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Supporting Mathematics Teacher Educators' Practices for Facilitating Prospective Teachers' Mathematical Explanations in Content Courses

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Abstract: The push for more student-generated explanations in K-8 mathematics classrooms, together with prospective teachers' well-documented weaknesses in both providing adequate explanation and appreciating what constitutes a sound mathematical explanation, points to the need for more experiences for prospective elementary teachers (PTs) to formulate explanations of mathematical ideas before asking them to facilitate students' explanations. Creating experiences for PTs to wrestle with challenging mathematics and learn to elicit, develop, and evaluate mathematical explanations is a responsibility of mathematics teacher educators (MTEs) who teach content courses for teachers. Our work considers MTEs' practices that encourage PTs to construct and critique explanations. Through examination of data from a professional development designed for MTEs, we identified aspects of PTs' mathematical explanations that MTEs attended to while observing and discussing a math content course for PTs. From this study we identified five MTE practices that provide opportunities to engage PTs in constructing mathematical explanations. Specific strategies for MTEs to implement these practices are shared and related dilemmas are discussed.

Keywords: teacher knowledge; professional development; laboratory classroom; mathematical explanations; mathematics teacher educators

Introduction

In recent years, national standards and research (Common Core State Standards Initiative [CCSSI], 2010; National Council of Teachers of Mathematics [NCTM], 2000, 2014) have placed increased emphasis on engaging K-8 students in “construct[ing] viable arguments and critique[ing] the reasoning of others,” (CCSSI, 2010, p. 6). Focusing teaching practice on student-generated explanation, rather than a teacher's explanation, provides students more

powerful learning opportunities (Rittle-Johnson, Saylor, & Swygert, 2008) to grapple with important mathematical ideas and provides teachers opportunities to assess, explore, and extend student thinking (Chapin, O'Connor, Anderson, 2003; O'Connor, 2001; Pirie & Schwarzenberger, 1988). Concomitant with the increased demands for K-12 students, standards for mathematics teacher learning and preparation (Association of Mathematics Teacher Educators [AMTE], 2017) have called on teacher education programs to prepare elementary (K-5) teachers to enact *ambitious instruction* for all learners (Lampert et al., 2013). Ambitious instruction includes a teacher's ability to "use [mathematical] language with precision and care" (AMTE, 2017, p. 9) and elicit and respond to student thinking (Lampert et al., 2013; Stein, Engle, Smith, & Hughes, 2008).

Prospective elementary teachers (those seeking certification to teach children ages 4-11), who have not experienced mathematics instruction focused on eliciting and responding to student thinking and explanations are likely to struggle in facilitating such instruction with their own students (Hallman-Thrasher, 2017). Because the mathematical experiences of prospective teachers (PTs) are critical to helping them support their students' conceptual understanding (Conference Board of Mathematics Sciences [CBMS], 2001; Greenberg & Walsh, 2008) mathematics teacher educators (MTEs) who teach mathematics content courses specifically designed for PTs have a responsibility to help them develop these pedagogical skills. This article shares what we learned from our study of a professional development program designed for MTEs who teach content courses for PTs. We identified five MTE practices that foster PTs' construction of mathematical explanations. To situate these recommendations, we first provide an overview of research on mathematical explanation.

Mathematical Explanation

Students learn more when they explain an idea to someone else (Rittle-Johnson et al., 2008), and students whose explanations go beyond explaining the steps of a procedure demonstrate stronger conceptual understanding than those whose explanations do not (Matthews & Rittle-Johnson, 2009). These findings are reflected in professional standards' emphasis on student-generated explanation (CCSSI, 2010; NCTM, 2014) and in practitioner recommendations (Banes, Lopez, Skubal, & Perfecto, 2017). Yet, classroom teachers typically introduce new content through teacher explanations (Perry, 2000). This discrepancy may occur because teachers concerned about content coverage prefer the short-term time-efficiency of teacher-generated explanations to the long-term time commitment of allowing students to construct their own explanations. Also, teachers may not have experienced specialized training in their teacher preparation programs to support students in constructing adequate explanations of their conjectures and comparing and critiquing different explanations (DeVilliers, 1990; Lampert 1990; Stylianides, 2007). One key venue to provide this specialized training in facilitating student explanations is content courses for PTs.

