The Mathematics Enthusiast

Volume 17
Numbers 2 & 3

Article 16

6-2020

Mathematics teacher educators' addressing the common core standards for mathematical practice in content courses for prospective elementary teachers: A focus on critiquing the reasoning of others

Brooke Max

Rachael M. Welder

Recommended Citation
Max, B. & Welder, R. M. (June 2020). Mathematics teacher educators' addressing the common core standards for mathematical practice in content courses for prospective elementary teachers: A focus on critiquing the reasoning of others. In A. Appova, R. M. Welder, and Z. Feldman, (Eds.), Supporting Mathematics Teacher Educators' Knowledge and Practices for Teaching Content to Prospective (Grades K-8) Teachers. Special Issue: The Mathematics Enthusiast, ISSN 1551-3440, vol. 17, nos. 2 & 3, pp. 843–881. ScholarWorks: University of Montana. Retrieve (open access) from: https://scholarworks.umt.edu/tme

Mathematics Teacher Educators' Addressing the Common Core Standards for Mathematical Practice in Content Courses for Prospective Elementary Teachers: A Focus on Critiquing the Reasoning of Others

Brooke Max
Purdue University, USA

Rachael M. Welder
Texas A&M University, USA

Abstract: *Over the last forty years, standards and recommendations for teachers and learners of K-12 mathematics in the US have evolved to highlight mathematical practices (e.g., Common Core State Standards of Mathematics, Standards for Mathematical Practice [SMPs]). Practice standards (i.e., SMPs) describe mathematical competencies that should be developed in learners of mathematics at all levels. National organizations (e.g., Conference Board of Mathematical Sciences) have specifically called for attention to be given to SMPs in collegiate mathematics content courses for prospective elementary (ages 5-12) teachers (PTs). The goal of this paper is to help instructors of such courses, especially those new to the field of mathematics education, gain familiarity with the organizations and documents that support the development of these practices and conceptualize ways in which they might engage PTs in their content courses in SMPs. First, we synthesize the evolution of mathematics standards for K-12 learners and teachers in the US. Second, we report results from an investigation into the ways in which mathematics teacher educators (MTEs) are addressing SMPs in their content courses for PTs. In this study, SMP3: Construct viable arguments and critique the reasoning of others was reported by MTEs as being addressed in their courses more than any other SMP. This finding precipitated a qualitative analysis of the ways in which PTs were being provided opportunities to engage in SMP 3 within the descriptions and samples of tasks provided by the MTEs. We will share and discuss example tasks that provided opportunities for PTs to analyze others' thinking. Lastly, we consider the potential benefits of leveraging children's thinking in SMP 3-related tasks for PTs and provide resources for MTEs who are interested in utilizing samples of children's thinking in their classes.*

Keywords: standards for mathematical practice, prospective elementary teachers, argumentation, children's thinking

Introduction

For years, researchers have called for prospective elementary (ages 5-12) teachers (PTs) to be afforded opportunities in teacher preparation programs to engage with the mathematical content they will be expected to teach and the mathematical ***processes*** and ***practices*** with which they will be expected to engage their future students (e.g., Association of Mathematics Teacher Educators [AMTE], 2019; Ma, 1999; Ferrini-Mundy, 2000). The most recent set of mathematics content and practice standards for K-12 learners, the *Common Core State Standards of Mathematics* (*CCSS-M*; National Governors Association Center for Best Practices and the Council of Chief State School Officer [NGA & CCSSO], 2010), specify eight Standards for Mathematical Practice (SMPs). In developing the SMPs, the authors referenced the *process standards* put forth by the National Council for Teachers of Mathematics (NCTM; 2000) and *strands of mathematical proficiency* identified by the National Research Council (NRC; 2001). The SMPs embody these ideas and “describe varieties of expertise that mathematics educators at ***all*** levels should seek to develop in their students” (NGA & CCSSO, 2010; p. 6, emphasis added). Table 1 provides a list of the eight SMPs as they are worded in the *CCSS-M* with a shortened name that we have assigned to each for the purpose of this article (detailed descriptions of each SMP can be found in Appendix A).

Although these standards “were written for K–12 students, they apply to all who do mathematics, including elementary teachers” (Conference Board of Mathematical Sciences [CBMS], 2012, p. 24). In fact, since the inception of the *CCSS-M*, national documents and organizations have recommended that the SMPs be an important consideration in elementary teacher preparation programs (e.g., *Principles to Actions*, NCTM, 2014; *Mathematical Education of Teachers II*, CBMS, 2012; *Standards for Preparing Teachers of Mathematics*, AMTE, 2019).

Table 1

CCSS-M Standards for Mathematical Practice (SMPs)

SMP	SMP Name
1. Make sense of problems and persevere in solving them	SMP1: Problem-Solving
2. Reason abstractly and quantitatively	SMP2: Reasoning
3. Construct viable arguments and critique the reasoning of others	SMP3: Argumentation
4. Model with mathematics	SMP4: Modeling
5. Use appropriate tools strategically	SMP5: Tools
6. Attend to precision	SMP6: Precision
7. Look for and make use of structure	SMP7: Structure
8. Look for and express regularity in repeated reasoning	SMP8: Regularity

As CBMS notes, "*engaging in mathematical practice takes time and opportunity* … coursework and professional development for teachers must be planned with that in mind" (2012, p. 24).In alignment with these recommendations, we argue that PTs need ample and well-designed opportunities in their teacher preparation courses to fully understand the nuances of the *CCSS-M* SMPs and the important role they will play in helping their future students learn mathematics. For this to take place within teacher preparation coursework, the mathematics teacher educators (MTEs) who teach courses for PTs must have extensive knowledge of the SMPs and the ability to develop meaningful opportunities within their classes for PTs to engage in such practices.

We argue that these practices, meant to support the acquisition and application of content knowledge (NCTM, 2000), should be embedded into courses specifically designed to support PTs in developing their mathematical content knowledge. As stated by the CBMS, "the features of mathematical practice described in [the *CCSS-M*] standards are not intended as separate from

mathematical ***content***. Teachers should acquire the types of mathematical expertise described in these standards as they ***learn mathematics***," (2012, p. 24, emphasis added). Opportunities to engage in SMPs within content courses can support PTs as learners, as they develop their own mathematical content knowledge, and as future teachers, as they gain insight into how such practices are implemented in the classroom to support content development. Thus, MTEs who teach content courses for PTs should have an understanding of the *CCSS-M* SMPs and how they can be effectively positioned within such courses.

However, novice MTEs and those who may be new to the field of mathematics education may not be familiar with the SMPs, the development of these practices, or the ways in which they can engage PTs in these practices in the content courses they teach. Research suggests that most content courses for PTs are taught in departments of mathematics (e.g., Greenberg & Walsh, 2008; Masingila, Olanoff, & Kwaka, 2012; Max & Newton, 2017) by instructors who have mathematical degrees/backgrounds, making them "well-qualified" to teach mathematical content (McCrory, Francis, & Young, 2008). However, since instructors who teach these content courses may not have a background in *mathematics education*, and tend to have little to no experience teaching elementary-aged children, they may lack familiarity with the practice standards and the ways in which PTs will be expected to engage their future students in such practices (Masingila, et. al., 2012; McCrory et al., 2008). Therefore, the goal of this paper is to support MTEs' understanding of mathematical practices and practice standards for K-12 learners and offer insight into the ways in which MTEs can engage PTs in SMPs in their content courses.

