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Teaching the Mathematics that Teachers Need to Know: Classroom Ideas for Supporting Prospective Elementary Teachers' Development of Mathematical Knowledge for Teaching

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Abstract: *The goal of this article is to articulate aspects of mathematical knowledge for teaching (MKT) that should be addressed in college mathematics content courses for prospective teachers (PTs), and to make research- and practice-based recommendations of class activities, materials, and strategies in support of developing this knowledge. We describe class activities that engage PTs in problem solving as a means of making sense of the content that they will later teach and that immerse them in the mathematics practices that they will later need to foster with their own students. We also provide examples of how mathematics manipulatives, school curriculum materials, and samples of children's thinking can be integrated as instructional tools in content courses. In developing these activities, we were guided by the research regarding the depth and breadth of knowledge required for the effective teaching of mathematics at the early childhood and upper elementary levels, and by the recommendations of professional organizations for math teacher education.*

Keywords: mathematical knowledge for teaching, mathematics teacher education, mathematics content courses for prospective teachers

Introduction

The purpose of mathematics content courses for prospective teachers (PTs) is to teach them the math that they will need to know as teachers. But what is the math they will need to know? Substantial research in the past decade has focused on the practice-based theory of Mathematical Knowledge for Teaching (MKT) (Ball, Hill, & Bass, 2005). As measured by items developed by the Learning Mathematics for Teaching Project at the University of Michigan, this construct has been associated with gains in both student achievement (Hill, Rowan, & Ball, 2005) and mathematical quality of instruction (Hill et al., 2008).

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Ball, Thames, and Phelps (2008) present evidence that MKT, the mathematical knowledge needed to carry out the work of teaching mathematics, is multidimensional and further refine the construct to include several inter-connected domains. *Common content knowledge* is the mathematical knowledge used in settings other than teaching. *Specialized content knowledge (SCK)* is the mathematics knowledge that is unique to teaching, such as knowledge of multiple interpretations and representations for mathematics concepts and the ability to appraise the validity of mathematical arguments and the generalizability of alternate methods and procedures. *Knowledge of content and students* is knowledge of student conceptions and misconceptions about the mathematics. *Knowledge of content and teaching* includes knowledge of models and manipulatives that are effective in promoting student understanding. *Knowledge of content and curriculum* includes knowledge of the school curricula and standards, and *horizon knowledge* deals with how mathematical topics are related throughout the span of the curriculum.

An important goal for their research on MKT and identifying these domains is to “inform the design of support materials for teachers as well as teacher education and professional development” and to “clarify a curriculum for the content preparation of teachers” (Ball, Thames, and Phelps, 2008, p. 405). They recommend that “if mathematical knowledge required for teaching is indeed multidimensional, then professional education could be organized to help teachers learn the range of knowledge and skill they need in focused ways” (p. 399).

In this paper, we propose a model for how mathematics content courses designed specifically for PTs could address several of these MKT domains in an integrated way. We argue that specialized content knowledge (SCK) can be developed by engaging in problem tasks designed to require PTs to create and appraise multiple representations, methods and arguments.

Furthermore, those same tasks can feature manipulatives and models that PTs can later use to teach that content (an aspect of knowledge of content and teaching), and can be followed with an exploration and discussion of examples from school curricula (knowledge of content and curriculum) and samples of children's thinking about that content topic (knowledge of content and students).

The authors of this article are mathematicians who each have more than fifteen years of experience teaching content courses for elementary PTs, including introductory courses in Number and Operations, Geometry and Measurement, and Probability and Statistics, as well as mathematics minor courses in Algebra, Geometry, Probability and Statistics, and Infinite Processes with a focus on the middle grades. We have also designed and taught professional development courses in mathematics content for practicing teachers, which have involved many hours of classroom observations of mathematics teaching in elementary and middle schools.

Content courses for PTs are often taught by instructors whose formal training, like ours, is in mathematics, rather than education (Masingila, Olanoff, & Kwaka, 2012), and our goal is to share our experience in offering some practice-based recommendations that are aligned with those from professional organizations (AMTE, 2017; CBMS, 2012). In what follows, we will present examples of class activities that we have found to be effective in supporting the teaching of MKT through problem solving, and we will suggest how manipulatives, school curriculum materials, and examples of children's thinking can be used to enhance PTs' learning of mathematics content. While we have not formally collected data to document their effectiveness, the activities we present are the result of a process of continual development and refinement over many semesters of implementation, informed by weekly meetings among a working group of the instructors in these courses.

Features of Content Courses to Promote Learning of MKT

In this section we articulate three recommendations for how content courses can promote learning of various interconnected domains of MKT:

1. Implement problem-solving tasks to motivate learning of content that require PTs to create and appraise multiple representations, methods and arguments,
2. Use manipulatives to build PTs' understanding of content, as well as their facility for using manipulative in their own classrooms, and
3. Make explicit connections among course content, school curricula and children's thinking.

Throughout this section, we will use the content area of fractions to illustrate our ideas and offer specific examples of class activities and instructional materials that can be used to integrate these three recommendations within a single task or unit in the content course.

Implement problem-solving tasks to motivate learning of content that require PTs to create and appraise multiple representations, methods and arguments. There is sizable and compelling research into the effectiveness of a problem-based inquiry approach to teaching mathematics at all levels, elementary and undergraduate alike (Lampert, 1990; Schoenfeld, 2016). In this format, classroom instruction on a content topic begins with a problem task that has been designed to spark exploration of a specific content topic. Good problems for developing MKT feature opportunities to unpack school mathematics, build connections among ideas, and explore multiple representations (Suzuka et al., 2009), as well as to anticipate unusual solution methods and evaluate conjectures (Superfine & Wagreich, 2009). In short, these problems should be aimed at addressing the *specialized content knowledge* (SCK) domain of MKT that we described in the introduction.