Broadly, a mathematical explanation refers to statements that “mathematically justify and help others understand why a statement is true” (Ball & Bass, 2005). Mathematical explanations not only describe how something was done (e.g., a solution obtained, an algorithm executed), but should also convey *why* an approach is mathematically valid. Explanation serves as justification of a mathematical argument and clarification of “one’s (mathematical) thinking that might not be apparent to others” (Yackel & Cobb, 1996, p. 467). A rich mathematical explanation should describe a procedure or strategy and justify why it is appropriate in ways that are both understandable to and mathematically precise for a given audience, and that are mathematically

valid to a more knowledgeable other. Such explanations are often supported by representations (diagrams, equations, graphs, charts).

The work of teachers to facilitate explanation can be complicated by the subjective nature of explanation; what constitutes adequate mathematical explanation “is interactively constituted by the students and the teacher” as they engage in mathematical activity (Yackel & Cobb, 1996, p. 469). The community in which the explanation is shared defines what constitutes an adequate explanation; whereas a diagram might count as “good enough” in a first-grade class, a college-level abstract algebra class would likely require a formal mathematical proof. This requires explicit attention to establishing and developing norms for explaining. MTEs face the added challenge of how to “develop preservice teachers’ awareness of how to connect what they are learning to teaching” (Superfine & Li, 2014, p. 305). PTs need to have mathematical experiences in their content courses that are similar to those they are expected to create in their future math classrooms. MTEs must not only ensure that PTs understand and can explain the mathematics content they will teach; they must also support PTs in learning to explain mathematics content in ways that are accessible to children, to familiarize themselves with children’s ways of thinking about mathematics, and to facilitate children’s construction of rich mathematical explanations. Achieving these goals places greater demands on MTEs and PTs than would the work of a typical college mathematics class. This article discusses five practices we have identified that would support MTEs in meeting these increased demands.

About Our Study: Professional Development Design for Mathematics Teacher Educators

The Center for Proficiency in Teaching Mathematics hosted an eight-day summer professional development institute for 65 MTEs. The institute focused on two guiding questions:

- (1) What mathematical knowledge and practices play a central role in the everyday work of

teaching? and (2) What are promising approaches for helping PTs learn mathematics for teaching and learn to use it in their work? To situate this professional development in the context of MTEs' daily teaching practice (Putnam & Borko, 1999), the institute's central feature involved daily observations (including pre- and post-observation debriefing sessions) of a mathematics content course specifically designed for PTs that was taught by a distinguished MTE. Hereafter, we refer to this class as *lab class* because it served as a laboratory for MTEs to develop, test, and reflect on hypotheses about teaching mathematics content to PTs. The MTE instructor was a leader in mathematics teacher preparation with over 20 years of experience in researching mathematics teacher preparation and teaching prospective elementary mathematics teachers. Her teaching in the lab class focused on developing conceptual understanding of mathematics content as it is needed specifically for teaching. The MTEs spent 2 hours daily observing the lab class and 2 hours in pre and post debrief sessions. The remaining 4 hours of their day were spent in sessions focused on particular mathematics content of institute.

To determine what practices MTEs deemed impactful for PTs explanations we examined video recordings of the lab class, MTEs' reflective notebooks, follow-up focus groups with MTEs, and research team field notes of MTE pre- and post-debrief sessions to identify recurring themes.

Our Findings: Facilitating Prospective Teachers' Explanations

We share the ways MTEs attended to PTs' explanations and the lab class MTE's actions during the lab class that supported PTs' mathematical explanations. Based on the recurring themes in the MTEs' focus, we identified five practices for engaging PTs in constructing mathematical explanations: 1) using explanation-worthy tasks with PTs, 2) co-constructing norms for explanation, 3) managing flawed representations and explanations, 4) asking PTs to

map between representations, and 5) making instructor's pedagogical decisions explicit. We discuss these practices, MTEs' perspectives and struggles with each practice, and recommendations for MTEs engaged in this work.