To provide background and context for the SMPs, we first synthesize the historical development of standards for elementary mathematics content and practices and related recommendations for the preparation of elementary teachers in the US. Next, we share the results

of a research study designed to explore the ways in which MTEs are currently addressing SMPs in their mathematics content courses for PTs. In this study, SMP3: Argumentation was reported by MTEs as being addressed in their courses more than any other SMP. This finding precipitated a qualitative analysis of the ways in which PTs were being provided opportunities to engage in SMP3: Argumentation through the mathematical tasks provided by the MTEs. We will share and discuss example tasks that provided opportunities for PTs to develop their understanding of and ability to apply the argumentation practice. SMP3 in its entirety is as follows:

> *CCSS.MATH.PRACTICE.MP3:* ***Construct viable arguments and critique the reasoning of others.***
>
> *Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the argument*s. (NGA & CCSSO, 2010, pp. 6-7)

Because of their direct application to the work of elementary teachers, our discussion will focus on the tasks that provided opportunities for PTs to engage in SMP3: Argumentation through analyzing others' thinking (i.e., comparing and responding to arguments of others, distinguishing between valid and invalid reasoning, etc. [NGA & CCSSO, 2010]). By "others' thinking," we refer to thinking that may be artificially constructed by the MTE or authentically generated by another individual (e.g., classmate or child). We will then discuss the potential

benefits of leveraging artifacts of children's thinking in tasks for PTs. We argue that such tasks can create meaningful avenues for supporting PTs in deepening their mathematical content knowledge, while gaining insight into the ways children think about such content, all the while engaging in an aspect of SMP3: Argumentation that is directly related to the work of teaching. Resources to support MTEs interested in implementing artifacts of children's thinking (through written work samples and videos) are provided in Appendix B.

Evolution of Standards for Learners and Teachers of Mathematics

The mathematical process and practice standards for K–12 learners have evolved over the course of four decades through the work of multiple professional organizations (NCTM, 2014). In 1980, NCTM began an initiative to move students beyond the development of procedural fluency towards conceptual understanding with a greater focus on the skills required to solve problems. This movement sparked the establishment of the Commission on Standards for School Mathematics in 1986 and the subsequent development of mathematical standards for K–12 learners and teachers (Research Advisory Committee of the National Council of Teachers of Mathematics, 1988), including the *Curriculum and Evaluation Standards* (NCTM, 1989), *Professional Teaching Standards* (NCTM, 1991), and *Assessment Standards* (NCTM, 1995). In 2000, NCTM's *Principles and Standards of School Mathematics* incorporated these three separate sets of standards into one cohesive collection, emphasizing the need for well-prepared teachers and learners of mathematics in the 21st century. This updated vision of K–12 mathematics included a set of five skills and practices that "highlight ways of acquiring and using content knowledge" (NCTM, 2000, p. 29), known as the *process standards*: Problem-Solving, Reasoning and Proof, Connections, Communication, and Representations.

The following year, the NRC published *Adding It Up: Helping Children Learn Mathematics* (2001), a set of research-based recommendations for K–12 teachers and learners of mathematics. In this work, the NRC describes mathematical proficiency as consisting of five interconnected strands: Strategic Competence, Adaptive Reasoning, Conceptual Understanding, Procedural Fluency, and Productive Disposition. Together, the mathematical processes and practices set forth by the NCTM's *process standards* (2000) and the NRC's s*trands of mathematical proficiency* (2001) guided the creation of the *CCSS-M* SMPs (NGA & CCSSO) in 2010. At the time this article was written, the *CCSS-M* was adopted by forty-one states and the District of Columbia (Common Core State Standards Initiative, 2019), making it the closest set of national standards ever adopted in the United States.

Initially, the CBMS's recommendations for the preparation of elementary teachers, in the *Mathematical Education of Teachers* (*MET*; 2001), were informed by NCTM's *Process Standards* (2000). However, with the release of the *CCSS-M* in 2010, CBMS updated their recommendations for PTs to reflect the newest standards in their 2012 report, *Mathematical Education of Teachers II* (*MET II*). The content strands in the *MET II*, deemed "Essential Ideas" for K–5 PTs, were consistent with the language of the *CCSS-M* content strands and emphasized the importance of providing opportunities for PTs and in-service teachers to actively engage with the SMPs throughout teacher preparation and professional development programs.
Collectively, these documents informed the most recent set of *Standards for Preparing Teachers of Mathematics* (2017), put forth by the AMTE, which references the *CCSS-M* and *MET II* throughout. AMTE is the largest professional organization for mathematics teacher education, whose embrace of the *MET II* and *CCSS-M* signified a unified push toward a focus on reasoning, sense-making, and communication in K-12 classrooms and teacher preparation programs.

Figure 1 shows the connections between the various standards documents for K-12 learners and how they influenced the standards developed for K-12 teacher preparation.

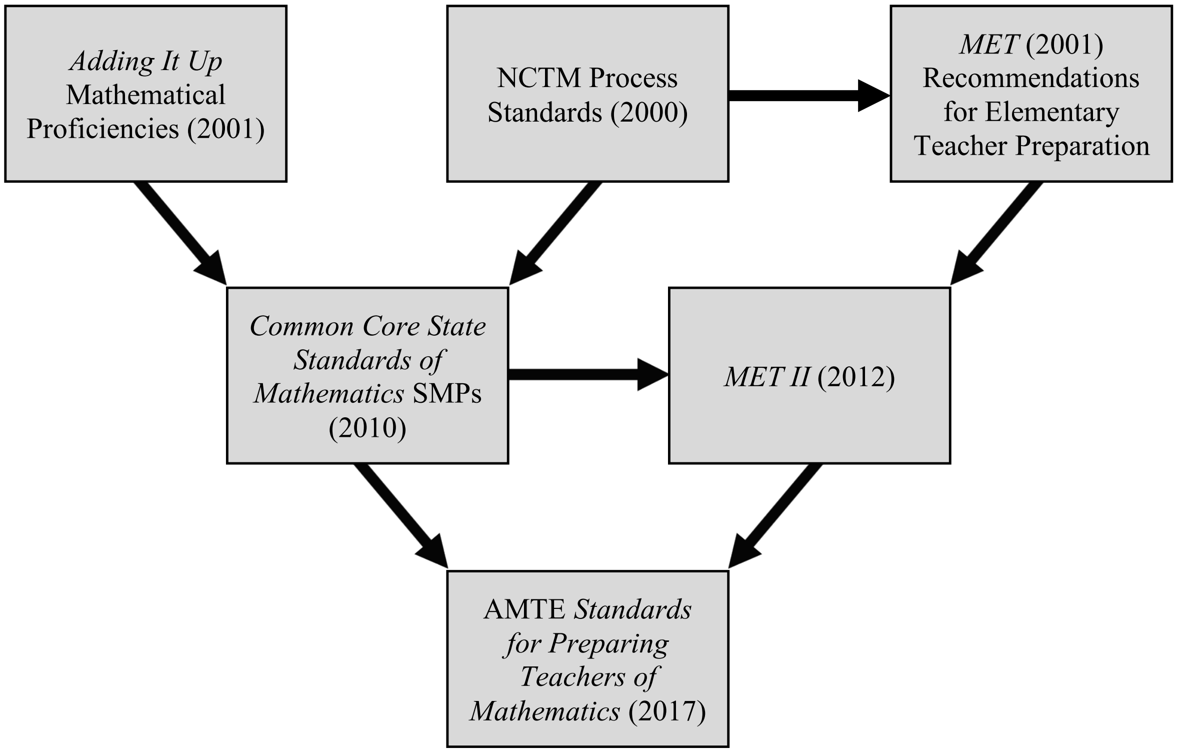

Figure 1: Flowchart illustrating the influence of standards and recommendations documents from distinguished organizations since 2000.

