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Using Transformative Learning Theory to Help Prospective Teachers Learn Mathematics That They Already "Know"

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Abstract: *Prospective Teachers often enter their mathematics content courses believing that they know enough mathematics to teach elementary school. However, research has shown that much of PTs' knowledge is procedurally based and lacks depth and conceptual understanding. One job of Mathematics Teacher Educators (MTEs) in mathematics content courses is to help PTs become more mathematically proficient, by relearning the mathematics that they believe they already know in deeper, more connected ways. We suggest that one way for MTEs to do this is to incorporate Transformative Learning Theory (TLT) into their mathematics content courses. TLT is an application of andragogy, which are the methods or techniques used to teach adults. Through TLT, learners participate in a process where they are presented with a disorienting dilemma that perturbs their prior understandings. Learners work through the dilemma by critically reflecting on their prior understandings and participating in rational discourse with others. Ultimately, learners are tasked with making connections between their prior understandings and their new knowledge. In this paper, we describe a cycle of transformative learning theory and give examples of incorporating TLT into mathematics content courses for PTs through lessons on proportional reasoning and whole number concepts. We conclude by discussing general considerations and resources for incorporating TLT into mathematics content courses and how this helps PTs develop the five strands of mathematical proficiency (National Research Council, 2001).*

Keywords: transformative learning theory, andragogy, preservice teacher education, mathematics teacher educators, mathematical proficiency

Introduction

Research (e.g. Thanheiser, Browning, Edson, Lo, Whitacre, Olanoff, & Morton, 2014) has shown that prospective elementary teachers (PTs) enter their mathematics content courses with predominantly procedural understandings of mathematics. However, in their work as mathematics teachers, PTs will be required to know and understand more than just *how* to solve mathematics problems (AMTE, 2017). They should possess a "robust knowledge of the

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mathematical and statistical concepts that underlie what they encounter in teaching" (AMTE, 2017, p. 6). However, PTs often believe that they already know the elementary mathematics that they will need to teach (Thanheiser, 2018). Therefore, it is the job of Mathematics Teacher Educators (MTEs) to help PTs see the need for and develop the specialized mathematics content knowledge that they will need to use in their work as teachers. Transformative learning theory (TLT) (Mezirow, 1991) provides a model through which MTEs can help PTs move from shallow perceptions of "knowing mathematics" to conceptual understandings with deeper procedural understandings of the mathematics they will teach. In transformative learning theory, the learner is presented with a disorienting dilemma where their preconceived ideas are challenged and perturbed. These disorienting dilemmas provide experiences where known procedures are not enough to solve the problem and they allow any lack of depth in PTs' understanding to surface. Learners are asked to work through the dilemma by reflecting on their prior assumptions through rational discourse. Discourse is *rational* when it seeks to create understanding with another, it is driven by objectivity, actions and statements are open to question and discussion, and understanding is arrived through the weighing of evidence and measuring the insight and strength of supporting arguments (Mezirow, 1991). Rational discourse involves learners justifying their solutions to peers in order to ultimately transform and deepen their knowledge. Finally, their new knowledge is put into action by making connections to why and how their previous assumptions were not enough to solve the problem.

In mathematics content courses, students (PTs) are asked to revisit mathematics that they learned in grades K-12. Although they have had prior experiences with the content, they often have underlying misconceptions about elementary mathematics concepts or lack the depth of understanding necessary to teach the content (Thanheiser et al., 2014). If these "gaps" in

understanding are not addressed in teacher preparation courses they will have to be confronted when PTs become inservice teachers. Through our research, (Johnson, 2013), we have evidence of PTs being able to *transform* their misconceptions with an intervention during individual, task-based interviews. Because of our success using TLT with individual PTs, we propose that transformative learning theory should be incorporated into mathematics content courses in order to help groups of PTs to transform their learning.

In this paper we describe how using TLT in mathematics content courses can enhance PTs' learning. We do so by first discussing the potential goals of mathematics content courses. Then we consider how adults learn differently from children and describe a cycle of transformative learning theory as it relates to mathematics content courses. We also give specific examples of how we incorporate TLT into our mathematics content courses through lessons on place value and proportional reasoning and offer implications for using TLT (by other MTEs) in mathematics content courses. We conclude with specific suggestions and resources for MTEs to help better incorporate TLT into their courses for PTs.

Theoretical Framework

Prospective Teachers and Mathematics Content Courses: The Need for Transformed and Reformed Learning to Develop Mathematical Proficiency

Although they are called mathematics content courses, we assume that PTs enter these courses already familiar with the majority of the content of the elementary curriculum. However, this familiarity is generally based on procedures with shallow understandings. Additionally, many PTs hold beliefs about what mathematics is that are based predominantly on these procedural understandings (Ball, 1990; Brady & Bowd, 2005; Johnson, 2019; Thanheiser, 2018). Therefore, we believe that a major goal for mathematics content courses is to help PTs become more "mathematically proficient" (National Research Council, 2001). In their 2001 book, *Adding*

it Up: Helping Children Learn Mathematics, the National Research Council (NRC) defines five interconnected strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. While the main intent of the book is to help children become mathematically proficient, the NRC contends that mathematical proficiency is also important for teachers: "If their students are to develop mathematical proficiency, teachers must have a clear vision of the goals of instruction and what proficiency means for the specific mathematical content they are teaching" (NRC, 2001, p. 369). The need for teachers to be mathematically proficient is echoed in policy documents regarding teacher knowledge and preparation (e.g., AMTE, 2017; Conference Board of Mathematical Sciences, 2012). Research suggests, however, that many practicing and prospective teachers still need to develop proficiency in each of the five strands (NRC, 2001). In this section, we will briefly discuss the five strands and what we know from research about PTs' abilities in each of these areas. In the following sections, we will examine ways to help PTs develop their mathematical proficiency using TLT.

Conceptual understanding refers to "comprehension of mathematical concepts, operations, and relations" (NRC, 2001, p. 5), whereas *procedural fluency* is defined as the "skill in carrying out procedures flexibly, accurately, efficiently, and appropriately" (NRC, p. 5). Research has shown that many PTs enter their teacher preparation programs with only shallow procedural fluency, and little conceptual understanding (Thanheiser et al., 2014). One reason for this could be based on the experiences that PTs have before they enter their preparation programs. The first author of this article conducted a survey of more than 300 PTs enrolled in their first mathematics content course (Johnson, 2019). In regards to their prior education experiences, 73% of the PTs said that they had been taught mathematics with a focus on

procedures, and 61% said that they were taught with an emphasis on memorizing a sequence of steps to solve problems, while only 38% reported being taught with an emphasis on understanding underlying concepts. It is essential that mathematics content courses help PTs move beyond memorizing procedures and develop both conceptual understanding and what Star (2005) calls *deep procedural knowledge*, which he defines as knowledge "associated with comprehension, flexibility, and critical judgement" (p. 408) in regards to choosing and using procedures to solve mathematical problems.