Using Explanation-Worthy Tasks with Prospective Teachers

Not all tasks are inherently 'good' for PTs. Explanation-worthy tasks for PTs are those that provide opportunities for PTs to explain and justify their thinking about important mathematical concepts and procedures *and* consider ways that students think about mathematics. These tasks share many of the same features of worthwhile tasks for K-12 students (Stein, Smith, Henningsen, & Silver, 2000): they have multiple solutions paths, are accessible yet challenging for learners, and connect to important mathematics concepts. However, explanation-worthy tasks for PTs must do more. Explanation-worthy tasks for PTs need to both be appropriately challenging and engaging for adults and relate to grades K-8 mathematics. They should challenge PTs' thinking about what it means to do and know mathematics, make evident to PTs their own understanding of mathematics content, challenge their misconceptions, and provide opportunities to explore K-8 student thinking and misconceptions. One such problem, the Cookie Jar Problem (Figure 1), was identified by MTEs as a particularly powerful task for promoting PTs' explanations.

There was a jar of cookies on the table. Kira was hungry because she hadn't had breakfast, so she ate half the cookies. Then Steve came along and noticed the cookies. He thought they looked good, so he ate a third of what was left in the jar. Niki came by and decided to take a fourth of the remaining cookies with her to her next class. Then Kayla came dashing up and took a cookie to munch on. When Pam looked at the cookie jar, she saw that there were two cookies left. "How many cookies were in the jar to begin with?" she asked Kira (lab class lesson plans).

Figure 1. Cookie jar problem used in the lab class.

The Cookie Jar Problem is an explanation-worthy task because, like worthwhile mathematical tasks for K-8 students, it has multiple solution methods and representations, a high level of accessibility (anyone could start with a guess and check approach), and a focus on several significant conceptual ideas regarding fractions (e.g., representing fractions in algebraic and geometric contexts and recognizing the effects of a changing whole). The multiple solution methods provided opportunities for PTs to continue engaging in the mathematics of the problem even after finding their own solution. Because all PTs could engage with content, more perspectives could be brought into the collective construction of an explanation. The underlying mathematics concept, was both significant to elementary school mathematics curriculum and the problem required more than a straightforward procedure, so it was worth the time invested to understand it and explain it well. Additionally, the Cookie Jar Problem was an explanation-worthy task for PTs because there is not an obvious, well-known procedure to use. The problem was constructed to prompt a mistake about what the fractions represented. As a consequence, the task challenged PTs' understanding of fractions and algebraic equations and provided PTs the opportunity to reflect on the mathematics they would teach (Ball & Bass, 2000).

MTEs' reflections show that some elements of the instructor's enactment of the task also made it explanation-worthy. The MTEs repeatedly referenced how the instructor repeatedly pressed for explanations even after a correct solution had been shared and in their writings expressed surprise that so much mathematics could be accomplished through one problem. The MTE instructor required the PTs to understand and be able to explain others' approaches and ways approaches were connected. The task challenged PTs' thinking about what mathematics for K-8 students can look like because the lab class encouraged them to justify solutions. The PTs had opportunities to consider what grade K-8 student explanations may involve and see that

elementary grade students *can* be held accountable for justifying solutions, not merely stating solutions. MTEs noted the mathematical work that PTs needed to undertake to facilitate K-8 student explanations of the Cookie Jar Problem. PTs had to consider what fractions in the problem represented, and how the whole changed as the problem progressed. One point of confusion for PTs that was discussed across MTE small groups was being clear if a fraction represented an amount of cookies that had been eaten or the amount of cookies that were remaining and how that should be conveyed in representing the solution. MTE participants claimed that recognizing that certain algebraic expressions actually referenced different parts of the problem helped PTs to realize the importance of being clear about defining variables and mapping parts of equations back to the original statement of the problem.

In choosing or developing explanation-worthy task for PTs, MTEs should consider tasks that are accessible yet challenging for children and adults, attend to important mathematical ideas at the K-8 level, allow for multiple solutions or a single solution that can be reached via multiple strategies, and address potential PT or K-8 student misconceptions. In the following sections we discuss how these features of the task play out in supporting PTs' rich mathematical explanations.

Co-Constructing Norms for Mathematical Explanations

MTEs should consider how norms for mathematical explanations, as well as criteria for good explanations (elements that strengthen argumentation), are established within their practices and communicated to PSTs (Yackel & Cobb, 1996). The PTs in our study expressed "I don't think I have a very good idea of what you think is a clear explanation." In their written reflections and small group discussions, the MTE participants focused on three instructor moves to establish what counted as a good mathematical explanation: co-developing norms for

explaining with the class; instructor moves to support those norms in the lab class; and reinforcing those norms with whole-class feedback.