PTs will be expected to help their future students engage in SMPs as they acquire and develop understanding of mathematical content, necessitating the need for MTEs to provide opportunities for PTs to engage in SMPs, as they acquire and develop understanding of mathematical knowledge for teaching, during teacher preparation programs. Below, we present the results of a study we conducted to illuminate the ways in which MTEs address SMPs in content courses for PTs. Our goal is to not only inform the community of these ways, but to also encourage MTEs to consider how SMPs might be addressed in their content courses.

Mathematics Teacher Educators' Development of Standards for Mathematical Practice in Content Courses for Prospective Elementary Teachers

Results of a Larger Study

A subset of data from a larger study on the mathematical content preparation of elementary teachers was analyzed to answer the question: "How do mathematics courses for elementary teachers provide opportunities for PTs to engage with *CCSS-M* SMPs while developing their content knowledge for teaching?" (see Max, 2018). The larger study focused on obtaining information via a questionnaire sent to MTEs regarding their programs, educational and professional backgrounds, and the content courses they teach for PTs (see Max & Newton, 2017).

Data collection. One-hundred and twenty MTEs provided information about 175 courses they had experience teaching, including courses that focus on content for PTs, methods for PTs, combined content/methods for PTs, and general mathematics. For the purpose of the investigation reported in this paper, we were interested in the ways SMPs can be addressed throughout the development of content. So, for each course for PTs, MTEs were asked to specify the degree to which the course focuses on content versus pedagogy using a Likert Scale of 1-7. Since, 73% of the combined content/methods courses reported having a content focus equal to or greater than their pedagogical focus, we included these in addition to the courses designated as purely content. We will refer to this combined set of courses as "content courses."

The set of content courses is comprised of 120 courses total (81 content, 39 combined content/methods), described by a total of 64 MTEs. Please note that MTEs were not required to respond to all prompts, thus some questions were answered by a subset of participants or for a subset of courses. We indicated these instances by reporting the relevant response sizes.

For each individual course, the participants were provided a list of the eight SMPs (see Table 1 for a list and Appendix A for detailed descriptions of each SMP) and asked to select all of the SMPs that are addressed in that particular course. Alternatively, they could select a response specifying that no SMPs were addressed in that course. To better understand the ways in which the SMPs were being addressed, participants were asked to select from the following list: reading the SMPs, creating lesson plans that reference the SMPs, facilitating lessons that reference the SMPs, and experiencing lessons planned with the SMPs in mind. They were also given the opportunity to provide alternatives or expand upon their selections in an open-response field. Participants were then asked to identify the single SMP they perceived to be addressed the most frequently in their course and to provide an example of how it was addressed.

Sample of courses and participants. The 64 MTEs in our sample represented 61 universities across 24 states. Of the 115 content courses for which student population was identified, 111 courses were designed to serve undergraduate students, with 4 courses belonging to post-baccalaureate programs. Of the 114 courses for which textbook use was indicated, MTEs reported using one or more textbooks in 95 (83%) of those courses. Over 20 different textbooks were identified; Table 2 displays the frequency of use for all textbooks reported in more than five courses.

Forty-four (69%) of the 64 MTEs reported having some affiliation with a department of mathematics, with 32 (50%) belonging solely to a mathematics department. Twelve (19%) MTEs were affiliated with both departments of mathematics and education, while the remaining 20 (31%) were solely based in schools of education. These 64 participants reported a mean of 9.7 years of experience teaching at the post-secondary level. Twenty-six (41%) reported holding an advisor role (for an average of 7.8 years), while 17 (27%) reported being a program coordinator

Table 2

Relative Frequency of Textbook Use in Content Courses

Author(s)	Textbook	Frequency (c = 95)
Beckmann	*Mathematics for Elementary Teachers with Activities*	21
Van de Walle, Karp, & Bay-Williams	*Elementary and Middle School Mathematics: Teaching Developmentally**	19
Billstein, Libeskind, & Lott	*A Problem Solving Approach to Mathematics for Elementary School Teachers*	12
Sowder, Sowder, & Nickerson	*Reconceptualizing Mathematics for Elementary School Teachers*	11
Bassarear	*Mathematics for Elementary School Teachers*	7
Musser, Burger, & Peterson	*Mathematics for Elementary Teachers: A Contemporary Approach*	6

* It should be noted that 16 of the 19 courses identified as using a version of *Elementary and Middle School Mathematics: Teaching Developmentally* by Van de Walle, Karp, and Bay-Williams were courses that combined content and methods. Had we not included these courses in our data, this book would not have met the criteria of being used in more than five courses to be included in this table.

(for an average of 6.9 years). Fifty-one (80%) of the MTEs reported having prior experience as a K-12 teacher, with a mean of 8.7 years of K-12 teaching experience. Of these 51 MTEs, 18 (35%) have taught at the elementary level, 29 (57%) at the middle level, and 35 (69%) at the secondary level, with the majority having taught at more than one of these levels. Fifty-six (88%) of the respondents reported mathematics education as being the focus of their research, while only four respondents (6%) indicated a research focus in mathematics. The four remaining respondents reported that research was not a significant component of their work.

Standards for mathematical practice in content courses. SMPs were selected by 60 of the 64 MTEs for 100 of the 120 content courses they described, indicating that they "intentionally address" one or more SMPs in these 100 courses. No information was provided for

15 courses while the remaining five were specifically identified as not addressing any SMPs. Table 3 displays the frequency with which each SMP (using the name identified in Table 1) was selected as being "intentionally addressed" in one of the 100 content courses. Table 3 also displays the frequency with which each SMP was selected as the "most addressed" SMP for each course. Although MTEs were asked to specify a single "most addressed" SMP for each course, this information was only reported by 47 MTEs for 79 of the 100 courses. We acknowledge that asking participants to choose a single "most addressed" SMP may have presented a challenge for them. For nine courses, two or three SMPs were reported as being the "most addressed," while equal attention to all eight SMPs was reported for six courses. One respondent indicated, "Each of them [SMPs] is addressed regularly. It is difficult to say which is addressed the most. I will, for the sake of answering the question, speak to the 'sense making' standard."

Table 3

Indication of Attention to SMPs in Content Courses

SMP Name*	Intentionally Addressed (c = 100)	Most Addressed (c = 79)
SMP1: Problem-Solving	92	26
SMP2: Reasoning	95	13
SMP3: Argumentation	92	36
SMP4: Modeling	86	15
SMP5: Tools	92	10
SMP6: Precision	89	8
SMP7: Structure	89	14
SMP8: Regularity	82	9

* For detailed descriptions of each SMP, see Appendix A.