In this section we will discuss task selection and problem-solving facilitation to promote learning of SCK. We have found that a rich starting problem, one that is non-trivial and engaging, can be quite motivating for PTs. Based on their prior experiences with learning mathematics, PTs tend to focus on getting correct answers, rather than on sense-making (Suzuka et al., 2009). In a course focused primarily on mathematics content at the elementary grade level, PTs often do not appreciate that there is anything new to learn. To address this, Suzuka et al. (2009) propose one feature of MKT tasks is to "provoke a stumble" that reveals PTs' superficial understanding of the content. For more information on developing tasks that create uncertainty, see Feldman, Wickstrom, Ghosh Hajra, and Gupta (2020) in this special issue of *The Mathematics Enthusiast.*

We might, for example, start a unit on fractions by posing the following variant of a classic problem:

> *The fourth graders in an elementary school challenge the fifth graders to a ping pong tournament. Each game will pit a fourth grader against a fifth grader. The school has several tables, and so several games can be played at the same time. When all the tables are being used, 1/3 of the fourth graders are playing against 2/5 of the fifth graders. What fraction of all of the fourth and fifth graders are playing at that time?*

Nearly all our PTs will add the two fractions given in the problem by finding a common denominator, and be satisfied with the solution that 11/15 of the children are participating. At this point, we will ask our PTs to justify that answer, and encourage them to draw a model to illustrate and make sense of their solution. As a result of this "push" for sense-making, some PTs in class-discussion will argue that 11/15 cannot be the correct answer, as this would mean over half of the children are playing ping pong, though less than half of each subgroup is playing at

that time. If anyone is having trouble seeing this contradiction, we repose the problem with the fractions 2/3 and 3/5; the PTs' initial method yields a more obviously nonsensical solution that 19/15 of the children are playing at that time.

Good motivating problems will allow the intended mathematics content to arise naturally during the exploration and solution process (Superfine & Wagreich, 2009). Ideally the problems will make it clear why it is necessary to understand the relevant mathematical concepts. As PTs explain their reasoning on the *Ping Pong Tournament* problem, our role is to ask them to precisely interpret (for this problem and in general) the contextual meaning of the fractions involved, the "whole" to which each fraction refers, the meaning of the denominators and the numerators, what it means to get a common denominator, and what it means to add fractions. Our PTs are surprised to discover that in this problem it is actually a common *numerator* that is desired, and that upon doing so, in this context it made sense to add those numerators together *and* the denominators together to find a solution.

What makes this *Ping Pong Tournament* problem especially motivating is that PTs quickly find that their prior understanding of fractions and fraction operations is inadequate to solve what seemed to be a straightforward problem, and so we take this problem as motivation for further investigating what is meant by the fractions given in the problem statement, and the need to make a more formal definition of a fraction in order to make sense of this problem and find a solution. Further, by introducing a counter-example for when adding two fractions does not make sense (in this case, it is because the two fractions are not referring to the same whole), it motivates further analysis of what it means to add two fractions and how this relates to the fraction addition algorithm of finding a common denominator. Our PTs know *how* to find a common denominator, but in general they do not know *why* they are finding a common

denominator. They will argue that by finding a common denominator, the two fractions then have the same whole. The *Ping Pong Tournament* problem makes apparent the need for precise definitions and distinction among the concepts of "denominator" and "whole".

To prepare them for the mathematical work of teaching, we argue content courses should give PTs experience in actually *doing mathematics.* By "doing mathematics", we mean engaging in the practices enumerated by the Common Core State Standards for Mathematical Practice (National Governors Association for Best Practices, Council of Chief State School Officers, 2010), which we view as consistent with Stein, Smith, Henningsen, and Silver's classification of tasks requiring exploration and understanding the nature of mathematical concepts, processes, or relationships (2000, p. 16). For more ideas on engaging PTs in mathematical practices, see Bernander, Szydlik, and Seaman (2020) in this special issue of *The Mathematics Enthusiast.*

Through work on problems such as the *Ping Pong Tournament*, prospective teachers can see that finding alternate solution methods, making conjectures, and justifying reasoning are key components of doing mathematics; this challenges PTs' prior concepts of what math is and how it can be learned (Superfine & Wagreich, 2009). The *Standards for Preparing Teachers of Mathematics* (AMTE, 2017) stipulate that well-prepared beginning teachers recognize that "engaging in mathematics is more than finding an answer" (p. 10), and that they "regard *doing mathematics* as a sense-making activity that promotes perseverance, problem posing, and problem solving. In short, they exemplify the mathematical thinking that will be expected of their students" (p. 9).

By featuring problem-based inquiry, content courses can model effective mathematics teaching for PTs to later use in their own classrooms. AMTE (2017) recommends settings where PTs are "provided challenging tasks that promote mathematical problem solving and are

provided opportunities to discuss their thinking in small and full-group discourse" so that they "experience learning mathematics using methods that are consistent with the methods they should use as teachers." (AMTE, 2017, p. 31). Further, courses should "immerse candidates in such mathematical practices as reasoning, sense-making, and problem solving while they learn content. Candidates learn to explain their thinking, recognize structures, and generalize" (AMTE, 2017, p. 69), while the role of the MTE is to "explicitly identify and address these mathematical practices for those learning to teach mathematics" (AMTE, 2017, p. 31).

Use manipulatives to build PTs' understanding of content as well as their facility for using manipulatives in their own classrooms. Returning to the domains of MKT identified by Ball, Thames, and Phelps' (2008), while problem tasks in a content course will naturally be targeted at specialized content knowledge, this can be linked with *knowledge of content and teaching* by having PTs use the same models and manipulatives in their work on problems in the content course that are effective in promoting children's understanding in elementary classrooms.