In collaboration with their PTs, MTEs should develop norms around what counts as adequate explanations, both in general and in the context of specific problems. Establishing these norms together engages PSTs in the important work of understanding what makes an explanation adequate and provides them needed opportunities to wrestle with important mathematical and pedagogical issues before they face them as teachers in their own K-8 classrooms. The lab class instructor facilitated a conversation with the PTs in which they generated a list of features of good explanations that was posted in the lab classroom (Figure 2). In response to PT questions about what constituted a “good” explanation, the MTE instructor turned the question back to students, and by asking clarifying questions and posing counter examples, she helped them elaborate further on their ideas. When she shared the completed list with the class, she worked to present the list close to language PTs used so that it was meaningful to them.

- Makes clear at the outset what is being explained, and why you start there, and carefully connects the explanation to the question or idea being explained
- Starts from the beginning, and traces the logical flow of the reasoning
- Should be logical and complete, makes conclusion clear and links back to the original question, claim, or problem
- Might number the steps if appropriate, or labels parts of a diagram
- Strives to be as simple and clear as possible
- Defines terms as needed, uses available definitions as needed
- Uses representation(s) accurately (algebraic, geometric, etc.) and combining representations
- Links the language and diagrams clearly to the steps of the argument
- Shows what something means or why [it] is true, and is convincing to the person to whom you are explaining
- Is calibrated to the context (considers the person to whom you are explaining, and what is already established as true and does not need more explanation)

Figure 2. “Features of ‘good’ mathematical explanations” generated by lab class.

To maintain norms, MTEs should focus class discussions on the solution process and its justification, not solely the answer. Questions posted must attend to both how and why a strategy works and require PTs to attend to others' explanations. PTs need also to be given the opportunity to question one another's work directly (not filtered through the MTE instructor). In addition, MTEs can provide feedback to PTs that emphasize features of explanations that are valued and further support the norms they have established. In the lab class, PTs' notebooks were reviewed daily by the MTE to see their thinking in progress and to provide individual written feedback. MTEs noted that the instructor took time at the beginning of each class to share feedback on the previous day's work. The instructor praised students for working to be clear in their written explanations, supporting written explanations with pictures, and their willingness to share their thinking in the whole group. While this might not be feasible in large classes or for MTEs with heavy teaching loads, the lab class instructor also gave general feedback to the whole class at the start of each class meeting. She commented on work she had observed during the previous class meeting to set expectations for productive behaviors and to encourage PTs to adopt these behaviors (Figure 3). She particularly praised students for behaviors that she knew were intimidating to enact, like sharing thinking even when it was still a work in progress.

Our Mathematics Work Yesterday

- Encountered some of the core ideas about fractions
 - Issues of what constitutes the unit
 - Different interpretations of fractions
- Demonstrated willingness to express your thinking, even when still formulating it
- Listened closely and responded to other people
- Used language and pictures to express ideas

Figure 3. Feedback given to the whole class after the first class meeting.

Several MTEs in our study wondered whether teachers should “lay out guidelines, a priori, or allow it [norms of a good explanation] to emerge?” They wondered if it was worth the time investment for PTs to develop norms on their own if the instructor could have explicitly stated them upfront, as well as how an instructor can ensure that mathematical ideas developed by PTs are valid, precise, and display integrity to mathematics. This up-front time investment to establish norms was invaluable to later work of the lab class because the list of co-constructed norms and instructor feedback served as important touchstones that were referenced throughout the class by PTs and throughout reflections and group discussion by the MTEs. The ability to quickly reference this common set of ideas saved time when issues of the clarity and completeness of explanations surfaced. In writings and discussions, MTEs were able to use this shared vocabulary and common set of norms as points of discussion.

MTE participants also pointed out the importance of providing sufficient time for PTs to grapple with developing explanations and reflecting on one another’s explanations. However, they wondered how such time-intensive work of establishing norms for student explanations could be possible in their own teaching. To support PTs in moving past simply restating an idea or procedure and move towards providing rich explanations, MTEs must take the up-front time to teach explicitly about good explanations. One strategy for doing so more efficiently can be providing PTs with examples of PT explanations of varying quality and completeness and ask them to critique those examples in light of the established features of good explanations (see Figure 2).