The data in Table 3 indicate that MTEs perceive they are attending to most, if not all, SMPs in their content courses for PTs, as every SMP was selected as being addressed in at least

82% of all courses. Table 4 provides data on the ways in which SMPs were addressed across the 100 content courses. Respondents who selected "other ways" described actions such as, "reflecting on their use in lessons in class," "discussion of how these standards play out in class lessons," and having a "poster of SMPs on [a] wall in [the] classroom."

Table 4

Indication of Opportunities for PTs to Engage with SMPs in Content Courses (c = 100)

Activity	Frequency
Experiencing lessons planned with the SMPs in mind	87
Reading the SMPs	51
Facilitating lessons that reference the SMPs	37
Creating lesson plans that reference the SMPs	27
Other ways	13

Although all eight SMPs were reportedly receiving attention in the vast majority of these classes, we found an uneven distribution across the individual SMPs when MTEs were asked to identify the one that they address the most. In fact, SMP3: Argumentation was selected as "most addressed" in 36 (46%) of the 79 courses. The ways in which this SMP is being addressed is not clear from these data, although the most frequent way in which MTEs reported addressing SMPs was through their "planning of lessons in ways that will allow for PTs to experience the SMPs." These findings precipitated the next stage of our investigation to get a better sense of how MTEs are providing PTs with opportunities to experience SMP3: Argumentation in content courses.

***CCSS-M* SMP3: Qualitative Data Analysis**

In addition to the survey items discussed above, participants of the aforementioned larger study were also provided a list of content domains taken from *MET II* (CBMS, 2012). They were

asked to select all of the content domains that are addressed in each of their courses and to "upload or describe an activity, assessment, or reading related to the most-addressed content domain." This information was provided by 49 of the 64 MTEs for 93 of the 120 content courses taught. We will refer to the collection of responses we received, including 65 written descriptions and 33 uploaded activities, as content activities (note that for five courses, both an uploaded document and supporting description were provided). Responses ranged from short descriptions such as, "Addition and subtraction in other bases using base 4 and base 5 blocks," to multi-page uploads containing several tasks, each with multiple individual questions or activities.

We qualitatively analyzed the 93 unique content activities to get a better understanding of how PTs were being provided opportunities to engage in SMP3 in their content courses. While respondents were not asked to consider the SMPs in regards to answering the questions specific to content domains, their responses were analyzed through this lens to investigate the ways in which PTs might be meaningfully engaged in SMP3 during activities designed to address content. Using the language provided in the *CCSS-M* description for SMP3, we coded opportunities within the provided activities for PTs to construct arguments, make and explore conjectures, use counterexamples, explain or justify their reasoning or conclusions, communicate reasoning or conclusions to others, analyze and respond to the work or arguments of others, and distinguish correct reasoning from flawed while being able to explain any flaws (NGA & CCSSO, 2010).

CCSS-M **SMP3: Findings of Content Activity Analysis**

Our analysis identified many opportunities for PTs to explain their thinking or justify solutions. However, in addition to explaining and justifying one's own reasoning, the *CCSS-M*'s description for SMP3 also states: "Students at all grades can listen or read the arguments of

others, decide whether they make sense, and ask useful questions to clarify or improve the arguments" (NGA & CCSSO, p. 7). When coding for this specific aspect, we only found 13 content activities that provided explicit opportunities for PTs to analyze or critique the reasoning of others.

It should be noted that analyzing others' thinking may be occurring more frequently in content courses than our data suggest because this type of activity could happen in any class, with any task, if the instructor facilitates discourse that allows PTs to share their ideas and respond to each other. However, we could only verify that this was happening in 13 of the 93 cases where it was explicitly stated in an activity description or uploaded task. Further analysis of these 13 content activities revealed that the activities provided PTs with opportunities to (1) critique, analyze, or apply correct reasoning and (2) critique, analyze, or explain flawed reasoning. Below, we provide examples of each of these types of tasks and discuss how MTEs afforded PTs opportunities to engage in multiple dimensions of SMP3.

Category 1: Critique, analyze, or apply correct reasoning. In 10 of the 13 content activities that provided opportunities for PTs to analyze the reasoning of others, PTs were asked to consider correct, albeit non-traditional, mathematical thinking. For example, one MTE shared an excerpt of a class activity from Mathematics for Elementary Teachers (Beckmann, 2014). In this example (see Figure 2), PTs are provided with the written work of a third-grade student and a written record of the student's thinking. In addition to analyzing the student's method for generalizability, the PTs are asked to apply the student's method to solve a new problem.

When asked to compute 423 – 157, Pat (a third-grader) wrote the following:

4–
30–
34–
300
266

"You can't take 7 from 3; it's 4 too many, so that's negative 4. You can't take 50 from 20; it's 30 too many, so that's negative 30; and with the other 4, it's negative 34. 400 minus 100 is 300, and then you take the 34 away from the 300, so it's 266."

Q1: Discuss Pat's idea for calculating 423 – 157. Is her method legitimate? Analyze Pat's method in terms of expanded forms.
Q2: Could you use Pat's idea to calculate 317 – 289? If so, write what you think Pat might write, and also use expanded forms.

Figure 2: An excerpt of class activity 3L: A third-grader's method of subtraction (Beckmann, 2014, p. CA-61).

Son and Crespo (2009) suggest using non-traditional strategies (like the one in Figure 2) as a productive way of assisting PTs in developing their understanding of mathematical content. Research by Thanheiser and her colleagues (2014) suggests that PTs' incoming knowledge is primarily, if not exclusively, procedural. This procedural understanding may support them in solving a problem, such as 423 – 157, but may be limiting in their ability to make sense of alternative strategies (Thanheiser et al., 2014). For example, in the task in Figure 2, PTs are asked to consider a child's use of a partial differences strategy, where the place value parts in each column are subtracted independently of the others, to solve a subtraction problem that would require regrouping if solved using the standard algorithm. PTs' procedural knowledge may support them in solving this problem most-likely by using the standard regrouping algorithm, but that knowledge alone may not be sufficient for making sense of this child's thinking.

Offering PTs the problem in this context affords them an opportunity to think more deeply about the underlying structure and notation used in our base-ten place value system, as well as the additive nature of our number system. These concepts are important ideas that children must make sense of as they prepare to learn standard algorithms (Thompson & Bramald, 2002). In asking PTs to engage in SMP3 by analyzing an unfamiliar, yet generalizable, method, they will likely have to reexamine their own mathematical understandings and consider other legitimate ways to operate on numbers.

Son and Crespo (2009) also recommend including tasks that require PTs to analyze nontraditional approaches as a way to help them anticipate strategies and "develop teaching practices that use those strategies meaningfully" (p. 259). Such tasks can be an effective way to help PTs appreciate the benefits of alternative algorithms, including student-generated algorithms (see Example 1 in Castro Superfine, Prasad, Welder, Olanoff, & Eubanks-Turner, 2020, pp. 376-382, in this issue) and those used by children who were born/raised outside of the US or by children whose parents/guardians were born/raised outside of the US (Ron, 1998). Using student-generated strategies, among others, "may assist [PTs] in developing the habits of mind necessary for successfully completing the mathematical tasks of elementary teaching" (Salinas, 2009, p. 33). Fischer and Davis (2005) note that alternative algorithms can support various learning styles and highlight the fact that problems do not have a single best solution path, thus helping learners value a multitude of employable strategies.

