, we said that really what we are solving is 50 plus 20 and 2 plus 5. When Elise said 5 plus 2 that was probably the 50 and 20. When she said she added both of them she automatically knew 5 plus 2 is the same as 2 and 5 because of the commutative property of addition. There's actually many important ideas in that one statement that we take for granted as adults.
MTE: Wow! Does someone want to repeat Lidia and her group's key idea in their own words? (*Repeating*)
Kaylee: Yes, they are saying that there are place value ideas, commutative property, and even equality ideas that are all within this what-seems-to-be-easy problem.
Whitney: I'll add to that, because I think when we talk about place value and looking at 50 and 20 combined with 2 and 5 we can better see the whole quantity. Like we remember that 50 and 2 is 52 not just isolated digits. It's funny because I didn't even do it that way. I immediately thought of 25s and 50s. Those numbers are easy to work with because of U.S. coins. So, I ignored the 2 in 52 and just saw 75 from 50 plus 2, then added the two back on to get 77. So, when Lidia brought that up I could see how holding on to the meaning of the number, like the place values is important. And, breaking numbers apart in different ways could make the problem easier to solve but you can do it in different ways.

Figure 7. Examples of talk moves in whole-class mathematical discourse.

In this vignette (see Figure 7), notice that the MTE gave a fairly simple mental mathematics problem to PTs, and by using talk moves, could guide PTs to talk about the complex ideas underlying the simplicity. Getting PTs to repeat and add to each other's thinking led them to see the mathematical meaning embedded in a problem that is challenging for young children. The MTE first used the repeating talk move in order to establish the norm that PTs should be listening to each other's ideas and to confirm that as a class they now have Elise's idea on the table for discussion. This is followed by the MTE's use of revoicing and adding on. The MTE used these moves to draw attention to the statements made by Elise and Marta (i.e., adding 5 and 2 to get 7), then pressed for further explanation about what is meant by 5 and 2 (i.e., 5 and 2 versus 50 and 20). Kaylee takes the bait and leads the class into a discussion about place value. This is where the MTE highlights this topic with a reasoning move (*The tens and ones places…do you agree or disagree and why?*) to further pursue the topic of place value and what is meant by tens and ones places.

There is not a prescriptive way to use the talk moves nor do they need to be used together all the time. Their use is strategic and are most effective when used intentionally to elicit PTs' ideas around particular mathematical ideas, such as place value. Notice at the end of the vignette, the MTE uses repeating again, this time to give the class another opportunity to summarize a key idea: *Wow! Does someone want to repeat Lidia and her group's key idea in their own words?* In this case, Lidia articulated some important concepts that the MTE felt the class needed to hear one more time. Further, notice the back and forth between the MTE and PTs at the beginning followed by an increasing amount of talk among PTs without the MTE. Talk moves can be used as pedagogical tools for shifting the responsibility for making sense of ideas onto PTs.

Small Group/Partner Mathematics Discourse

One disadvantage to whole-class mathematical discourse is that time constraints can limit how many PTs have an opportunity to share their thinking. Pausing whole-class discourse to turn and talk to a partner or in a small group is one way to increase PTs' opportunities to share their thinking. Small group mathematics discourse allows for more PTs to participate and engage in mathematics discussions (Charalambous & Litke, 2018; Coakley, 2018; Hung, 2015). A "Think-Pair-Show-Share" model can be used after PTs have had an opportunity to engage in individual productive struggle with a mathematics tasks (Hunt, MacDonald, Lambert, Sugita, & Silva, 2018). In this model, PTs create a representation to model their thinking and share this model with a partner or small group. Delaying the discussion until after students have an opportunity to work individually can provide PTs with sufficient individual time to process and write down ideas they may want to share (Walter, 2018). One misconception PTs may have it that "mathematics questions only require a short time to solve" (Horn, 2012, p. 5). Allowing time for PTs to engage with problems that demand a greater depth of thinking can help them focus on the tasks and provide opportunities to make important connections (Horn, 2012).

MTEs can also provide opportunities to PTs to work in small cooperative groups while solving mathematics tasks. In cooperative learning, each PT is a valued member of the team as they bring their own expertise to the group. Sometimes during group work, one PT emerges as the leader or takes over the majority of the work. This can lead to inequality in status and framing of certain PTs as experts. MTEs can support PTs who struggle in mathematics by helping the group recognize the competence that each PT brings to their group discussion and collaboration (Cohen, Lotan, Scarloss, & Arellano, 2009). PTs who recognize that everyone can bring their own competence and contribution to a group will be better prepared to participate in professional learning communities with teachers of varying levels of knowledge and teaching experience as well as work with students in their own classrooms.

MTEs will want to avoid “stepping into” a conversation without fully understanding the extent of the full conversation. When an MTE does not know the full extent of the conversation, they may be tempted to share an idea or strategy, only to discover that the group has already talked about that strategy and is currently evaluating other possible solutions. Some groups may get stuck and ask the MTE for help. The MTE should first evaluate whether this is a group question (i.e., everyone in the group is confused) or an individual question (i.e., one person is confused). Individual questions should be directed back to the group first, as other members of the group may already have an answer. When answering a group question, the MTE should first use questioning techniques to understand the full nature of the confusion before providing one possible idea or scaffold to help the group succeed.

Mathematics discourse can have many benefits for both MTEs and PTs. It allows PTs to cognitively evaluate their own understanding of the lesson as well as opportunities for both MTEs and PTs to evaluate alternative strategies put forth by other PTs in the class, and evaluate a variety of examples and counterexamples (Georgius, 2014; Williams, 2010). Additionally, classroom discussions can provide MTEs with an informal assessment of PT’s understanding of mathematics concepts and procedures which can help MTEs adapt their instruction.

Conclusion

In this article, we recommended two strategies to promote active learning: 1) Selecting tasks that promote reasoning and problem solving, and 2) facilitating meaningful mathematical discourse. When MTEs use active learning strategies with PTs in their mathematics content courses, PTs can develop productive images for what it means to teach and learn mathematics

(CITE). As shown in the previous sections, active learning does not mean adding content to an already limited schedule. Rather, MTEs select or adapt tasks that help to promote PTs' reasoning and problem solving. MTEs can help PTs construct viable arguments, critique the reasoning of others, and facilitate meaningful mathematical discourse to enhance their PTs' content knowledge while also modeling the instructional strategies these future teachers will be expected to use in their mathematics classrooms. By offering PTs valuable active learning experiences during their mathematics content courses and highlighting how these experiences can enhance both PTs' own understanding and their future students' mathematical understanding, MTEs will provide a valuable foundation for PTs to meet the expectations for teaching and learning in mathematics classrooms.

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