To support their work on the *Ping Pong Tournament* problem that we presented above, we introduce our PTs to "fraction tiles". A set of fraction tiles (see Figure 1) includes one long piece (blue), two pieces (brown) that are each half the length of the blue, three pieces (black) that are each one-third the length of the blue, and so on. Often these pieces are labeled (1 for the blue, ½ for each brown, ⅓ for each black, etc.), but we advocate for using unlabeled ones, since the value of a fraction is not absolute, but relative to the whole to which it refers. These fraction tiles are commonly used to represent arithmetic with fractions; they are especially good for the "measurement" interpretation of division. For example, $\frac{2}{3} \div \frac{1}{6} = 4$, since four of the orange tiles

(each 1/6 of the whole blue) would fit into two of the black tiles (which are each 1/3 of the whole blue).

Figure 1. A partial set of "fraction tiles".

In our *Ping Pong Tournament* problem, given prior to our working with fraction arithmetic, a PT might represent ⅓ of the fourth graders with a black tile (in which case all of the fourth graders would be represented by 3 black tiles, or by 1 blue tile). Since the fourth-grade players are paired up with the fifth-grade players, this black tile must be the same length as the collection of tiles representing 2/5 of the fifth graders; 2 orange tiles suffice (in which case 5 orange tiles would represent all of the fifth graders). During our class discussions, we ask our PTs to decide whether there are more fourth graders or fifth graders in the elementary school (or equal numbers), and the fraction tile model provides our PTs with an effective way to argue that there are 5/6 as many fifth graders as fourth graders. The actual numbers of students remain undetermined, as each orange tile can represent any chosen number of fifth graders, and each black tile represents twice that number of fourth graders; if the PTs do not point this out themselves, we might ask how many fourth graders and fifth graders there are, and the PTs will arrive at multiple solutions.

The proper use of tangible manipulatives is well established to effectively support mathematical problem solving (Cope, 2015). "Well-prepared (beginning teachers) know when to use different manipulatives and various technologies to support students in developing understanding of mathematical concepts and to create opportunities for collective consideration of mathematical ideas such as multiplication, fractions, areas, volume, and coordinate geometry" (AMTE, 2017, p. 84). However, just because students are using manipulatives does not mean that students will interpret them in the same ways or automatically draw the same desired mathematical conclusions from them (Ball 1992). So simply using manipulatives in content courses is not enough to ensure PTs develop the knowledge of how to use them effectively in their own teaching. They must also "understand that meaning is not inherent in a tool or representation but that it needs to be developed through a combination of exploration, carefully orchestrated experiences and explicit dialog focused on meaning-making" (AMTE, 2017, p. 84). We propose that PTs need to experience this sense-making themselves and can do so in their mathematics content courses. Analyzing and discussing the use of manipulatives to represent mathematical procedures and concepts will further add to the complexity of the mathematics in their content course but can also to prepare them to promote thoughtful and effective use of manipulatives in their future classrooms.

Make explicit connections among course content, elementary school curricula, and children's thinking. Courses for PTs to develop MKT should be "structured around tasks that intertwine attention to content, students, and pedagogy" (Ball, Sleep, Boerst, & Bass, 2009, p. 462), and MTEs should attend to thinking and make explicit connection to teaching (Suzuka et al., 2009) to connect the mathematics content of the course to PTs' future work as teachers. One way to do this is to ask the PTs to reflect on the effectiveness of manipulative use in their own

problem solving as described in the previous section. Two further ways of connecting the mathematics in their content course to teaching is to bring in examples from elementary textbooks and to show samples of children's work solving mathematics problems.

The *knowledge of content and curriculum* domain relates mathematics topics in the content courses to school curricula and standards, and beginning teachers need to be able to "read, analyze, interpret and enact mathematics curricula" (AMTE, 2017, p. 10). Our PTs sometimes perceive the content of our college courses to be too deep or abstract for their needs; when we make direct connections to new standards-based elementary school curricula, they are often surprised at the complexity and scope of the problems posed, and at the reasoning capabilities of elementary aged children.

The Bridges in Mathematics University Program (https://www.mathlearningcenter.org/resources/university) is an excellent resource for MTEs to incorporate school curricula into their content courses. The *Bridges in Mathematics* elementary materials are aligned with the Common Core State Standards, and all of the content of this curriculum is available online or as PDF downloads for free for MTEs and PTs. As part of our unit on fractions, we assign our PTs to work through the activities "Picturing Fraction Multiplication" (p. 184), "Grouping or Sharing?" (p. 190-1), and "Do-It-Yourself Story Problems" (p. 196) from the *Bridges in Mathematics* Grade 5 Student Book (The Math Learning Center, 2017b), and we ask them to make explicit connections between these activities and our content coursework on the area model of multiplication and the partitive and measurement models of division with fractions.

The MKT *knowledge of content and students* domain regards students' thinking about mathematics, including alternate conceptions and misconceptions. Well-prepared beginning

teachers "try to see mathematical situations though their students' eyes rather than immediately correcting mathematical errors or demonstrating their approaches. This ability to decenter and understand students' thinking is not only useful in planning for instruction, but also provides a resource for making sense of moment-to-moment interactions with students" (AMTE, 2017, p. 19).

To provide opportunities for our PTs to analyze students' thinking, we occasionally show video clips of children thinking aloud as they work on mathematics problems, allowing for a discussion of children's conceptions and misconceptions about mathematics. We ask our PTs to appraise the children's methods and discuss whether they are correct and generalizable; our role as MTEs during this discussion is to relate the children's thinking and our PTs' analysis to the mathematics content, models and representations in our course. In what follows, we highlight three sources of video clips of children's thinking that we have used in our content courses. For an extensive list of resources for MTEs to find examples of children's thinking please see the article by Max and Welder (2020) in this special issue of *The Mathematics Enthusiast*.