To learn more about historically and culturally relevant alternative algorithms, we suggest reading Philipp (1996) and Fischer and Davis (2005). Both provide several alternative algorithms for whole number operations. Fischer and Davis (2005) provide detailed explanations

of algorithms commonly used outside of the US, while Philipp (1996) discusses the value of sharing some of the historical development of algorithms with PTs.

Category 2: Critique, analyze, or explain flawed student reasoning. Other activities provided by MTEs use student work as a way of addressing flawed reasoning, an additional dimension of SMP3: Argumentation. Utilizing examples that introduce flawed reasoning can support PTs' learning of children's common misconceptions and offer opportunities to reconsider their own understanding of challenging mathematical content (e.g., Ashlock, 2010).

The following activity provided by a respondent to our survey (discussed in Tobias, et al., 2014) takes analyzing flawed thinking in a unique direction by asking PTs to consider the types of conceptions or misconceptions that could lead a student to provide incorrect answers. This activity was designed around a video of a fifth-grade student, Ally, who is asked to compare various sets of fractions (Video 11: Ally; San Diego State University Research Foundation, Philipp, Cabral, & Schappelle, 2012). First, PTs are asked to solve three of the six fraction comparison problems that were posed to Ally in the video, given the exact same instructions that Ally had received:

> For each set of fractions below, circle the fraction that is greater (or if the fractions are equivalent, write "=" in between them).
>
> a. $1 \quad \frac{4}{3}$ b. $\frac{3}{6} \quad \frac{1}{2}$ c. $\frac{1}{7} \quad \frac{2}{7}$

After solving these problems using any chosen strategy, PTs are asked to consider the type of mathematical thinking that could lead a student to incorrectly answer each of these problems. In the facilitation guide for this task (Task Masters, 2014), the researchers note that although PTs are able to anticipate some common misunderstandings, they "often struggle to think of a reason why students would incorrectly conclude that the $\frac{1}{7} > \frac{2}{7}$" (p. 7).

The activity then proceeds with PTs watching a portion of the video where they witness Ally's verbal reasoning while she answers the six problems posed by the interviewer (her answers are displayed in Figure 3, which contains a screenshot from the video). PTs are then asked to work in small groups to discuss their thoughts about Ally's understanding and describe any misconceptions that Ally may have.

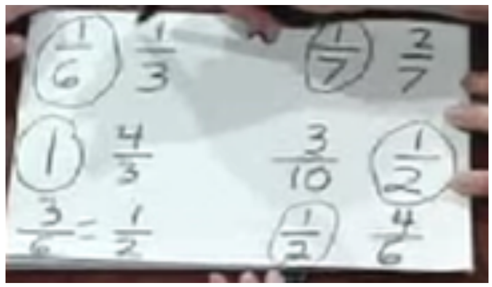

Figure 3: Ally's initial answers to the six fraction comparison problems (Video 11: Ally; San Diego State University Research Foundation et al., 2012).

This activity provides a genuine opportunity for PTs to make conjectures about the ways in which children's reasoning might lead to specific incorrect orderings and compare their conjectures to an actual child's discussion of her reasoning. Furthermore, PTs are tasked with drawing conclusions about Ally's misconceptions regarding fractions in general. The problems posed to Ally and the design of this task help PTs see how Ally's flawed reasoning is connected to several common fraction misconceptions, including the idea that all fractions are less than one (e.g., $1 > \frac{4}{3}$), fractions whose denominators are closer to one are greater (e.g., $\frac{1}{2} > \frac{3}{6}$) (Tobias et al., 2014), fractions with the larger numerators and/or denominators are greater (e.g., $\frac{3}{6} > \frac{1}{2}$, because 3 > 1 and 6 > 2; Task Masters, 2014), and "smaller is bigger with fractions" (e.g., $\frac{1}{7} > \frac{2}{7}$; Tobias et al., 2014, p. 186).

The MTE facilitates class discussions focused on these and other ways in which children and teachers' knowledge of fractions has been identified as limited by researchers (e.g., Behr, Wachsmuth, Post, & Lesh, 1984; Tobias, 2009; Zazkis and Chernoff, 2008). For example, the task facilitation guide suggests attention to the concept of "gap thinking," which Ally demonstrates in several examples where she calculates the differences between each fraction's numerator and denominator (Task Masters, 2014). PTs are asked to consider the mathematical validity of such reasoning and how it might lead to the misconception that the fraction "missing" the least number of pieces is greater (e.g. $\frac{1}{2} > \frac{3}{6}$, because 2 – 1 = 1 and 6 – 3 = 3) (Task Masters, 2014). Clarke and Roche (2009) note that such whole-number reasoning can lead to the conclusion that fractions "missing the same number of pieces" are always equivalent (e.g., $\frac{7}{8} = \frac{8}{9}$ because 8 – 7 = 1 and 9 – 8 = 1, so they both have a "gap of one"). Recognizing that this reasoning is flawed can serve as a springboard for considering a fraction as a certain number of equal-sized pieces (i.e., $\frac{a}{b}$ as *a*-pieces of size $\frac{1}{b}$). Thus, considering ways in which their future students might incorrectly reason about fractions provides an opportunity for PTs to reconsider their own understanding of fractions as numbers and question any potential misconceptions they may have.

Furthermore, the facilitation guide poses a potential extension to this task by asking PTs to consider "what other questions they would want to ask Ally (if they could interview her) and why" (Task Masters, 2014, p. 7). This exercise places PTs in the role of a teacher and requires them to think both mathematically and pedagogically about Ally's misconceptions and how they might be uncovered and addressed.

Discussion

In our analysis of a set of curricular activities used by responding MTEs in content courses, we found that the vast majority of opportunities for PTs to engage in justification asked them to explain their own reasoning. Pressing PTs to justify and construct viable arguments can offer MTEs deeper insight into PTs' conceptual understanding of content than an answer alone would typically provide. However, additional dimensions of SMP3: Argumentation include verifying and critiquing the reasoning of others and identifying flawed reasoning. Our investigation uncovered a small subset of tasks engaging PTs in these practices through the analysis of another person's mathematical reasoning. Examples were most often presented as work samples coming from children, including written work provided by a child, written records of children's thinking, and videos of children solving problems and explaining their reasoning. These tasks offered opportunities for PTs to consider correct, yet nontraditional, strategies, as well as flawed reasoning. Analysis of the activities grouped according to the correctness of the work analyzed allowed us to identify mathematical and pedagogical affordances of each type of task.

Based on our findings, we suggest that MTEs consider adding examples of others' thinking as an avenue for engaging PTs in the multiple dimensions of SMP3: Argumentation and deepening their conceptual understanding of content. We posit that adding artifacts of others' thinking can be a meaningful way to modify tasks MTEs already use in their content courses. Below, we provide concrete suggestions for MTEs who are interested in exploring this approach.

First, MTEs can provide opportunities for PTs to critique and respond to the work of their classmates using instructional strategies that support mathematical discussions. See the work of Saylor and Walton (2018), who extended a framework for creating math talk communities with

children (Hufferd-Ackles, Fuson, & Sherin, 2004; 2015) to their classes with PTs. This work suggests that MTEs may find it beneficial to apply other mathematical discussion-promoting strategies designed for children (e.g., Anderson, Chapin, & O'Connor, 2012; Chapin, O'Connor, & Anderson, 2009; Lamberg, 2013) to their content courses with PTs. Since the opportunities afforded to analyze others' thinking will rely on the strategies used by the PTs at that time, this practice may not necessarily lead to authentic opportunities to address a predetermined set of ideas, algorithms, or misconceptions. However, additional strategies and ideas can be added by the instructor as needed. One way to preemptively guide this process is by providing work samples that have been intentionally selected to target particular learning objectives. MTEs can carefully select examples of work produced by their or others' PTs in previous courses or create examples based on their knowledge of PTs' incoming conceptions and the common errors exhibited in their classrooms. This practice can help illuminate and address PTs' often limiting incoming conceptions (e.g., Thanheiser et al., 2014).