While much emphasis in the past has been placed on the importance of developing conceptual understanding and procedural fluency (e.g., Hiebert, 1986; Star, 2005), the NRC contends that the other three strands are equally important in the development of mathematical proficiency. These strands are newer ideas based on research in mathematics education, cognitive psychology, and the researchers' personal experiences in teaching and learning mathematics (NRC, 2001). *Strategic competence* is defined as the "ability to formulate, represent, and solve mathematical problems" (NRC, 2001, p. 5). The AMTE (2017) and NRC (2001) emphasize that students need experience not only with solving problems but also with posing problems to deal with real life situations. Other research has shown that success in problem posing can lead to better problem-solving skills (e.g., Abu Elwan, 2016; Silver, 1994). PTs will need to know how to pose problems as part of their work as teachers. However, research has shown that PTs often struggle to pose problems, (Işık, Kar, Yalçin, & Zehir, 2001), but that this skill can be improved upon with opportunities to explore what makes a good problem and practice with both posing and solving problems (Crespo, 2003; Crespo & Sinclair 2008). Mathematics content courses for PTs must provide them with opportunities to develop

their strategic competence by posing, representing, and solving mathematical problems (AMTE, 2017).

Adaptive reasoning is the "capacity for logical thought, reflection, explanation, and justification" (NRC, 2001, p. 5). This involves being able to think through a problem and understand whether or not one's solution is valid based on using valid reasoning. Adaptive reasoning also involves being able to explain one's solution to others, another important task for teachers. The AMTE (2017) recommends that PTs be given opportunities to develop their skills in reasoning, explaining and justifying. Research has shown that throughout their teacher education programs, PTs often struggle with reasoning and justification (Courtney-Clarke & Wessels, 2014; Son & Lee, 2016; Thanheiser et al., 2014). Therefore, it is important that PTs have opportunities to develop their adaptive reasoning skills in their mathematics content courses.

Productive disposition fits in with other research that looks at beliefs, orientations, or mindsets towards mathematics (e.g., Boaler, 2013, 2016; Fennema & Franke, 1992; Grossman, 1990, 1991) and is defined as the "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (NRC, 2001, p. 5). Thus, part of being knowledgeable in mathematics means finding the topic worthwhile and also believing that one is capable of learning and doing mathematics. Boaler's (2013, 2016) work on fostering a "growth mindset," where one believes that one's intelligence can be learned and can grow, coincides with believing that one is capable of learning and doing mathematics or fostering a productive disposition.

Despite the fact that many PTs believe that they know enough mathematics to teach it effectively (Thanheiser, 2018), many PTs have been found to hold negative beliefs towards

mathematics (Barrantes & Blanco, 2006; Brady & Bowd, 2005; Johnson, 2019; Jong & Hodges, 2015; Uusimaki & Nason, 2004). Additional research suggests that teachers' productive dispositions will influence their students' dispositions towards mathematics (e.g. Boaler, 2013; Philipp & Siegfried, 2015). We believe that helping PTs develop a productive disposition toward mathematics, while simultaneously helping them to discover that they need to understand mathematics content in deeper, more meaningful ways should be one of the goals for MTEs of mathematics content courses.

An important note about the NRC's five strands of mathematical proficiency is that the researchers look at these strands as "interwoven and interdependent" (NRC, 2001, p. 5). They depict the strands as being intertwined and emphasize that each of the strands works together to support one another. Thus, the five strands must all combine for true mathematical proficiency to exist. PTs enter mathematics content courses with varying degrees of proficiency of each of these strands; however, we believe that our jobs, as MTEs, is to help PTs develop each of the five strands in order to help them become mathematically proficient and pedagogically competent to teach mathematics to children. In the next section, we will discuss how incorporating TLT into mathematics content courses may help MTEs to meet these goals.

Prospective Teachers as Adult Learners: The Importance of Andragogy

Prospective teachers are adults and, therefore, learn differently than children. Malcom Knowles researched this idea in the 1980s and proposed a model of adult learning called *andragogy*. (Andragogy, the methods of teaching adults contrasts with pedagogy, the methods of teaching children.) Knowles' definition of andragogy involves four interconnected assumptions about adult learners: changes in self-concept, the role of experience, the readiness to learn and

their orientation to learning (Knowles, 1984). Below, we discuss each of the four assumptions, and the implications that they involve for MTEs and mathematics content courses.

1. **Self-concept.** This assumption is that as a person grows and matures, their self-concept moves from one of total dependency (as is the reality of the infant) to one of increasing self-directedness. Adults may be more likely to resist or resent instances when others impose their will upon them (Merriam & Bierema, 2014). In a mathematics content course this can mean giving students opportunities to direct and be responsible for their own learning (Springer, Stanne, & Donovan, 1999). An example of how this would work in a mathematics content course is by providing time for PTs to critically reflect on their own misconceptions and assumptions and discuss and debate with others' pathways to resolve their misconceptions.
2. **Role of experience.** Adults enter a learning situation with a wealth of experience. This may serve as a resource to make learning meaningful. As MTEs we should take advantage of students' prior knowledge about mathematical topics because they can act as building blocks for the development of new knowledge (Whitacre, 2013). However, we must also be aware of how prior assumptions can act as barriers to this development (Kagan, 1992; Slotta, Chi, & Joram, 1995), and determine ways to challenge the assumptions that are providing the barriers. It is additionally important for MTEs to honor PTs' prior experiences with mathematics because, unlike children, adults tend to identify themselves based on their previous experiences, and rejecting those experiences usually means rejecting the person (Harkness, Ambrosio, & Morrone, 2007).
3. **Readiness to learn.** Unlike many children, adults need to know the utility and value of the content they are learning and how it applies to them and their future careers

(Knowles, Swanson & Holton, 2005). Tough (1979) argues that the first task of a teacher of adult learners is to help them become aware of the need to know. An example in a mathematics content course would be helping PTs gain perspectives of how children view mathematics and how their own understandings of mathematics will directly affect their future students (Karp, 1991). For PTs this means that they need to see the application of the mathematics content they will be learning. Therefore, MTEs should be explicit with PTs about the importance and rationale of knowing more than procedures. MTEs should help PTs to become motivated to learn by emphasizing this necessity of the knowledge in their future careers.

4. **Orientation to learning.** Adults are life-centered and/or problem-centered in their desire to learn. Thus, adults learn best and are motivated more when knowledge skills and attitude are presented in the context of real-life problem solving. (Knowles, et al, 2005). MTEs can build upon this motivation by focusing mathematics content courses around presenting problems that emphasize learning mathematics in deeper, more conceptual ways than they may have previously understood it. For example, research has shown that PTs struggle to justify (Courtney-Clarke & Wessels, 2014; Son & Lee, 2016; Thanheiser et al., 2014). Asking PTs to explain their thinking orients them to the difficulties they face when explaining their thinking to future students.