In our unit on fractions, we typically show our PTs the video clips of children's mathematical thinking from *IMAP (Integrating Mathematics and Pedagogy)* (Philipp & Cabral, 2005). The *IMAP* videos are a searchable set of clips of children in grades K-5 reasoning aloud about problems posed by an interviewer, including problems about comparing, converting, adding, subtracting, multiplying and dividing fractions. Especially enlightening to our PTs are clips showing children being unsuccessful in solving problems when attempting to rely on procedure learned by rote, while others exhibit strong conceptions and sense-making about the meaning of the fractions and the operations involved. There are also clips (#510-512) of children trying to give meaning to fraction manipulatives and struggling to see that the same piece of

plastic or same piece of a donut could represent different fractions depending on what is considered to be one whole.

Other video resources we use include the *Math Time Videos* (Richardson, 1997), which show kindergarten, first and second grade classes engaged in a "number talk." Viewers have the opportunity to see how children who have never been taught standard algorithms solve problems using various mental strategies, which allows us to later present PTs with tasks where they are asked to appraise children's invented algorithms and justify their validity using concepts of place value, commutative, associative and distributive laws. The *Thinking Mathematically Videos* (Carpenter et al., 2003) show early elementary children reasoning about algebraic concepts such as the meaning of the equals sign, and making conjectures and arguments about arithmetic properties such as commutativity, associativity, identities, and inverses.

The *Cognitively Guided Instruction Videos* (Carpenter et al., 1999) show elementary children solving addition, subtraction, multiplication, and division problems using direct modeling strategies. We find these particularly effective in helping our PTs understand various interpretations of the standard operations, such as recognizing the "take-away" concept of subtraction when watching a child build one number out of cubes, then remove a number of cubes from that set, versus the "comparison" concept when watching the same child build two numbers out of cubes and find their difference by comparing them side by side. Similarly, the distinction between partitive and measurement division concepts can be made clearer by watching a child solve one story problem for 15 divided by 3 by making a set of 15 cubes, then forming them into three equal sized groups, and then contrasting this with a different story that same child solves instead by making a set of 15 cubes and then forming groups of size 3 and counting the number of groups made.

Additional Examples of Problem Tasks to Promote MKT

In the narrative that follows, we present several more sample activities to further illustrate how the goals for content courses identified in the previous section might be accomplished. As with the *Ping Pong Tournament* task described earlier, each activity is designed to engage PTs in problem solving as a way of "doing mathematics", and to initiate an extensive exploration into the mathematics we are developing, as well as our PTs' future work of teaching and children's thinking as related to that content. We will focus our discussion of these tasks on how the task might be implemented to help PTs develop specialized content knowledge and make use of manipulatives, and we will also suggest how the task can be connected to children's thinking and school curricula.

The activities are intended to introduce a content topic and thus be accessible prior to a formal study of the ideas they motivate. Our class time is structured so that PTs work together in small groups on the activity for 30-60 minutes (depending on the activity), with the MTE visiting groups to gauge their progress and clarify the task as needed, and leading whole-class discussions when appropriate. The activities we describe in this section are all presented in the authors' texts (Beam et al., 2019a; 2019b).

Rocky Math

Variations of this non-written algorithm for multiplication are attributed to Egyptians, Ethiopians, and Russians, among others (Nelson, Joseph, & Williams, 1993). The algorithm is carried out using a large collection of pebbles (we use counters in our classrooms).

> *Consider this ancient method for multiplying: Dig two parallel columns of holes in the ground. Into the topmost holes of the two columns, place the numbers of pebbles corresponding to the two factors (one factor in each column). In one column, place half*

as many pebbles into successive holes. (If the halving process results in a remainder, discard the remainder.) Stop when you get to 1 pebble. In the other column, place twice as many pebbles into successive holes.

Here is an example, for the product of 25 and 9:

Halves Column	Doubles Column
25 pebbles	9 pebbles
12 pebbles	18 pebbles
6 pebbles	36 pebbles
3 pebbles	72 pebbles
1 pebble	144 pebbles

Next, if an even number of pebbles appeared in the halves column, eliminate that whole row:

Halves Column	Doubles Column
25 pebbles	9 pebbles
~~12 pebbles~~	~~18 pebbles~~
~~6 pebbles~~	~~36 pebbles~~
3 pebbles	72 pebbles
1 pebble	144 pebbles

The product is equal to the number of pebbles remaining in the doubles column. (In this example, 25 × 9 equals 9 + 72 + 144). Why does this procedure work?

One of the main topics in a content course for PTs is base ten algorithms. College students are of course already very familiar with base ten and with these algorithms, such as addition and subtraction with regrouping, and standard algorithms for multi-digit multiplication

and long division. However, PTs lack understanding of why these algorithms work (Thanheiser (2009). We argue that this procedural familiarity in fact creates a major obstacle to their conceptual understanding, in that they often fail to recognize that there is anything to be understood.

The manipulatives in this activity do not directly provide a means of visualizing multiplication. Their purpose, rather, is to free the minds of the PTs from the algorithms with which they are familiar, and to allow them to see that algorithms need not be written and that the concept of multiplication is not defined by any algorithm. Both of the processes of the "Rocky Math" algorithm, halving and doubling, can be done without performing any calculations: when halving, remove one of every two pebbles, and when doubling, substitute two pebbles for every one.