Additionally, since PTs are preparing for a future working with children, we argue that they should not only be involved in critiquing the reasoning of their peers but also analyzing the thinking of children. Researchers suggest that artifacts of children's thinking can add meaningful motivation for PTs in content courses, because PTs care deeply about children (Philipp, 2008). Furthermore, PTs who studied children's thinking have shown greater improvements in their own mathematical understandings than PTs who did not (Philipp et al., 2007).

One way that MTEs can use artifacts of children's thinking to support the development of PTs' content knowledge is by providing authentic opportunities for PTs to see the need to reconsider their own understanding of and *relearn* mathematical content (e.g., Castro Superfine et al., 2020, in this issue). PTs in content courses often feel confident in their mostly procedural

understanding of lower-level elementary content (Thanheiser et al., 2014), therefore, "an important aspect of the work of MTEs becomes creating opportunities for PTs to question the basis of their current knowledge and see an authentic need to restructure their understanding of seemingly "simple" content" (Castro Superfine et al., 2020, p. 376). By exposing PTs to a nontraditional, yet valid, child-generated strategy for subtraction (see Example 1, pp. 376-382) Castro Superfine and her colleagues helped PTs realize that their incoming understanding of elementary content, and often superficial use of algorithms, may not be sufficient for the varied tasks of teaching, which motivated them to learn about subtraction more deeply.

The analysis of children's thinking, however, is not likely to occur naturally in content courses for PTs, like the critiquing of their peers might, since content courses are not often related to practicum experiences where PTs might experience children's reasoning in person. Thus, the use of this instructional strategy requires initiative and planning on the part of the MTE. A variety of samples of children's thinking should be considered, including examples of strategies that are flawed (e.g., Ashlock, 2010), as well as those that are correct yet nontraditional, incorporating student generated strategies and culturally relevant algorithms (e.g. Philipp, 1996; Ron, 1998). We also suggest that MTEs consider sharing children's mathematical thinking through the use of video, in addition to providing samples of written work, as it affords PTs the opportunity to analyze children's in-the-moment thinking (Philipp, 2008).

Resources Offering Artifacts of Children's Thinking

Since the majority of instructors of mathematics content courses for PTs do not have experience teaching elementary-aged children (Masingila et al., 2012), MTEs may not have examples of children's work at their disposal to share. However, there are many free and commercially available resources that MTEs can utilize for this purpose. In fact, many of the

textbooks designed for use in content courses for PTs, including several of the textbooks used by MTEs in our study, provide at least some written work and/or videos involving children. For example, the newest editions of textbooks by Sowder et al. (2017) and Beckmann (2018) provide examples of children's written work and ask PTs to consider the mathematical structure of various approaches while making sense of what the work may mean in terms of the child's mathematical understanding. Others provide videos that can be accessed on the textbook publisher's website or by clicking links embedded into e-text versions (e.g., Van de Walle et al., 2019).

Additionally, there are many (non-text) books that provide printed copies of children's work. Some also provide videos of children's thinking through associated DVDs or online content (e.g., Carpenter, Fennema, Franke, Levi, & Empson, 2015; Parrish, 2010). Other video collections exist independently (e.g., Integrating Mathematics and Pedagogy to Illustrate Children's Reasoning [IMAP], San Diego State University Foundation et al., 2012; Learning and Teaching with Learning Trajectories [LT][2], Clements & Sarama, 2017/2019). Appendix B includes a compilation of available resources for MTEs to consider. This list is not meant to be exhaustive but to instead serve as a starting point for MTEs' exploration.

Conclusion

National standards for teacher preparation recommend that PTs be provided opportunities in teacher preparation programs to develop their understanding of the *CCSS-M* SMPs and how such practices can meaningfully support the learning of mathematical content (CBMS, 2012). The results of our study indicated that MTEs intended to address all eight of the SMPs in over 80% of their content courses for PTs, mostly through planning lessons in ways that allow for PTs to engage in the practices. Furthermore, SMP3: Argumentation was selected by MTEs as the

most addressed practice in more classes than any other SMP. Our analysis of content descriptions and class activities provided by the MTEs revealed substantially more opportunities for PTs to engage with SMP3: Argumentation through explaining their reasoning than by analyzing the reasoning of others. By sharing concrete examples of tasks that ask PTs to critique, analyze, or apply correct reasoning, and to critique, analyze, or explain flawed student reasoning, we aim to support MTEs in providing meaningful opportunities for PTs' to engage in multiple dimensions of SMP 3: Argumentation.

We note that the findings of this work are based on MTEs' textbook use and descriptions and samples of activities from their content courses and hypothesize that more, and potentially more explicit, attention to the SMPs might be identified through observing such activities being enacted with PTs. Furthermore, we encourage increased transparency around all of the SMPs in MTEs' work with PTs. By "being more verbal about our instructional and curricular decisions with our own [PTs]," (Yow, Eli, Beisiegel, McCloskey, & Welder, 2016, p. 63) we can help them become aware of the reasoning behind some of our actions, such as selecting artifacts of flawed thinking to highlight a common misconception . By helping PTs become more aware of *how* they are meaningfully engaging in SMPs while developing their own mathematical knowledge, MTEs may be able to better prepare PTs for engaging their future students in such practices as well. Additional research is needed to explore these and other ways MTEs can foster PTs' working understanding of the SMPs and the valuable role each plays in their learning, and future teaching, of mathematics.

References

Anderson, N. C., Chapin, S. H., & O'Connor, C. O. (2012). *Classroom discussions in math: A facilitator's guide to support professional learning of discourse and the Common Core.* Sausalito, CA: Math Solutions.

Ashlock, R. B. (2010). *Error patterns in computation: Using error patterns to help each student learn* (10th ed.). Boston, MA: Allyn & Bacon.

Association of Mathematics Teacher Educators. (2017). Standards for preparing teachers of mathematics. Available online at http://amte.net/standards

Bassarear, T. (2016). *Mathematics for elementary school teachers*. (6th ed.). Boston, MA: Cengage Learning.

Beckmann, S. (2018). *Mathematics for elementary teachers with activities*. (5th ed.). Boston, MA: Pearson Education.

Behr, M. J., Wachsmuth, I., Post, T. R., & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education*, *15*(5), 323–341.

Billstein, R., Libeskind, S., & Lott, J. (2013). *A problem solving approach to mathematics for elementary school teachers*. (11th ed.). Boston, MA: Pearson Education.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2015). *Children's mathematics: Cognitively guided instruction* (2nd ed.). Portsmouth, NH: Heinemann.