In her dissertation, Deborah Zopf (2010) also discussed andragogy, although she did not name it as such. She discussed three main differences between the work of mathematics teachers and the work of mathematics teacher educators: "First the mathematical content is different" (p. 5). While the job of teachers is to teach students mathematics, the job of the MTE is to teach mathematical knowledge for teaching teachers. "Second, the learners are different" (p. 5). Zopf

points out the difference between teaching children and adults. Each group brings different experiences and prior knowledge to the classroom and it is the job of their teacher to work with these experiences and help their students to construct new knowledge. “Third, the purposes of instruction are different. Children learn mathematics for their own use; teachers learn mathematical knowledge for teaching to teach mathematics to students” (p. 6). Helping teachers unpack mathematics in a way that will help them make sense of it to present it to students requires different work than helping students make sense of mathematics; thus the role of the MTE is to provide opportunities for PTs to experience such activities.

Transformative Learning Theory

Transformative learning theory (TLT) is a model of andragogy that attempts to reveal and clarify a learner’s prior assumptions and then transform these assumptions into new understandings (Mezirow, 1991, 2012). TLT was developed by Jack Mezirow in the mid to late 1980s and early 1990s. He based his initial theory on a study of 83 women returning to college in 12 different reentry programs in 1975. Mezirow’s initial transformation process included ten phases:

1. Experiencing a disorienting dilemma;
2. Undergoing self-examination;
3. Conducting critical assessment of internal assumptions and feeling alienation from traditional social expectations;
4. Relating discontent to the similar experiences of others. In other words recognizing the that problem is shared;
5. Exploring options for new ways of acting;
6. Building competence and self-confidence in new roles;

7. Planning a course of action;
8. Acquiring the knowledge and skills for implementing the new action;
9. Trying new roles and assessing them;
10. Reintegrating into society with the new perspective (Cranton, 2006, p. 20).

Mezirow wrote about and amended his theory of adult learning and development in articles beginning in the mid-1980s and continuing for over 30 years. While the initial theory included the ten phases above, later versions used by Mezirow (e.g., Mezirow, 1991; 2000) and others (e.g., Dirkx & Smith, 2009; van Halen-Farber, 1997) included a subset of these phases.

Key ideas in TLT include the notions that "we transform our frame of reference through critical reflection on assumptions" (Mezirow, 1997, p. 7), and that "rational discourse through communicative learning" is a key concept in understanding (Mezirow, 1991, p. 78). Mezirow says that this reflection and discourse often take place within the context of problem solving (Mezirow, 1994). His theory claims that only after learners are aware of their assumptions can they develop strategies to transform these assumptions (Mezirow, 2012). This suggests that in order for PTs to make shifts in their understanding and become proficient at the mathematics content, they need to have their reasoning challenged in ways that encourage reflection on their prior assumptions about mathematics.

Since the introduction of TLT, it has been utilized and studied in a variety of contexts ranging from individuals to classrooms to whole communities. Publications have included descriptions of interventions in disciplines such as teacher education for environmental studies (Burns, 2009; Caldarelli, 2004; Hashimoto, 2007), and professional preparation for teacher education (van Halen-Faber, 1997). Other key professional education curricular interventions have focused on medical and nursing education (Hanson, 2010; Morris & Faulk, 2007), school

administration (Brown, 2006; Donaldson, 2009), and social work (Lee & Greene, 2004). The First International Transformative Learning Conference was held in 1998 at Teachers College, Columbia University, with the purpose of inviting those interested in transformative learning to share their research and also to provide learning activities and conversations around TLT with participants. This conference has been held annually or bi-annually since then. "Each conference has contributed to Mezirow's initiating vision of transformative learning theory as a 'theory in progress' by developing diverse perspectives on transformation learning theory, research and practice" (Kasl, 2014, p. 20).

Kasworm and Bowles (2012) reviewed 250 published reports on TLT in higher education settings including faculty development (Gravett and Peterson, 2009), mentoring settings (Mandell & Herman, 2009), and experiential learning (Deeley, 2010). They note that success in higher education is a natural site for transformative learning theory to occur because "ideally, higher education offers an invitation to think, to be, and to act in new and enhanced ways. . . These learning environments sometimes challenge individuals to move beyond their comfort zone of the known, of self and others" (Kasworm & Bowles, 2012, p. 389).

Dirkx and Smith (2009) explored the use of TLT in adult, online classrooms. They identified the following six design and instructional strategies that appeared to foster "deep" or transformative learning: "(1) use of messy, ill-structured practice-based problems as the central pedagogical focus; (2) interactive and collaborative learning; (3) use of consensus group writing teams; (4) individual and team debriefings; (5) reflective activities; and (6) journal writing" (p. 60).

Van Halen-Farber (1997) designed transformative learning experiences for PTs to get them to become "reflective practitioners" (Schön, 1987). The basis for these experiences

revolved around *critical reflection*, which “occurs when patterns of a person’s beliefs, goals, or expectations are put to the test by means of thoughtful questioning” (van Halen-Farber, 1997, p. 51-52). In a four-step process, prospective teachers listened to experienced teachers’ stories, reflected on these stories, wrote about significant incidents in their own practice, and then wrote a personal reflective essay where they were asked to “summarize and synthesize their thoughts on what it means to describe one’s own learning journey in becoming an intentionally thoughtful teacher” (p. 55). Van Helen-Farber noted that through the process, the PTs were able to increase awareness of their own learning processes and also their self-confidence and ability to become more self-directed learners. Additionally, the researcher, who was also the teacher educator, was able to transform her own learning about how to help PTs become reflective practitioners.

Chapman and Guerra (2016) used TLT in college-level developmental mathematics courses to help their students transform how they saw themselves as learners and doers of mathematics. They noted that many students who take developmental mathematics “associate math with feelings of anxiety or shame, and see themselves as not only ‘bad at math’ but also ‘bad at learning’ or intellectually deficient” (p. 694). If students believe that they are not capable of learning mathematics, this becomes a self-fulfilling prophecy, so it is imperative for instructors of developmental mathematics courses to work to transform their students’ negative self-identities. In order to facilitate this transformation, Chapman and Guerra work to develop a safe classroom environment where the teacher plays a role of “learning companion,” who shares the transformative learning experiences with her students. In order to develop trust, instructors share personal experiences about their own mistakes and how they were able to learn from them. Then the instructors provide students with activities that focus on problem solving rather than computational ability. “These problem solving activities often invite vigorous debate, creative

solutions, and teamwork...All of this serves to create an atmosphere of safety where risk-taking is encouraged and creativity is valued" (p. 696-697).

While mathematics content courses for PTs are not generally categorized as remedial or developmental, a significant percentage of PTs do report experiencing math anxiety or negative experiences with mathematics (Brady & Bowd, 2005; Johnson, 2019; Karunakaran, 2020; Rayner, Pitsolantis, & Osana 2009). Therefore, our work as MTEs involves helping PTs' develop productive dispositions by transforming their understandings of their own abilities to learn and do mathematics. Additionally, we must help PTs transform their ideas about what mathematics is, in order to move from procedurally-based understandings to deeper understandings based on a knowledge of concepts.