This task of explaining why this method should work is quite a challenge for our PTs, but they find the problem intriguing enough to persist with it. Once they begin to interpret each row as a product, many will recognize that sometimes the product is preserved from one step to the next, and sometimes the product is not preserved. The product is preserved from one step to the next precisely when the factor in the Halves Column is *evenly* halved: 12×18 is the same as 6×36, which is the same as 3×72, since in each case one factor was halved and the other doubled. Once they see this, the PTs will begin to realize that in the rows that were not crossed out, a part of the original product was left behind; it is those pieces that must accumulate toward the original product. As some PTs will have more trouble than others in observing these things, the MTE can ask for observations to be shared and discussed within small groups, and ultimately with the entire class.

The Mathematical Education of Teachers II (*MET II*), by The Conference Board of the Mathematical Sciences (CBMS, 2012) emphasizes that prospective elementary teachers must understand "[h]ow efficient base-ten computation methods for addition, subtraction, multiplication, and division rely on decomposing numbers represented in base ten according to the base-ten units represented by their digits and applying (often informally) properties of operations... to decompose the calculation into parts" as well as "[h]ow to use math drawings or manipulative materials to reveal, discuss, and explain the rationale behind computation methods" (CBMS, 2012, p. 27).

The *Rocky Math* task does not overtly appear to be related to a place value system. However, the algorithm does in fact involve a base-two form of reasoning: the number in the first row of the Halves Column is odd if and only if there is a 1 in the ones' place of a binary representation; the number in the second row of the Halves Column is odd if and only if there is a 1 in the twos' place; and so on. In our example, 25 has been broken down into 1 one, 0 twos, 0 fours, 1 eight, and 1 sixteen. The parts of the product appear in the Doubles Column. Although our PTs are familiar with representing numbers in other bases prior to working on this activity, the connection between the Rocky Math algorithm and base two is not something we expect them to see. We would, rather, point out this connection, with the main intention of conveying that our various multiplication algorithms are predicated on the ability to break down one of the two multiplicands (in this example, 25) into parts, and to distribute the overall product accordingly.

After our PTs have worked with the activity enough to recognize how the product has been broken into parts, we ask them how to identify how this same idea is used in the various multiplication algorithms with which they are familiar in base ten (such as the "standard",

"partial products", or "lattice" algorithms). This discussion is enriched by showing examples from the *IMAP* videos of children thinking through standard and invented multiplication algorithms (Philipp & Cabral, 2005). For example, clip #124 shows a fifth-grade girl named Brooke explaining how she can multiply 15×12 by thinking that four 15s is sixty (by thinking that 15 minutes is a quarter of an hour (60 minutes), and so eight 15s is 120, and twelve 15s is 180). We follow this with clip #150 that shows Gilberto, a third-grade boy, finding the same product by thinking aloud that ten 12s is 120 and another five 12s is 60, so 15×12 is the sum, 180. We ask our PTs to compare all of these methods, identifying how each method decomposes one or more of the factors and which property of arithmetic (the distributive property) justifies the ability to do this.

To further motivate our PTs and address further aspects of MKT, we use examples from school curricula that show the variety of multiplication strategies and algorithms that they will need to be fluent with in their future work as teachers. For example, in the *Bridges in Mathematics* Grade 5 Student Book (Math Learning Center, 2017b, p. 148), children are asked to compute 25×64 using five different methods, one of which is the "doubling and halving" strategy in which the child would compute 25×66 = 50×32 = 100×16 = 16,000. This method of doubling one factor while halving the other is a key feature of the algorithm in the *Rocky Math* task.

Even Numbers in Base Five

In base ten, we can tell whether a numeral represents an even number of objects just by looking at the ones' place. If the ones' place holds a 0, 2, 4, 6 or 8, the numeral represents an even number of objects. How can you tell whether a base-five numeral represents an even number of objects (without converting to base ten)?

This task is designed to elicit questioning about which properties of numbers are dependent on the way they are written as numerals. When we implement this class activity, our PTs are already familiar with representing numbers in other bases. Similar to Yackel, Underwood, and Elias (2007), we have found it effective to teach place value, arithmetic algorithms and number theory concepts by using different bases other than ten. Our purpose is not to develop proficiency in another base, but rather to illuminate the thinking that children must do in order to learn place value and do arithmetic using a place value system. We propose that PTs can see and understand base ten by experiencing a system that is not base ten.

Using other bases also removes PTs' ability to rely on memorized procedures, so that they can have the experience of creating algorithms through their own logical reasoning. This usually requires a significant struggle, but in the end, we feel that our PTs exhibit a much stronger understanding of the concepts and how they can be taught. Asking our PTs to perform subtraction with regrouping of a number written in, say, base five, is an example of a task that "provokes a stumble", one that allows our PTs to realize they hadn't really understood base ten or the subtraction algorithm in a way that would prepare them to teach these concepts.

We use interlocking cubes as our main manipulative when working with other bases (see Figure 2). Bases three, four and five are most efficient, requiring fewer cubes to build two- and three-digit numerals. Interlocking cubes have the advantage of being easily ungrouped and re-grouped into different "base blocks" when modeling addition or subtraction with regrouping. They can also be used to make sense of division algorithms in another base, since the base blocks can be broken apart and shared into groups and can be used to make rectangular arrays to illustrate the partial products algorithm for multiplication in other bases.