Castro Superfine, A., Prasad, P. V., Welder, R. M., Olanoff, D., & Eubanks-Turner, C. (2020). Exploring mathematical knowledge for teaching teachers: Supporting prospective teachers' relearning of mathematics. In A. Appova, R. M. Welder, and Z. Feldman, (Eds.), *Supporting Mathematics Teacher Educators' Knowledge and Practices for Teaching Content to Prospective (Grades K-8) Teachers. Special Issue: The Mathematics Enthusiast*, ISSN 1551-3440, *17*(2&3), pp. 367-402. ScholarWorks: University of Montana. Retrieve (open access) from: https://scholarworks.umt.edu/tme

Chapin, S. H., O'Connor, C. O., & Anderson, N. C. (2009). *Classroom discussions: Using math talk to help students learn.* Sausalito, CA: Math Solutions.

Clarke, D. M. & Roche, A. (2009). Students' fraction comparison strategies as a window into robust understanding and possible pointers for instruction. *Educational Studies in Mathematics, 72*, 127-138.

Clements, D. H. & Sarama, J. (2017/2019). Learning and teaching with learning trajectories [LT]^2. Retrieved from Marsico Institute, Morgridge College of Education, University of Denver, http://LearningTrajectories.org

Common Core State Standards Initiative. (2019). Standards for mathematical practice. Available online at http://www.corestandards.org/Math/Practice/

Conference Board of Mathematical Sciences. (2001). The mathematical education of teachers. Providence, RI: American Mathematical Society and Mathematical Association of America.

Conference Board of Mathematical Sciences. (2012). *The mathematical education of teachers II*. Providence, RI: American Mathematical Society and Mathematical Association of America.

Empson, S. B. & Levi, L. (2011). Extending children's mathematics: Fractions and decimals. *Mathematics Education*, *27*(4), 403-434.

Erikson Institute Early Math Collaborative. (2019). idea library. Retrieved from https://earlymath.erikson.edu/ideas

Ferrini-Mundy, J. (2000). Principles and standards for school mathematics: A guide for mathematicians. *Notices of the American Mathematical Society*, *47*(8). 868–876.

Fischer, J. F. & Davis, J. F (2005). *Algorithms: Through the ages and around the world*. Texas University: Texas University Press.

Fosnot, C. T. & Jacob, B. (2010). *Young mathematicians at work: Constructing algebra*. Reston, VA: National Council of Teachers of Mathematics.

Fosnot, C. T. & Dolk, M. (2001a). *Young mathematicians at work: Constructing multiplication and division*. Portsmouth, NH: Heinemann.

Fosnot, C. T. & Dolk, M. (2001b). *Young mathematicians at work: Constructing number sense, addition, and subtraction*. Portsmouth, NH: Heinemann.

Fosnot, C. T. & Dolk, M. (2002). *Young mathematicians at work: Constructing fractions, decimals, and percents*. Portsmouth, NH: Heinemann.

Graybeal, C. D. (n.d.). Graybeal videos. Retrieved November 27, 2019 from https://graybealmath.weebly.com/graybeal-videos.html

Greenberg, J. & Walsh, K. (2008). *No common denominator: The preparation of elementary teachers in mathematics by America's education schools*. Washington, DC: National Council on Teacher Quality.

Hufferd-Ackles, K., Fuson, K. C., & Sherin, M. G. (2004). Describing levels and components of a math-talk learning community. *Journal for Research in Mathematics Education*, *35*(2), 81–116.

Hufferd-Ackles, K., Fuson, K. C., & Sherin, M. G. (2015). Describing levels and components of a math-talk learning community. In E. A. Silver & P. A. Kenney (Eds.), *More lessons learned from research: Volume 1: Useful and usable research related to core mathematical practices* (pp. 125–134). Reston, VA: NCTM.

Illustrative Mathematics. (2019). Retrieved from https://www.illustrativemathematics.org

Inside Mathematics. (2019). Classroom videos. Retrieved from Charles A. Dana Center, University of Texas at Austin, http://www.insidemathematics.org/classroom-videos

Koestler, C., Felton, M., Bieda, K., & Otten, S. (2013). *Connecting the NCTM process standards & the CCSSM practices*. Reston, VA: National Council of Teachers of Mathematics.

Lamberg, T. (2013). *Whole class mathematics discussions: Improving in-depth mathematical thinking and learning*. Upper Saddle River, NJ: Pearson

Lamon, S. J. (2012). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. New York: Routledge.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Masingila, J. O., Olanoff, D., & Kwaka, D. K. (2012). Who teaches mathematics content courses for prospective elementary teachers in the United States? Results of a national survey. *Journal of Mathematics Teacher Education*, *15*(5), 347–58.

Max, B. & Newton, J. (2017). Mathematics preparation of elementary teachers: Results of a national survey. *AMTE Connections*. Retrieved from https://amte.net/connections/2017/09/mathematics-preparation-elementary-teachers-results-national-survey

Max, B. (2018). *The mathematical content preparation of elementary teachers*. Unpublished doctoral dissertation. Indianapolis, IN: Purdue University. Retrieved from Dissertations Abstracts International 80-01A(E).

McCrory, R., Francis, A., & Young, S. (2008). *Resource use by instructors of mathematics classes for future elementary teachers: Results of a survey*. Paper presented at the meeting of the International Committee on Mathematics Instruction (ICMI-11) in Monterrey, Mexico.

McNamara, J. (2015). *Beyond invert and multiply, grades 3-6: Making sense of fraction computation*. Sausalito, CA: Scholastic Inc.

McNamara, J. & Shaughnessy, M. M. (2015). *Beyond pizzas & pies: 10 essential strategies for supporting fraction sense, grades 3-5* (3rd ed.). Sausalito, CA: Math Solutions.

Musser, G. L., Peterson, B. E., & Burger, W. F. (2013). *Mathematics for elementary teachers: A contemporary approach*. Hoboken, NJ: Wiley.

National Council of Teachers of Mathematics (1980). *An agenda for action: Recommendations for school mathematics of the 1980s*. Reston, VA: Author.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics (1991). *Professional standards for the teaching of mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics (1995). *Assessment standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: Author.

National Council of Teachers of Mathematics (n.d.). Principles to actions professional learning toolkit. Retrieved from https://www.nctm.org/PtAToolkit

National Governors Association Center for Best Practices and the Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington, DC: Authors.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: The National Academies Press. https://doi.org/10.17226/9822

Parrish, S. (2010). *Number talks: Helping children build mental math and computation strategies, grades K-5*. Sausalito, CA: Math Solutions.

Parrish, S. (2016). *Number talks: Fractions, decimals, and percentages*. Sausalito, CA: Math Solutions.

Philipp, R. A. (1996). Multicultural mathematics and alternative algorithms. *Teaching Children Mathematics*, *3*(3), 128-134.

Philipp, R. A., Ambrose, R., Lamb, L. C., Sowder, J. T., Schappelle, B. P., Sowder, L., Thanheiser, E., & Chauvot, J. (2007). Effects of early field experiences on the mathematical content knowledge and beliefs of prospective elementary school teachers: An experimental study. *Journal for Research in Mathematics Education*, *38*, 438-476.

Philipp, R. A. (2008). Motivating prospective elementary school teachers to learn mathematics by focusing upon children's mathematical thinking. *Issues in Teacher Education*, *17*(2), 7-26.

Research Advisory Committee of the National Council of Teachers of Mathematics. (1988). NCTM curriculum and evaluation standards for school Mathematics: Responses from the research community. *Journal for Research in Mathematics Education*, *19*(4), 338-344.