Using Transformative Learning Theory in Mathematical Situations with Prospective Elementary Teachers

While there are numerous examples of using TLT in university settings and with PTs (see above), there has yet to be published literature on using TLT with PTs in mathematics. In her dissertation study, the first author interviewed PTs in individual settings in relation to solving proportional reasoning problems (Johnson, 2013). The interview questions were designed to disorient PTs' thinking about proportions and cause them to question their prior assumptions about the topic. The interviews also provided the PTs an opportunity to engage in rational discourse with the interviewer as well as to critically reflect on their prior assumptions which led them to revise these assumptions. That is, the process of participating in the interviews helped PTs to transform their understandings of proportions.

As a result of her dissertation work, and based on the TLT literature, Johnson (2016) designed a model to illustrate the TLT cycle as it relates to transforming PTs' knowledge. In Figure 1 below, we include the cycle and then we describe the four stages in the cycle.

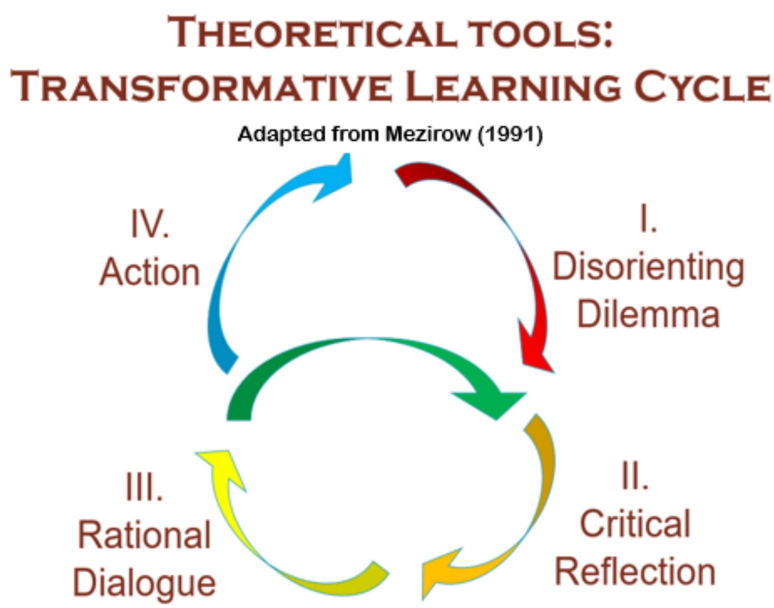


Figure 1. The cycle of transformative learning theory.

The cycle begins with a *disorienting dilemma*, such as a mathematical task that makes PTs aware of and challenges their prior assumptions. This task could be one where their procedural knowledge is not enough to solve a problem or where their conceptual understanding is challenged. The role of experience in andragogy directly correlates with the disorienting dilemma since the purpose of the dilemma is to bring PTs' prior assumptions to the surface. Work on the disorienting task sets up the second part of the cycle, *critical reflection.* Critical reflection involves PTs becoming aware of their previous assumptions and then challenging the validity of this prior knowledge. The critical reflection phase correlates with PTs' readiness to learn and orientation to learning, as they realize that their prior knowledge often is not enough for them to be able to explain mathematics in ways necessary to teach it to others, which should be the ultimate goal for PTs. The third part of the cycle provides PTs with time to engage in *rational discourse* with their peers as they resolve the dilemma. This phase of the cycle is where PTs seek to create understanding through their discussions with each other. The rational

discourse part of the cycle gives an opportunity for the PTs to justify their reasoning about a misconception or discuss how the task is challenging their conceptual understanding of the topic. Often, time must be spent moving back and forth between the second and third phases of the cycle, allowing PTs to engage in discourse and then critically reflect on the discussion. As the PTs become aware of different aspects of the mathematics and their prior assumptions are challenged, they will need to reflect on and participate in more discourse in order to help them resolve their disequilibrium. Both the second and third phases are critical in helping PTs develop their strategic competence and adaptive reasoning (NRC, 2001), as they work to solve the problems and reflect, explain, and justify their solutions.

The fourth and final part of the cycle is *taking action on the previous assumptions*. It involves providing time for PTs to discuss how their prior assumptions fit with their transformed understanding. Explicitly making the connections between the prior knowledge and the deeper concepts allows the PTs to become more mathematically proficient. It is important that this final step not be skipped or overlooked in teaching mathematics where the temptation could be to move on once one gets a correct answer. Without explicit discussion on the connections between concepts, PTs' equilibrium may not be reached. Working through the cycle may also help to improve PTs' self-concepts, as they can come to realize that they are capable of thinking mathematically and solving problems on their own. Figure 2 helps to illustrate the connections between TLT and the four assumptions of andragogy and five strands of mathematical proficiency.

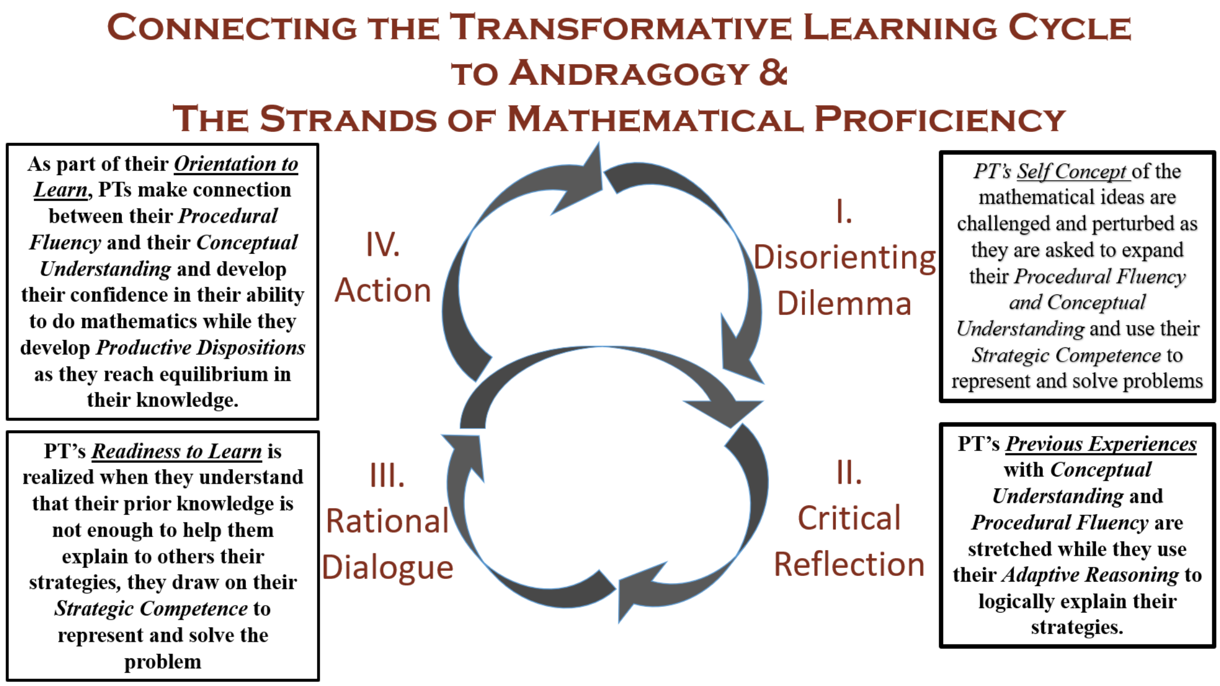


Figure 2. Model connecting TLT cycle to assumptions of andragogy and five strands of mathematical proficiency.