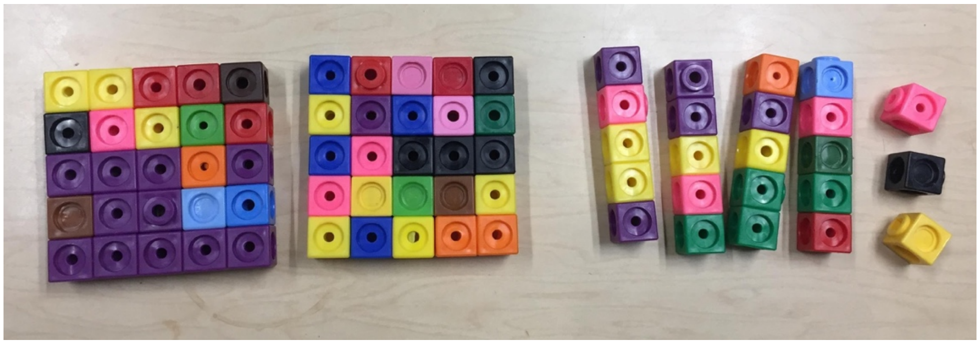

Figure 2. Seventy-three interlocking cubes grouped into base blocks to represent the base five numeral 243.

In our experience, PTs often have “ends with a 0, 2, 4, 6, or 8” as their *definition* of an even number, and they will typically first conjecture that a number written in base five is even if it ends in 0, 2 or 4. At some point the MTE will need to discuss with the PTs what is meant by an “even” number, why it could be problematic to define such a property in a way that is dependent on its representation, and agree upon the more mathematically powerful definition of even number being divisible by two. Hence this activity highlights the role of the mathematical definition and the power it gives one to make arguments. Video clip 8.1 in *Thinking Mathematically* (Carpenter et al., 2003) shows a fourth grader named Allison at a whiteboard making an argument for why the sum of two odd numbers should be even, explaining that odd numbers are made of pairs with one extra, so that when adding two odd numbers you will get all those pairs that are in the odd numbers, and two extras that will make another pair, resulting in a number that is made up of pairs with no extra. Using this clip is a great way to further connect this task to the *knowledge of content and students* domain of MKT.

To demonstrate that divisibility is independent of the base, we might give each group a pile of approximately 30-40 individual cubes and ask them to determine whether that number of cubes is even (or divisible by three, or four, etc.) without enumerating them or forming them into

base blocks. The PTs will easily determine this by separating the cubes into groups of size two or into two equal sized groups, which the MTE should then ask the PTs to relate to either the measurement or partitive division concept.

To facilitate progress on the *Even Numbers in Base Five* problem, we suggest PTs collect data and look for patterns. A list of numbers written in base five numerals, with the even numbers circled will show that the last digit of an even number can be 0, 1, 2, 3 or 4 when written in base five. PTs will see patterns such as for two-digit numbers, if both digits are even, or both are odd, the number is will be even, or, for larger numbers, even numbers must have an even number of odd digits, or the sum of the digits must be an even number. We write these conjectures on the board as they arise, and ask our PTs to make arguments to justify these conjectures based on the concept of base five place value, and use the base block representations to help make these arguments. This problem can then be compared to a similar divisibility rule in base ten for determining if a number is divisible by 3.

Tiling with Equilateral Triangles

> *You want to tile two rooms with ceramic tiles in the shape of equilateral triangles with 1-foot sides. One room is in the shape of an equilateral triangle with 10-foot sides. The other is in the shape of a square with 10-foot sides. How many tiles do you need to buy in order to be able to cover each room?*

Like Menon (1998), we find our PTs struggle to conceptualize areas and tend to see the area of a geometric figure as being a formula, or the result of a calculation stemming from a formula, rather than a measure of how many of some unit it would take to cover a figure. When asked to define “area”, they most commonly respond, “length times width”, not realizing that

there is a conceptual distinction between the concept of area and a formula to calculate it, or that the formula "length times width" only works for a small subset of figures, namely rectangles.

The *MET II* document (CBMS, 2012, p. 29) specifies the need for prospective elementary teachers' "[u]nderstanding what area and volume are and giving rationales for area and volume formulas that can be obtained by finitely many compositions and decompositions of unit squares or unit cubes". A key component of understanding why an area formula works, such as why the area of a rectangle can be computed by multiplying its length times its width, is to understand how the formula is counting the number of unit squares it would take to cover the figure.

The problem of covering the equilateral triangular room is rather easy if instead of square units, one uses the equilateral triangle as the unit of area (see Figure 3, built of triangular pattern blocks). As they work to solve this part of the problem, our PTs tend to begin by creating simpler cases (a practice we have repeatedly used in class): a triangular room with 1-foot sides would be covered by one tile; a room with 2-foot sides would be covered by four tiles; a room with 3-foot sides would be covered by 9 tiles; etc. They quickly conjecture that a room with n-foot sides would be covered by n^2 tiles. We give them time to work in groups to form an argument establishing this conjecture to be true, and they are able to come up with a variety of arguments; most commonly, they see that by producing two copies of the triangle and rotating one, they could form a rhombus with n rows of $2n$ tiles.

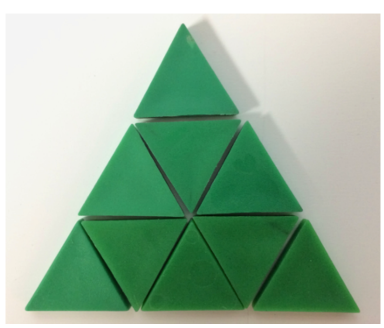

Figure 3. Tiling a triangular room with triangles.

The problem of covering the square room is more challenging. Our PTs almost always initially speculate that an n-foot-by-n-foot square could be covered by n rows of triangles, each row of length n feet and containing $2n$ triangles. They mistakenly assume that the height of an equilateral triangle is the same as its base length. By comparing a triangle with a square from a set of pattern blocks, they will see that this is not true.

By exploring how one might cover a figure using a different unit of area, such as equilateral triangle units, and even deriving a formula for the area of a figure using this unit, PTs are forced to confront their previous conception of area as being a formula. Discovering, when using equilateral triangle units rather than square units, that the area of an equilateral triangle of side length n would be calculated as $A = n^2$, rather than by applying the familiar formula $A = \frac{1}{2}bh$ (which assumes square units), helps to develop PTs' understanding that area formulas merely count the number of some standard unit it takes to cover a figure.