Ron, P. (1998). My family taught me this way. In L. J. Morrow and M. J. Kenney (Eds.), *The Teaching and Learning of Algorithms in School Mathematics* (pp. 115-119). Reston, VA: National Council of Teachers of Mathematics.

Salinas, T. M. (2009). Beyond the right answer: Exploring how preservice elementary teachers evaluate student-generated algorithms. *Mathematics Educator*, *19*(1), 27-34.

San Diego State University Research Foundation, Philipp, R., Cabral, C., & Schappelle, B. (2012). *IMAP: Integrating mathematics and pedagogy: Searchable collection of children's mathematical thinking video clips and facilitator's guide*. Boston, MA: Allyn & Bacon.

Saylor, L. L. & Walton, J. B. (2018). Creating a math-talk learning community with preservice teachers. *School Science and Mathematics*, *118*(8), 348-357.

Schifter, D., Bastable, V., & Russell, S. J. (1999). *Developing mathematical ideas*. Parsippany, NJ: Dale Seymour Publications.

Son, J. W. & Crespo, S. (2009). Prospective teachers' reasoning and response to a student's non-traditional strategy when dividing fractions. *Journal of Mathematics Teacher Education*, *12*(4), 235-261.

Sowder, J., Sowder, L., & Nickerson, S. (2017). *Reconceptualizing mathematics for elementary school teachers*. (3rd ed.). New York, NY: W. H. Freeman and Company.

Task Masters (2014). Comparing fractions using reasoning and sense-making: Facilitation guide. Retrieved from https://mathtaskmasters.com

Teaching Channel. (2019). Videos. Retrieved from https://www.teachingchannel.org/videos

TeachingWorks. (2019). Video resources. Retrieved from School of Education, University of Michigan, http://www.teachingworks.org/support-resources/video-resources

Thanheiser, E., Browning, C., Edson, A. J., Lo, J., Whitacre, I., Olanoff, D., & Morton, C. (2014). Prospective elementary mathematics teacher content knowledge: What do we know, what do we not know, and where do we go? *The Mathematics Enthusiast*, *11*(2), 433-448.

Tobias, J. M. (2009). Preservice teachers' conceptualization of fraction multiplication. In S. L. Swars, D. W. Stinson, & S. Lemons-Smith (Eds.), *Proceedings of the 31st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 1276–1283). Atlanta, GA: Georgia State University.

Tobias, J. M., Olanoff, D., Hillen, A., Welder, R. M., Feldman, Z., & Thanheiser, E. (2014). Research-based modifications of elementary school tasks for use in teacher preparation. In K. King (Ed.), *Annual Perspectives in Mathematics Education: Using Research to Improve Instruction* (pp. 181-192). Reston, VA: National Council of Teachers of Mathematics.

Thompson, I. & Bramald, R. (2002). *An investigation of the relationship between young children's understanding of the concept of place value and their competence at mental addition* (Report for the Nuffield Foundation). Newcastle upon Tyne: University of Newcastle upon Tyne.

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2019). *Elementary and middle school mathematics: Teaching developmentally* (10th ed). Upper Saddle River, NJ: Pearson.

Video Mosaic Collaborative. (n.d.). Retrieved from Robert B. Davis Institute for Learning, Rutgers University Libraries, https://videomosaic.org/

Youcubed at Stanford University. (n.d.). Retrieved from http://youcubed.org

Yow, J., Eli, J., Beisiegel, M., McCloskey, A., & Welder, R. M. (2016). Challenging transitions and crossing borders: Preparing novice mathematics teacher educators to support novice K-12 mathematics teachers. *Mathematics Teacher Education and Development*, *18*(1), 52-69.

Zazkis, R. (2011). *Relearning mathematics: A challenge for prospective elementary school teachers.* Information Age Publishing: Charlotte: NC.

Zazkis, R. & Chernoff, E. (2008). What makes a counterexample exemplary? *Educational Studies in Mathematic*s, *68*(3), 195-208.

Appendix A: Standards for Mathematical Practice (NGA & CCSSO, 2010)

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

CCSS.MATH.PRACTICE.MP1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

CCSS.MATH.PRACTICE.MP2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

CCSS.MATH.PRACTICE.MP3 Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

CCSS.MATH.PRACTICE.MP4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

CCSS.MATH.PRACTICE.MP5 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they

know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

CCSS.MATH.PRACTICE.MP6 Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

CCSS.MATH.PRACTICE.MP7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

CCSS.MATH.PRACTICE.MP8 Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Appendix B: Resources for Artifacts of Children's Mathematical Thinking

(Resources in each category are listed alphabetically by title.)

Commercially available books that offer examples of children's written work:

Error Patterns in Computations: Using Error Patterns to Help Each Student Learn (10th edition) (Ashlock, 2010)

Extending Children's Mathematics: Fractions and Decimals: Innovations in Cognitively Guided Instruction (Empson & Levi, 2011)

Teaching Fractions and Ratios for Understanding: Essential Content Knowledge and Instructional Strategies for Teachers (Lamon, 2012)

Young Mathematicians at Work: Constructing Algebra (Fosnot & Jacob, 2010)

Young Mathematicians at Work: Constructing Fractions, Decimals, and Percents (Fosnot & Dolk, 2002)

Young Mathematicians at Work: Constructing Multiplication and Division (Fosnot & Dolk, 2001a)

Young Mathematicians at Work: Constructing Number Sense, Addition, and Subtraction (Fosnot & Dolk, 2001b)

Commercially available books that offer video components, in addition to examples of children's written work:

Beyond Pizza & Pies: 10 Essential Strategies for Supporting Fraction Sense (2nd edition) (McNamara & Shaughnessy, 2015)

Beyond Invert & Multiply: Making Sense of Fraction Computation (McNamara, 2015)

Children's Mathematics: Cognitively Guided Instruction (2nd edition) (Carpenter, Fennema, Franke, Levi, & Empson, 2015)

Developing Mathematical Ideas (Schifter, Bastable, & Russell, 1999)

Number Talks: Helping Children Build Mental Math and Computation Strategies (Parrish, 2010)

Number Talks: Fractions, Decimals, and Percentages (Parrish, 2016)

Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School (Carpenter, Franke, & Levi, 2003)

Whole Class Mathematics Discussions: Improving In-Depth Mathematical Thinking and Learning (Lamberg, 2013)

Commercially available video collections:

IMAP: Integrating Mathematics and Pedagogy to Illustrate Children's Reasoning (San Diego State University Foundation, Philipp, Cabral, & Schappelle, 2012)

Free online video resources:

Erikson Institute Early Math Collaborative: https://earlymath.erikson.edu/ideas

Graybeal Videos (Graybeal, n.d.): https://graybealmath.weebly.com/graybeal-videos.html

Illustrative Mathematics: https://www.illustrativemathematics.org

Inside Mathematics: Classroom Videos: http://www.insidemathematics.org/classroom-videos

Learning and Teaching with Learning Trajectories [LT]^2 (Clements & Sarama, 2017/2019): http://LearningTrajectories.org

Teaching Channel: https://www.teachingchannel.org/videos

Video Mosaic Collaborative: https://videomosaic.org

Youcubed at Stanford University: http://youcubed.org

Online video resources available with a paid subscription:

NCTM Principles to Actions Professional Learning Toolkit: https://www.nctm.org/PtAToolkit

TeachingWorks: http://www.teachingworks.org/support-resources/video-resources