Applying Theory into Practice

The first challenge of incorporating TLT into mathematics content courses involves finding tasks that will create disorienting dilemmas. In this section, we outline three ways to provide disorientation, and then discuss in-depth examples of that use TLT with PTs. One way to cause disorientation for PTs is to ask them to solve tasks with which they may be familiar, but restrict their use of familiar procedures. This will encourage their development of conceptual understanding, as they must make sense of other ways to solve the problems. A second way of causing disorientation is to change aspects of a familiar situation to take away PTs' automaticity. Many early topics in elementary mathematics like counting and adding, are things that adults, including PTs, do without much thought about what they are doing. MTEs can cause disorientation by removing this automaticity and asking PTs to look at something familiar in an

unfamiliar way. A third way to cause disorientation is to introduce PTs to an unfamiliar way to do a familiar thing. This could include asking PTs to interpret non-standard algorithms that are either used in other countries or other contexts or invented by children. In the examples below, we detail two examples of how we use disorienting dilemmas to help PTs transform their learning of proportional reasoning and whole number concepts.

Classroom Example: Transformative Learning Theory and Proportional Reasoning

We present a specific example of incorporating TLT into mathematics content courses that disorients PTs by restricting their use of familiar procedures. This task involves proportional reasoning. We chose this topic because proportional reasoning is an area where PTs often have significant difficulty but feel that their understanding is sufficient in order to teach. For example, research shows that PTs' understanding of proportions is often based on the use of procedures without understanding (Hillen, 2005; Johnson, 2013; Simon & Blume, 1994; Smith, Stein, Silver, Hillen, & Heffernan, 2001). It is essential that we as MTEs help PTs recognize their prior assumptions about their understandings in order to help them transform and develop a deeper knowledge of proportional reasoning. We begin with a task that provides a disorienting dilemma. See Figure 3 below. As mentioned above, this task is disorienting for PTs because it asks them to solve the problem without using the procedures with which they are familiar. Our experience has

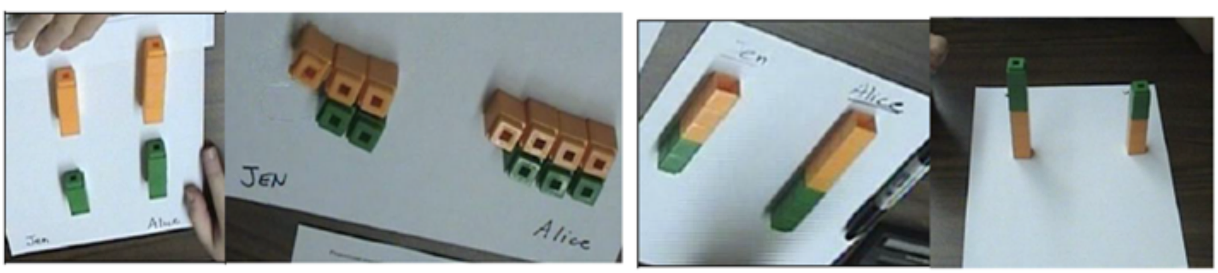
Lemon/Lime Problem

Jen and Alice are making lemon-limeade. Jen mixes 3 cups of lemon juice with 2 cups of lime juice. Alice mixes 4 cups of lemon juice with 3 cups of lime juice. Whose mixture is more lemony?

Justify your conclusion WITHOUT using calculations. Use the cubes to help you demonstrate which one has more lemon taste or that they taste the same.

Figure 3. The lemon/lime problem.

shown that most PTs can solve this problem quantitatively by changing the ratios to decimals or percentages, but by removing these procedures as an option, we disorient them and require them to reason qualitatively. Qualitative reasoning is defined as the ability to express conceptual knowledge using models as a means to externalize the context of a problem (Liem, Beek, & Bredeweg, 2010). In this problem, we provide PTs with yellow and green cubes to use to stand for the cups of lemon and lime juice. Qualitative reasoning is utilized when PTs interpret the manipulatives in the context of cups of lemon and lime juice and then explain how to compare the mixtures (Heller, Post, Behr, & Lesh, 1990). By not allowing PTs to use procedures (such as setting up a proportion to compare the mixtures), we disorient their thinking and require them to build on their conceptual understanding of the meaning of ratios and proportions. This task also can help them develop their *strategic competence and adaptive reasoning,* as they must discover new ways to represent the situation and justify their reasoning. Figure 4 shows pictures of some PTs' cubes. It is interesting to note that while the PTs on the left separated the yellow and green cubes, the PTs on the right put them together in one line.



(a) (b) (c) (d)

Figure 4. PTs' work with unifix cubes on the lemon-lime problem.

As PTs move into the second phase (critical reflection) of the TLT cycle with this task, they reflect on how they need to explain and justify their solutions (*adaptive reasoning*). At this point, most students have created the models seen in figure 3, and either concluded that the

mixtures are the same or that one of the mixtures is "more lemony" but cannot explain why. We help motivate PTs to determine a justification for their solution by reminding them that when they are teachers, explaining will be part of their job (i.e., we are encouraging their readiness and orientation to learn). Despite the motivating factor, many PTs continue to struggle to understand *why* their solution is true or have difficulty putting their understandings into words, which is essential in the rational discourse step and in developing their adaptive reasoning.

While PTs engage in critical reflection, they begin rational discourse with their peers. Often misconceptions are brought to the surface as they discuss their ideas with others. For example, it is common for some PTs to initially believe that the mixtures are the same because they each have one more lemon than lime (Johnson, 2013). This is an example of using additive reasoning: *the difference is one in each*, versus multiplicative reasoning: *the first mixture has 1 ½ times as much lemon as lime, while the second has 1 ⅓ times as much*. As students debate why the first mixture is more lemony than the second mixture, a deeper understanding of the problem involving these proportional reasoning ideas emerges. Those who had focused on additive reasoning begin to see why it is not appropriate in this situation. Through their discussions, PTs start to gain *conceptual understanding* and expand their *adaptive reasoning* skills through logical thought, explanation and engaging in justification, as well as develop the necessary vocabulary to talk about proportions The discussions also help them recognize how difficult it is to explain their reasoning despite this being something that they will need to do in their work as teachers. These realizations provide extra motivation to learn the material in this way, provoking PTs' *orientations to learning*.