Managing Flawed Explanations and Representations

A teaching practice focused on student-generated explanations requires a different approach to managing flawed explanations than merely correcting them. Flawed representations

can support fruitful discussions (Hallman-Thrasher, 2011). Exploring incorrect strategies can help students determine why a strategy failed, a key process in learning (Boaler, 2016). MTEs in the study focused on how the instructor responded to mathematical flaws in PTs' representations; she paid particular attention to the incongruencies between representations and PTs' explanations of them. For example, in the context of the Cookie Jar Problem misconceptions about what the fractions represented (what was eaten or what was left behind in the jar) and the changing whole were errors or misconceptions worth further exploration. Two flawed solutions were shared. The first was a series of several incorrect algebraic equations (Figure 4). The MTE instructor asked multiple PTs to explain the equations and map them back to the problem statement. The PTs' equations and explanations aligned, but neither addressed the different meaning of $\frac{1}{2}$ in the first term (the half eaten by Kira) and the second term (the half remaining after Kira had eaten). The propagation of this confusion extended to rest of the problem (being unclear when the fractions need to represent what was eaten and what was remaining) and was the critical piece to unpacking and understanding the algebraic solution. An MTE observer noted the power of the instructor getting others to explain "why the solution is so problematic" as a way to encourage PTs to continue "exploring misconceptions that others have not resolved." The MTE participants agreed that PT Melissa needed to clarify her algebraic equation so that the terms accurately represented either cookies eaten or cookies left behind, and that the instructor continuing to press this point was needed to ensure PTs' understanding of the problem.

<p>Melissa's Equation</p> $x = \frac{1}{2}x + \left(\frac{1}{2}x\right)\frac{1}{3} + \left[\left(\frac{1}{2}x\right)\frac{1}{3}\right]\frac{1}{4} + 1 + 2$ $x - 3 = \frac{17}{24}x + \frac{72}{24}$ $x = \frac{17}{24}x + \frac{72}{24}$ $x\left(\frac{24}{17}\right) = \frac{72}{24}$ $x \approx 2.12$	<p>Melissa's Explanation</p> <p>So that's--x is the total number of cookies, Kira is responsible for eating one half of x. Then Steve came along and saw cookies, so he ate one third of what was left [pointing to $\left(\frac{1}{2}x\right)\frac{1}{3}$] after Kira ate. Does that make any sense? Then Niki came by to share one fourth of the cookies [points to $\left[\left(\frac{1}{2}x\right)\frac{1}{3}\right]\frac{1}{4}$]. So this is what we have left after Kira and Steve got there, so that's Niki taking a fourth of it. Then Kayla came and she took one [points to the 1] and then there were two left [points to the 2]. So we have the feeling that if you added those all together, you would come up with the correct answer for x. But, solving, it doesn't really work.</p>
<p>Algebraic equation mapped to the original problem</p> $\frac{1}{2}x_{\text{Kira}} - \frac{1}{3}\left(\frac{1}{2}x\right)_{\text{Steve}} - \frac{1}{4}\left(\frac{1}{3}\right)\left(\frac{1}{2}x\right)_{\text{Niki}} - \frac{1}{4}\left(\frac{1}{3}\right)\left(\frac{1}{2}x\right) - 1_{\text{Kayla}} = 2_{\text{Pam}}$	

Figure 4. PT's algebraic solutions to the cookie jar problem.

A second incorrect solution presented was a non-proportional geometric representation (Figure 5), which was intensely debated by participants. Questions arose during discussion, such as whether the non-proportional representation implied that the PT did not understand that equivalent fractions should be represented by an equal area, or whether the flaw in her representation was irrelevant given that her explanation made sense. The lab class instructor's choice to not address the non-proportional partitions of the circle was intentional; the verbal explanation was sensible and complete and that is the aspect on which she wanted PTs and MTE observers to focus. Additionally she wanted to leave space for PTs to notice and debate this representation. One MTE noted, "There's such a tendency to bring closure to the idea as a teacher. [The instructor] continues to push their [the preservice teachers'] thinking though." This scenario suggests that MTEs should consider the need to balance pushing for evidence of PTs'

understanding and precision in a mathematical explanation. MTEs should recognize that, in some cases, PTs may have a developing understanding of a mathematical idea but not yet be able to craft a precise explanation.

Figure 5. PT's geometric solutions to the cookie jar problem.