Pattern blocks manipulatives come with equilateral triangles, squares, rhombi, and hexagons all with the same side length, which make them ideal manipulatives for exploring non-standard units of area and the effect of the choice of unit on an area formula. Given a plane figure, PTs can be tasked with finding its area using any one of the pattern blocks figures as the unit of area. When converting between different units of area, they will discover the issue of non-commensurability: figures with rational number areas using square units will have irrational areas when using triangular units, and vice-versa, and finding the precise relationship between these two units of area makes for a good application of the Pythagorean Theorem.

A study of the *Bridges in Mathematics* Grade 3 Student Book (Math Learning Center, 2017a) helps to connect this activity with the *knowledge of content and curriculum* domain of MKT. As per convention, squares are used as the units when third-graders are introduced to the

concept of area; but when asking for the areas of rectangles, the problems require the children to create more than one equation that would count the squares (p. 178). Further, before the concept of area has been formally introduced, a lesson on fractions relative to "one whole" makes implicit use of hexagonal units for area: "If the hexagon is 1, the triangle is ______ because…" (p. 130).

Building to Scale

Step A: Build something out of 10 small cubes. We'll call this thing "Object A."

Step B: Using 10 large cubes, build a larger version of Object A. Call this "Object B."

Step C: Using as many small cubes as necessary, reproduce Object B. (It should be the same size and shape as Object B, but built out of small cubes instead of large ones.) We'll call this thing "Object C."

Now answer these questions:

1) *How much taller is Object B than Object A?*
2) *How much bigger is the surface area of Object B compared to that of Object A?*
3) *How much bigger is the volume of Object B compared to that of Object A?*
4) *What is the relevance of Object C to answering these questions?*
5) *Suppose we had given you larger cubes that are 3 times the length of the small cubes in order to build Object B. How would your answers to questions 1) - 3) change? What if they were 4 times the length?*

We begin this activity by defining the volume of a three-dimensional object to be the amount of space enclosed by that object and observing that one way to measure the volume of an enclosed space is to count the number of equal-size cubes that it takes to fill the space. For the

task we provide a supply of small wooden cubes with half-inch edges and large wooden cubes with one-inch edges (see Figure 4).

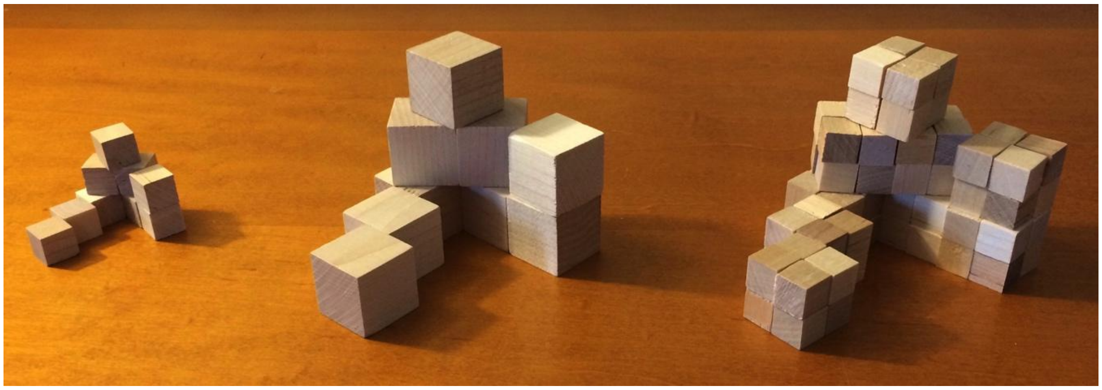

Figure 4. Sample objects A, B, and C for the building to scale task.

The *MET II* document (CBMS, 2012) asserts that prospective elementary teachers must understand proportional relationships as this type of thinking applies to the areas of algebra, measurement and data, and geometry. Specifically, in the area of geometry, it states that they should learn to "[r]eason about proportional relationships in scaling shapes up or down" (CBMS, 2012, p. 30). An ability to distinguish proportional relationships from other relationships is a major area of weakness among PTs (Izsák & Jacobson 2017). We use this activity in order to provide experience with proportional and non-proportional relationships within the same context, as well as to further develop the concepts of length, area, and volume, and their relationships to one another.

Prior to this activity, our PTs, including upper-level mathematics majors, almost always carry the misconception that lengths, areas, and volumes share the same scale factor when an object is enlarged. They are surprised when working this activity to discover that such a proportional relationship is not true. After completing this activity, they are able to explain why: in doubling the length of a cube, each face is replaced by 4 copies of the smaller face, and each

cube itself is replaced by 8 copies of the smaller cube. They are able to connect these discoveries to their understanding of areas and volumes, and they are able to argue that if the scale factor for length is n, then the scale factor for surface area is n^2 and the scale factor for volume is n^3. A direction this problem can be taken for further exploration is to investigate whether this relationship among scale factors is true for all three-dimensional shapes, or whether there is something special about cubes.

To make connections to children's thinking, PTs could be assigned to read Riehl and Steinthorsdottir's 2014 article in *Mathematics Teaching in the Middle School* on the Mr. Tall and Mr. Short problem, which discusses and gives examples of middle schoolers' thinking about proportional relationships. In the real world, there are many implications of this non-proportionality of scale factors. For instance, an engineer who likes the appearance of a bridge that crosses a small river cannot simply build a larger version of the bridge to span a bigger river; the strength of the larger bridge would be proportional to its cross-sectional contact area, but its weight would be proportional to its volume. Many other illustrations of these ideas are presented in a classic article by J. B. S. Haldane (Haldane, 1926). Several further exploratory activities are described in (Kluger-Bell, 1995).