Lastly, we enter the most important part of the cycle, taking action. This involves making connections and subsequently help PTs arrive at equilibrium in their thinking. At this stage, we

encourage the PTs to use procedures or complete any calculations they wish to determine which mixture is lemonier. For those students who are still not convinced by the arguments with the cubes, the numbers seem to have an important impact. However, it is essential to have the students not just "check" their qualitative reasoning with procedures but connect these calculations to the arguments made with the cubes as well as to the discussions about proportional reasoning.

For example, one way PTs solve this problem using calculations is by creating fractions with "common denominators" to compare the mixtures. In this example, the denominator could represent the amount of lime, so the "common denominator method" equates to making the amount of lime in each mixture equal. PTs would say, *Jen's mixture, 3/2, becomes 9 lemons /6 limes, while Alice's mixture, 4/3, becomes 8 lemons/6 limes. Since 9/6 is greater than 8/6, then the first mixture is more lemony*. Connecting back to the problem, this means that when there are 6 cups of lime in both mixtures Jen's mixture will have 9 cups of lemon and Alice's mixture will only have 8 cups of lemon. Nine cups of lemon is more than 8 cups of lemon, so Jen's mixture is more lemony. We can model this situation with the cubes by making three batches of the first mixture and two batches of the second mixture. See Figure 5 for an illustration of this example.

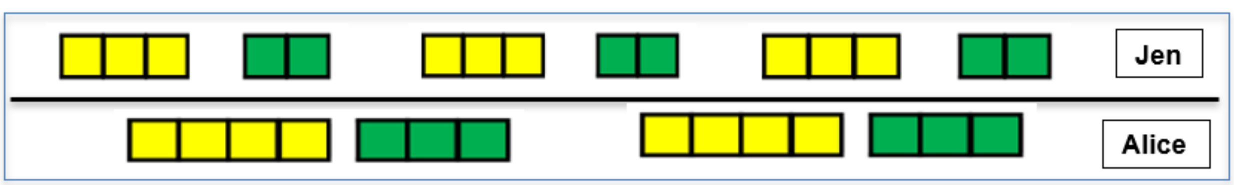


Figure 5. Three copies of mixture 1 and two copies of mixture 2. each have 6 green cubes, but mixture 1 has 9 yellow cubes, while mixture 2 only has 8.

It is by closing the loop of the cycle that the PTs are able to arrive at some form of equilibrium, where they can build connections between their new knowledge and their prior

understandings to deepen their proportional reasoning. It is also where the connections between the various strands of the mathematical proficiency become explicit. As the PTs were involved in activities of problems solving, they were able to use logical thought to reflect and justify their thinking (*adaptive reasoning*) as well as to explain their representation of the situation using blocks (*strategic competence*). They also illustrated their comprehension of the mathematical relationship between the two proportions (*conceptual understanding*) through the explicit justifications made to determine the lemonier mixture. As the loop is closed, they reflect back to their use of calculations to justify (*procedural fluency*) the concept and hopefully recognize the beauty and usefulness of mathematics in the world (*productive disposition*). They have gone from disorientation and disequilibrium to discussion and debate and finally to equilibrium, transforming their previous assumptions about proportional reasoning in the process (Johnson, 2013). Though the PTs may have initially been able to solve the problem with a procedure, most were not able to explain how or why the procedure worked in the context of the situation. Keeping the focus on the practical need for their knowledge of this topic in their future teaching, and stressing how it is necessary for them to develop a deeper understanding so that they can help to develop their future students' aptitude in proportional reasoning provides the necessary motivation and readiness to learn for adult learners.

Classroom Example: Transformative Learning Theory and Whole Number Concepts

Our second example focuses on disorienting PTs by taking away their automaticity in terms of counting in a positional number system. While research has shown that PTs have struggled with proportional reasoning and multiplicative thinking (Simon & Blume, 1994; Smith et al., 2001), we assume that all PTs enter their teacher preparation programs already knowing how to count, add, and subtract. However, for most PTs, these processes are so automatized that

they apply them without considering why our number system works the way that it does. Additionally, Thanheiser (2009; 2010; 2018) found that the majority of PTs that she studied had difficulty with place value concepts and the meaning of regrouped digits. Of more concern, Thanheiser (2018) found that in surveys of 97 PTs enrolled in mathematics content and/or methods courses, 67% believed that they had sufficient knowledge to teach mathematics to students in grades K-3, including 64% of the 81 PTs whose conceptions of place value were categorized as incorrect. If these difficulties and beliefs are not addressed during their teacher preparation, PTs will continue to hold these views as teachers. It becomes important for MTEs to help PTs become aware of their prior assumptions about counting and whole number operations that they may not be conscious of so that they are able to teach these topics to children.

McClain (2003) and Yackel, Underwood and Elias (2007) describe success in helping PTs develop conceptual understandings of place value by working in bases other than base-10. (In the case of these examples, both groups of researchers used base-8.) We describe another TLT activity that uses a base-5 place value system called Alphabitian[1].

We begin the Alphabitian activities with the disorienting dilemma of introducing a new numeration system. This system is disorienting, because PTs must not only use a different base system than they are used to, but also, the digits in the new system are 0, A, B, C, and D, rather than 0, 1, 2, 3, 4, as would be used in a traditional base-5 system. After being introduced to the new system, PTs are asked to count using Alphabitian numbers. We ask PTs to count in Alphabitian to help them become aware of how their future elementary students will learn to count in base 10. Changing the base system and digits forces PTs to critically reflect on the meaning of place values and why we move to a new column when we run out of digits in one

[1] Note that these activities were adapted from Bassarear, 2005, Section 2.3.

column. The rational discourse stage comes when PTs work together to figure out how to count in the new number system. The first four numbers are relatively straightforward, but the discourse revolves around what happens when you have D in one place value column and you add A more. PTs must figure out that you need to write an A in the next place value column and put a 0 in the original place value column. The MTEs ask PTs additional questions at this point, such as, "What number comes directly after DDD? What number comes directly before A0?" During their work on the activities, we introduce PTs to different manipulatives that will be helpful in their work as they learn this new number system, such as base-5 blocks and 5-frames (illustrated below figure 6). Finally, PTs are provided with an activity where they must estimate and count multiple bags full of large numbers of cubes. This forces them to unitize the sets of 5, 25, 125 in order to determine the total number of cubes rather than attempt to write out all the Alphabitian numbers between one and the total. The connections can be made to how we group by 10s in our base system but in Alphabitian we group by 5s.