Leveraging PT-created representations made PTs' thinking evident to the lab class instructor and the observing MTEs and exposed misconceptions that would have been otherwise hidden. To identify their own errors or misconceptions and support them in constructing arguments that are convincing to their peers, MTEs can ask PTs to explain one another's strategies, compare correct and flawed strategies, or compare different enactments of the same strategy. By posing clarifying questions, offering counterexamples, or playing the role of a skeptic, MTEs can support PTs in constructing arguments that convince their peers. MTEs must determine which student ideas to follow to achieve instructional goals and which ideas are less relevant to the conceptual core of a task. One MTE clarified that it was not just about the instructor identifying PTs' "wrong answers but the instructor purposefully bringing the lab class's attention to an idea that would contrast with one that was already presented for discussion." MTEs should purposefully choose which incorrect ideas to share and invite students to share them in an order that highlights key take-aways.

Asking the class to unpack the meaning behind a flawed representation or questioning a flawed explanation provided PTs the opportunity to strengthen their conceptual understanding and construct more convincing arguments. Exploring common misconceptions with PTs prepares them for noticing, interpreting, and responding to those misconceptions in their own classrooms.

Asking Prospective Teachers to Map Between Representations

Representations refer to both the process and product, what the student produces—diagram, chart, or idea—and how the student makes sense of it—what reasoning led to its creation and how the student explains it (Pape & Tchoshanov, 2001). This is an important distinction, as the representations PTs (and students) construct as they are solving a problem may not be the same polished final version that they share to convince others of their approach. Representations cannot stand alone as explanations because they do not “display an explicit path through that information and... would leave the viewer to establish what is important (and what is not) and in what order the dependencies should be assessed” (Hanna, 2000, p.16).

To support PTs in constructing rich mathematical explanations, MTEs should not only ask PTs to explain representations they have created, but also map their representation to those created by others. Doing so supports the norms of PTs engaging in understanding one another’s thinking and PTs engaging deeply with the conceptual underpinnings of problems. MTEs should consider the sophistication of particular representations and which representations highlight key concepts. For example, in the Cookie Jar Problem, the lab class instructor repeatedly asked individuals and small groups “to show the relationship between the algebra and the picture” because the picture was more accessible and highlighted errors and misconceptions that were more prevalent in the algebraic strategy, a strategy many considered more sophisticated. She also asked students to map those representations to a numerical, working backwards strategy (e.g., if

3 cookies remained after Nikki ate one-fourth, then there had been four cookies before Nikki ate any) The solution from picture and algebraic equation were in fact constructed in reverse order from the working backwards strategy and comparing the two approaches helped students notice that the working backwards strategy was equivalent to solving the algebraic equation. Hearing multiple students map the representations may have helped the lab class collectively construct a richer, clearer explanation that addressed misconceptions.

Using representations can help support PTs' explanations in several ways. In classroom settings, representations can catalyze discussion that leads to PTs' explanations. Asking PTs to connect representations can support their understanding of content by helping them identify inconsistencies and revise their thinking. Though errors or misconceptions can serve as teachable moments that can develop and deepen PSTs' conceptual understanding, it is also important to attend to valid solutions and their accompanying explanations which can be connected, extended, and generalized in order to enhance teachers' understanding (NCTM, 2014).

PTs also need particular guidance from MTEs in considering how visual representations can support written or spoken explanations. Visual representations are especially important to consider because they provide something for PTs to adhere to in articulating their ideas (Hallman-Thrasher, 2011) and can make PTs' thinking evident to an instructor in ways words alone may not. For example, with the Cookie Jar Problem Terri's diagram served as a record of her thinking and as she shared her explanation she re-drew the diagram describing her process as she went. Had she shared the explanation without the diagram her confusion about representing fractions with an area model would not have been apparent to the MTE instructor or those MTEs observing the lab class. The visual model also gives fellow classmates and MTE instructor something they could easily question even if they did not understand her initial explanation.