Recommended Resources for Problem Tasks, Manipulatives, Children's Thinking and Curricula

While writing one's own problem for a class activity is an enjoyable and rewarding aspect of teaching through problem-based inquiry, we also commonly adapt problems we find in other texts or online resources for use in our classrooms. There a numerous online sources for problems aimed at school mathematics, such as *NCTM Illuminations* website (http://illuminations.nctm.org), *Inside Mathematics* website (http://www.insidemathematics.org),

YouCubed website (https://www.youcubed.org), and *NRICH* website (https://nrich.maths.org) that can be adapted for use in the content courses for PTs.

A number of textbooks for math content courses offer supplementary "activities" or "explorations" manuals that are designed for classroom implementation, including those by Beckmann (2008b) and Bassarear (2005). Unfortunately, we found that often textbooks can do more to hinder PTs' learning of MKT rather than support it. Our informal survey of review copies of textbooks for content courses, sent by publishers to the authors, revealed the focus of most are still (largely) on developing skills and procedures, rather than problem-based inquiry, sense-making, and mathematical reasoning behind the skills or procedures. When concepts and models related to MKT are presented, they tend to be moved to the background and not the primary focus of the content. In other words, the PTs are expected to be able to solve problems, not understand or analyze the models. We found only two of the reviewed copies, the one by Beckmann (2008a) and one by Parker and Baldridge (2004), to address the partitive and measurement concepts for division of fractions *and* ask PTs to make sense and write examples of stories for each. The Parker and Baldridge text is also notable for making explicit connections to school curriculum materials, namely the *Singapore Math* series (available from https://www.singaporemath.com).

A concern we have with the typical textbooks available for mathematics content courses, specifically designed for prospective elementary teachers, is they give away solutions to the problems they pose and provide worked examples. This puts PTs in the position of being consumers of the mathematics, rather than doers and creators. Since well-prepared beginning teachers "endeavor to position students as authors of ideas, students who discuss, explain, and justify their reasoning using varied representations and tools" (AMTE, p. 16), we argue that

content course textbooks should endeavor to position PTs as *the authors* of the mathematical ideas of the course. Consequently, we do not use a standard textbook in our content course, but have instead developed our own series of texts for prospective elementary and middle school math teachers (Beam et al. 2019a, 2019b). These *Creative Commons* licensed texts are available for free by contacting the authors. Each section of these books begins with a *Class Activity*, designed for small-group work in class, which is then followed by material for the student to read and work through outside of class. We have also prepared an instructor's guide for many of the activities with our commentary on the big ideas they are intended to introduce and suggestions for how these ideas can be developed through class discussion.

We note that in our content courses for PTs, for many activities we do not use manipulatives. Our choice to use them is guided by whether they might contribute in a substantive way to the PTs' ability to represent information or otherwise understand and think about a problem. Manipulatives that we have found to be useful on a regular basis include fraction tiles, red and black counters, interlocking cubes, pattern blocks, interlocking polygons, fillable solids, dice, and mathematics balances. We recommend that mathematics departments acquire at least a collection of these items. There are several companies with extensive online and print catalogs of mathematics manipulatives, including EAI Education (http://www.eaieducation.com), Nasco (https://www.enasco.com), and Didax Educational Resources (http://www.didax.com).

As mentioned previously in this article, the materials from *Bridges in Mathematics* (https://www.mathlearningcenter.org/resources/university) offer the opportunity for MTEs to freely access elementary grades curricula materials that are aligned with the Common Core State Standards. The collection of videos from *IMAP* (*Integrating Mathematics and Pedagogy*)

(Philipp & Cabral, 2005) provides a valuable way for PTs to gain insight into children's mathematical thinking in the content area of number and operations. The article by Max and Welder (2020) in this special issue lists more resources for MTEs to find examples of children's thinking.

Conclusion

We argue that a well-designed content course can help prospective teachers to develop a specialized content knowledge that includes an understanding of children's conceptions and misconceptions and a recognition of what types of problems, models and manipulatives are effective in building mathematical understanding. We have described the components for such a course, centered around problem-solving tasks that build an understanding of multiple representations, methods, and arguments. We shared several sample problem tasks, and have illustrated how manipulatives can be effectively employed in the process of solving and understanding those problems. And we have described how a variety of resources can be studied within the context of those problems in a way that strengthens the prospective teachers' understanding of children's thinking and gives them a familiarity with elementary curricula. These problem-solving experiences can help prospective teachers to develop not only a proficiency in mathematical thinking, but the multi-dimensional components of the mathematical knowledge required for teaching.

As mathematicians, we want to provide the PTs in our classes with experiences of *doing mathematics* (e.g. making conjectures, justifying reasoning, making sense of solution methods) so that they are prepared to offer their future elementary students the same opportunities. We want to show our PTs that they can reason like a mathematician, or "begin to see themselves as 'mathematical'" (Suzuka et al., p. 10), so that they can show this to their students too. We feel

that this way of reasoning is just as important as any content they will teach. At the same time, we want our PTs to become experts in the elementary mathematics curriculum and in the models, manipulatives, and representations that are the most powerful in helping children learn mathematics.

These two goals of preparing better mathematicians on the one hand, and better mathematics teachers on the other, are not disjoint. The good news is that the content domain of elementary mathematics is full of deep and interesting mathematics. We can engage prospective teachers in solving problems in this domain that necessitate an examination and discussion of elementary mathematics concepts and procedures and ask students to make sense of and make connections among a variety of methods and models. In so doing, our mathematics courses for elementary education can develop proficiency in mathematical thinking and problem-solving as well as the multi-dimensional knowledge required for the mathematical work of teaching.

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