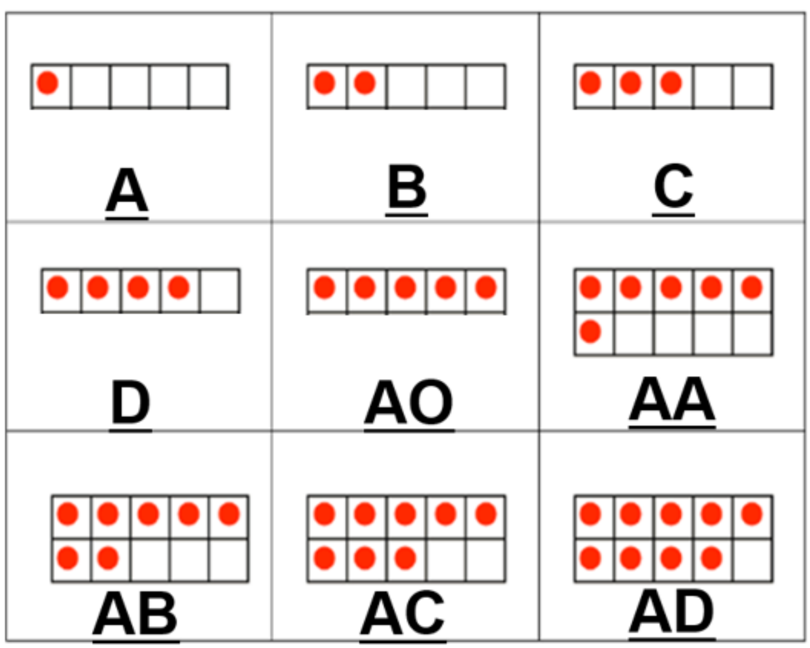


Figure 6. Using 5-frames to count in Alphabitian.

In the action phase, MTEs facilitate discussions about the connections between counting in Alphabitian and learning to count in base-10. We ask PTs about similarities and differences between the two number systems. We ask them to think about the relationship of A0 in Alphabitian with 10 in our number system. The discussion is mainly focused around transitional numbers, as these are where difficulties can arise. We also discuss how PTs' future students will struggle with learning about counting, place value, and re-grouping when adding and subtracting, just as they found the Alphabitian system to be challenging. A firm foundation of our base-10 positional system involves more than just rote counting and writing; it involves a rich understanding of what the digits mean and how these digits change based on position. Only after children have fully grasped these concepts can we move onto helping them develop an understanding of the role of place value in addition and subtraction (NCTM 2008; NRC 2009). The Alphabitian tasks allow PTs to move out of their comfort zone to struggle through learning a new counting system to explore the meaning of place value in our positional counting system. Work on this task allows PTs to develop in each of the strands of mathematical proficiency in a variety of ways. PTs are work to gain a deeper *conceptual understanding* of place value and operations using the Alphabitia activity. As the PTs use manipulatives to represent and solve math problems (*strategic competence*) they also come to justify and explain (*adaptive reasoning*) the usefulness of these tools in learning a number system. Eventually through their diligence and own self-efficacy (*productive disposition*) they are able to accurately and efficiently carry out procedures (*procedural fluency*) in this new number system. Ultimately, PTs understand that learning to count in a positional number system is not easy and as adults we take these ideas for granted, as a teacher we need to be aware of the difficulties our future students will face when we teach them.

A third way to cause disorientation is to introduce PTs to an unfamiliar way to do a familiar thing. This could include asking PTs to interpret traditional algorithms for subtracting that might be "standard" in other countries but not the same as the U.S. standard algorithm (see figure 7 b). For example, figure 7a shows a standard algorithm often taught in Mexico/Latin America. Since most PTs have not encountered this algorithm and their place value understanding is not sufficiently developed (Thanheiser, 2009; Thanheiser 2010; Thanheiser 2018) this example is disorienting for students. PTs know how to subtract whole numbers (i.e. it is a familiar thing) and usually know how to use the US standard algorithm shown in figure 7b but asking PTs to explain and justify the mathematical validity of this algorithm (*adaptive reasoning*) is challenging for them. Asking PTs to compare the mathematics of the two algorithms and discussing the similarities and differences between them develops their *strategic competence* and *conceptual understanding* of the operation of subtraction. Finally asking PTs to solve a similar problem like 936 – 478 using the alternative standard algorithm develops their *procedural fluency* and *productive disposition* as they apply their knowledge to a new situation and discover how mathematics can be done in a variety of ways.

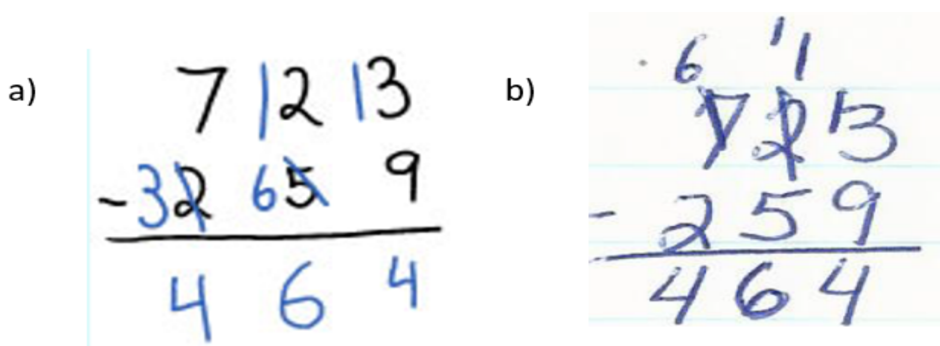


Figure 7: Example of standard algorithm for subtraction often taught in Mexico.

This type of disorienting dilemma allows PTs to make connections to something they already know in a new way. Asking them how the U.S. standard algorithm is related to the

alternative algorithm is useful in developing PTs mathematical proficiency with understanding subtraction and reminding them that as teachers they may have students in their classrooms that are from other countries and who have learned these algorithms. As such, it is essential that they understand how alternative algorithms work so that they can help their future students and not simply dismiss their work because it is different from their own thinking.

Considerations, Suggestions, and Resources for Incorporating Transformative Learning Theory into Mathematics Content Courses

Incorporating TLT into mathematics content courses can provide challenges for MTEs beyond those involved with teaching a more traditional mathematics course. The most important part of using these types of lessons in a mathematics content course is to start the term by creating an environment where students feel comfortable making mistakes and discussing misconceptions that they have about mathematics (Cranton, 2006). Many PTs are embarrassed by making mistakes in mathematics (Johnson, 2019, Harper & Daane, 1998). As MTEs, we need to allow PTs to feel comfortable with making mistakes as part of the learning process (Boaler, 2016). Beginning on the first day of class, it is important to establish norms for the classroom that focus on discussion and respect for multiple strategies to solve problems. Reminding PTs that, as teachers, they will need to listen to students' various strategies and assess the validity of their methods without being judgmental, is one way to motivate the process.

TLT can be incorporated into mathematics content courses by using it as a template for planning lessons. This involves initially choosing tasks for PTs that will provide sufficient levels of disorientation with the mathematical content that is being taught. This should be done with knowledge of the understandings that PTs bring to their content courses, including common misconceptions and areas where they struggle. (See *The Mathematics Enthusiast*, Volume 11, No 2, August, 2014 for information on PTs' knowledge of mathematics content.) The task

should allow PTs to become aware of their prior assumptions about the content and provide opportunities for them to critically reflect on their mathematical proficiency in the content area.