Making Pedagogical Decisions Explicit

MTEs need to carefully consider the “why” behind their teaching practices and recognize that even in courses intended to focus strictly on content, discussions of pedagogy can motivate PTs to reconsider their understanding of content. In the lab class, MTEs noticed that the lab class instructor not only taught content, but also implicitly and explicitly communicating about pedagogy. She implicitly communicated to PTs what she valued by modelling pedagogical strategies she wanted them to adopt in their own teaching and by making particular instructional decisions. By asking the class to evaluate one another’s strategies rather doing so herself, she communicated that the responsibility for learning and mathematical authority rested with the PTs. By questioning correct and incorrect solutions, she communicated to PTs that process was more important than solution and that errors were opportunities to examine content more deeply. Additionally, at times, the lab class instructor explicitly discussed her pedagogical decisions and the rationale for making them with PTs. She also took care to address pedagogical decisions that she knew might be different from her PTs’ prior experiences by explaining their rationale. For example, she discussed “not working to closure” to successfully solve a task, which was frustrating for PTs and MTE participants who wanted to resolve all confusion and questions each day. The instructor explained the rationale for this was to see how students thought about concepts and “expose you to different thinking.” At times, the instructor went even further, connecting her pedagogical decisions to PTs’ future work as teachers. In explaining why she would call on PTs to contribute to the conversation, even if they had not volunteered, she pointed out that to get the work of the class done, that she needed to ensure that different perspectives were shared. She connected this to K-8 classroom teaching, saying, “If you start teaching second grade and you don’t think about these things [how to ensure equitable participation] pretty soon

you'll have three people doing all the talking. That's another reason to tell you about it [her plans for managing participation] because it's an idea you'll need to be thinking about too." When the instructor made large and small pedagogical decisions explicit to her PTs and connected those decisions to the PTs' future decisions it seemed to foster PTs' positive disposition towards the challenging work of the lab class.

When commenting on the instructor's pedagogical moves, MTEs often drew comparisons to their own teaching practices. One noted that, "I often tend to just sort of state what's happening without explaining, giving a rationale for it. I kept noticing how [the instructor] was so good at doing that and I could see the [PTs'] responses." Another described the lab class instructor's explicit attention to explanation, "We need to be able to explain things and communicate in a clear process for our students and then be able to understand all of the different strategies that there were and just that explanation of always" (notebook).

When MTEs make their decisions explicit, they invite PTs to analyze and reflect on teaching practice and what it means to teach and learn mathematics. This can help PTs connect their mathematical learning to their future teaching practice. MTEs' observations of the instructor's pedagogical decisions related to explanations brings attention to the powerful impacts of MTEs making their instructional decisions and the intentions behind those decisions explicit for PTs.

Conclusion

Teachers draw on several kinds of mathematical knowledge for teaching school mathematics (Ball, 2003) when facilitating students' mathematical explanations. Although it some have posited that mathematical knowledge needed by MTEs is similar to mathematical knowledge needed for teaching (Mason 1998, 2010), recent thinking has suggested there is more

to the mathematical knowledge needed for teaching teachers, particularly related to helping PTs make connections between what they are learning to what they will teach (Castro Superfine et al. from this issue; Superfine & Li, 2014). The mathematical knowledge for teaching teachers goes beyond the mathematical knowledge for teaching. MTEs have the added demands of making their pedagogical decisions and the rationale for them explicit and making connections to ways that mathematics teaching and student thinking of the elementary classroom. Our study highlights ways MTEs can best support PTs' mathematical explanations in math courses. The observations and related discussions of the MTEs in our study provided insights into five ways that MTEs can engage PTs in mathematical explanations: 1) using explanation-worthy tasks for PTs, 2) co-constructing norms for mathematical explanation, 3) managing flawed representations and explanations, 4) asking PTs to map between representations, and 5) making pedagogical decisions explicit. MTEs can use these practices within their instruction in a variety of ways to help PTs develop strong mathematical explanations.

The idea that background content knowledge is a prerequisite to developing explanations, as MTEs in this study wondered, misses the potential power of constructing and critiquing explanations. PTs can engage in the work of developing mathematical explanations while they are developing their content knowledge. In fact, through doing so and critiquing the explanations they and their peers develop serves to enrich and strengthen their content knowledge. While some norms and practices for teaching math content to PTs have been developed (e.g., University of Michigan's Teaching Works), we as a community of mathematics educators we have not reflected on and widely adopted a set of shared norms and practices for teaching mathematics content to PTs. For example, while constructing rich mathematical explanations is a key component of mathematics courses for teachers, the MTEs in our study had different

perspectives on how PTs should best articulate mathematical explanations or how MTEs should respond to particular classroom situations. Differing views on what constitutes a “good” explanation indicate the need for researchers in the field to devote attention to what is an adequate mathematical explanation in mathematics courses for PTs. Perhaps the features of good mathematical explanations generated from this study (see Figure 2) may serve as a starting point for future research.

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