As discussed above, three ways of disorienting PTs are 1) restricting their use of familiar procedures, 2) removing their automaticity in performing tasks, and 3) introducing unfamiliar methods to do familiar things. The proportional reasoning task described above is an example of the first way of providing disorientation. Another example of a task that provides PTs with disorientation by restricting the use of familiar procedures can be found in Thanheiser and colleagues' fraction comparison tasks, where they designed the task in a way to discourage PTs from using common denominators. (see Thanheiser, Olanoff, Hillen, Feldman, Tobias, & Welder, 2016, and www.mathtaskmasters.com for more details of these tasks.)

Using tasks that require PTs to work in different bases is a way to remove their automaticity in counting, adding, and subtracting. Most mathematics textbooks for PTs contain activities that involve using different bases. The task that we presented above using Alphabitian is adapted from Bassarear, 2005. Additional research articles such as McClain (2003); Roy and Safi, (2008); and Yackel, Underwood, and Elias (2007) discuss activities for PTs in base-8. In these activities, the idea is for PTs to develop the automaticity in base-8 in the same way that their students will develop this automaticity in base-10.

Multiple research articles include tasks that require PTs to look at non-standard algorithms or student thinking. Castro Superfine, Prasad, Welder, Olanoff, and Eubanks-Turner (2020, this issue) provide an example of a task where PTs are asked to explain a child's non-standard method for subtraction. Philipp (1996) includes many different alternative algorithms for adding, subtracting, multiplying and dividing that are different from the U.S. standard algorithm.

Additional resources for task design include Feldman, Wickstrom, Ghosh Hajra, and Gupta (2020, this issue) as well as Tobias, Olanoff, Hillen, Welder, Feldman, and Thanheiser (2014). Some resources for finding tasks in general include:

- NCTM illuminations https://illuminations.nctm.org/
- http://metamorphosistlc.com offers mathematics tasks for students,
- Dan Meyer has a math blog, where he shares mathematical tasks that help students learn concepts on a deeper level (blog.mrmeyer.com),
- https://www.illustrativemathematics.org/ includes a section on tasks that are open ended and free.
- https://www.youcubed.org/tasks/ includes tasks that are open ended and free for various levels of challenge.

Another challenge for incorporating TLT into the mathematics content courses involves facilitating the task and its implementation. When implementing such tasks with PTs, it is acceptable to have students individually work on a task prior to discussing it with others. This allows them time to become aware of their own knowledge about the concepts. Many PTs believe that they are proficient in most elementary mathematical content, but when asked to elaborate on concepts, they are often confronted with information that contradicts their current understandings (Thanheiser, 2018). Implementing tasks that allow students to experience disorientation provides an opportunity for them to transform and deepen their mathematical proficiency by questioning their prior assumptions. During the critical reflection part of the cycle, students should be encouraged to represent, formulate, and solve problems in a way that makes sense to them. This can help them in developing their self-concept and strategic competence by learning the mathematics with more conceptual understanding and procedural

fluency (Hiebert & Wearne, 1996, Rayner, 2009). MTEs might have PTs discuss with a partner or small group to begin the critical reflection of the prior assumptions in the next part of the cycle.

Once PTs are presented with tasks that contradict and/or challenge their previous assumptions, they will need time to critically reflect on the inconsistencies. This reflection is then followed by rational discourse. MTEs need to ask questions that allow students to discuss and argue their current understandings with each other. MTEs should be cautious about judging solutions or giving PTs too much information about the answer they might be looking for, as the key to the reflection is giving PTs an opportunity to work through the dilemma on their own (Heibert, et. al, 1996; Heibert, et. al, 1997). Conversations with their peers may support PTs in developing their capacity for logical thought as they justify and explain their reasoning (*adaptive reasoning*). This rational discourse is not only important for PTs to transform their understandings of the mathematical concepts; MTEs should also emphasize that, as teachers PTs will need to engage in rational discourse with their students on a regular basis. This will help encourage PTs' *readiness to learn*. Listening to their peers and justifying their own reasoning can support PTs in developing the necessary vocabulary for being future teachers. If PTs are asked only to listen to and apply mathematics, they are missing out on opportunities to develop a crucial component for teaching. Namely, they need to develop the language of mathematics. The development of accurate mathematical language and talk helps develop PTs' orientation to learn because it is a necessary condition for communicating knowledge to others (Shulman, 1986).

Finally, the last piece of the cycle of TLT involves action, where MTEs must help PTs make connections between their prior knowledge and their new understandings. This is often accomplished through questioning. Weimer (2012) notes that questioning is transformative when

the questions offer learners the chance to figure things out for themselves. This action includes connecting the mathematical ideas to other mathematical ideas and to why this concept is meaningful. Facilitating this part of the cycle involves asking questions such as: "How is this different from how you previously thought about this concept? What is the relationship between these ideas? How has this discussion transformed your ideas about the mathematics?" Being able to think about these questions is essential for making the transformation complete. The action phase can help PTs with the final piece of the development of mathematical proficiency, productive disposition. It is essential that PTs develop a habitual inclination to see mathematics as sensible, useful, and worthwhile. These beliefs will be carried into their future practice and passed on to their future students in the classroom (Karp, 1991). Understanding the connections that exist between different ways of seeing and understanding a problem or mathematical concept is essential in developing mathematical proficiency. Closing the loop of the cycle, by discussing the connections that exist between PTs' prior knowledge and this new set of knowledge, can allow them to deepen their understanding of the mathematics that they will someday be responsible to teach others.

There are many resources on discourse and how to facilitate productive discussions in mathematics classes. Some of these include:

- https://pll.asu.edu/p/system/files/lrm/attachments/Lets%20talk.pdf (this article by Catherine Stein focuses on levels of discourse and how to set up the environment and assess the discourse in your own classroom),
- http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.1033.1430&rep=rep1&type=pdf In this article Walshaw and Anthony summarize the research that has been done on discourse in mathematics classrooms, and

- http://www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/CBS_AskingEffectiveQuestions.pdf. This article, from the Canadian government, provides tips and prompts for teachers to use to ask effective questions during discourse.
- Litster, MacDonald, and Shumway (2020, in this issue).

Conclusion

In this paper, we have outlined a cycle of transformative learning theory and its connection to andragogy, the ways in which adults learn that are different from children and mathematical proficiencies. Because PTs enter mathematics content courses already having knowledge of much of the content of the elementary curriculum, it is important for MTEs to provide experiences for them to build upon their existing knowledge, and to develop the five strands of mathematical proficiency. By challenging PTs with problems that disorient their thinking, and then pressing them to resolve dilemmas using critical reflection and rational discourse, MTEs can support PTs in enhancing their knowledge in all five of the strands. Although incorporating TLT into mathematics content courses provides some extra challenges for MTEs, we believe that the benefits for PTs' learning outweigh the extra work required.

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