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Fostering and Modeling the Common Core Standards for Mathematical Practice in Content Courses for Prospective Elementary Teachers

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Abstract: *What do prospective elementary teachers (PTs) need to learn about the nature and practice of mathematics? What pedagogy is powerful in achieving those learning outcomes? This article focuses on ways to foster and model a classroom culture of doing mathematics with PTs. We describe ways to support them in solving problems, using and valuing precise language and notation, making sense of definitions, conjecturing, generalizing, and in communicating and justifying their mathematical thinking. We assert that participation in authentic mathematical practices provides PTs with the mathematical disposition and the tools to continue to learn mathematics long after they leave your classroom; it helps them to make sense of the Common Core State Standards (CCSS) for Mathematical Practice (CCSSI, 2010); and it supports them in embracing a culture of doing mathematics with children in their own mathematics classrooms. While our advice is situated in theory and supported by research, it is also practical. We use examples of classroom dilemmas and dialogue to illustrate our suggestions.*

Keywords*:* mathematical practice, inquiry-based learning

Introduction

> *"I have always been told by a teacher or textbook how to solve problems. This is the way it was – no questions and no discussion. Of course what I was told always worked, but I did not know why, and as a result I did not appreciate or understand math … until now. Through the class activities, we, the students, figured out theorems. We went into each problem cold, without knowing what would come out of it. We were able to see patterns develop, create hypotheses and solve problems." PT Paula*

The University of Wisconsin Oshkosh is a comprehensive four-year school in the US serving 11,000 students, almost 600 of whom are prospective elementary teachers (PTs). We require PTs to complete nine credits of mathematics content focused on number systems, geometry and data. Our three 3-credit content courses specifically designed for PTs are housed in the mathematics department and taught by a cohort of mathematics faculty who do mathematics

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education research and spend significant time in the schools. The classes are designed so that PTs do mathematics each day in class. PTs work together on rich problems and activities designed to provide a context for reasoning, modeling, conjecturing, generalizing, debating and justifying their ideas. Because we cannot address every facet of the vast content domain PTs need for the mathematical work of teaching, we try to help them learn to *learn* mathematics. In short, we focus on fostering and modeling mathematical practices.

Traditionally, mathematics teachers provide students with explanations, procedures, examples and exercises. The student's job is to learn to solve problems as presented and to understand why ideas and procedures make sense. In this model of teaching and learning, the mathematics comes mainly from the instructor, often via lecture or demonstration (Hiebert, et. al, 2003). However, research suggests that students and their instructor must *negotiate* shared language through discussion and debate if students are to give adequate meanings to explanations and representations (Schipper, 1982). Furthermore, standard textbook examples and procedures often suppress mathematical complexity and context, so that students deal with over-simplified or overly-defined problems. "In effect, all too many students develop routines and strategies that are viable for coping with only reduced complexities. These impoverished and nonadaptive strategies dampen the development of strong mathematical models" (Bauersfeld, 1995, p. 140). In short, many PTs learn very little from traditional pedagogy (Schoenfeld, 1988; Boaler, 1997), and the more mathematically inexperienced the students, the *less* likely they are to make sense of lectures or textbook explanations.

Equally troubling is the fact that traditional pedagogy does not honor the practice of *doing* mathematics – at least not by the students. The students rarely have an opportunity to see what mathematics is all about and so they do not learn to *think* mathematically. Research reveals that PTs believe mathematics is a collection of facts and procedures that they are to memorize and apply (Szydlik, J., Seaman, Szydlik, S., & Beam, 2005). Most do not believe that they can do mathematics or make sense of it, and many believe that there *is* no sense to be made of it (Seaman & Szydlik, 2007). As a result, many PTs memorize their mathematics. Furthermore, they are likely to provide these same impoverished experiences to children and to have few expectations for mathematical learning beyond the rote execution of algorithms.

The suggestions we present in this article arise from a philosophy of learning that asserts that humans construct meaning within a cultural context (von Glasersfeld, 1995). We view the mathematics classroom as a culture of doing mathematics and learning as a process of acculturation. Mathematics is not formulas, procedures and theorems; it is a way of thinking about idealized patterns and structure that help us to build those very things. "Numbers and shapes and motions and arrangement, and also thoughts and their order, and concepts such as 'property' and 'relation' -- all such things are the raw material of mathematics" (Halmos, 1968). Mathematicians create precise definitions for ideal objects. We value and use careful language and notation. We imagine simple examples that illuminate complexity. We build mental (and sometimes physical) models for objects and relationships. While mathematics is often inductive in its creation, the mathematical community demands a deductive argument for justification (Halmos, 1968). We want to convey these values and practices to our PTs both covertly and explicitly.

In our classrooms, we provide PTs normative definitions, notation and language, and raise problems, questions and issues in a pedagogically reasonable order to support a classroom culture of *doing* mathematics. Rather than demonstrate techniques, we pose questions or problems for debate. In short, we attempt to model classroom practices advocated by Ball (1993) and beautifully rendered by Lampert (2001). The vignettes and dialog we use to illustrate classroom interactions are real examples from our work with PTs (student names are pseudonyms).

Our focus is on mathematical practice aligned with the National Council of Teachers of Mathematics *Principles and Standards for School Mathematics* (2000) recommendations, and more specifically in accord with the Common Core State Standards (CCSS) for Mathematical Practice (NGA Center and CCSSO, 2010). Relevant research literature regarding classroom culture and student learning and understanding of specific topics also informs our writing (e.g., Hattie, 2017; Boaler, 2016; Carpenter, Fennema, Loef Franke, Levi, & Empson, 2015; NCTM, 2014; Carpenter, Loef Franke & Levi, 2003; Ma, 1999; and Driscoll, 1999).

We intend for this article to provide suggestions regarding how to create a culture of mathematical practice. But, in truth, we intend much more than that: we hope to support you in doing *transformative teaching*, the kind of teaching where you challenge the assumptions PTs bring to class. We intend to help you change PTs' minds about the very nature of mathematics and their views of themselves as mathematicians.

CCSS Mathematical Practice 1: Make Sense of Problems and Persevere in Solving Them

> *Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. (NGA Center and CCSSO, 2010)*

Creating a Problem-Based Course

The constructivist learning philosophy led us to adopt pedagogies that include collaborative work on rich mathematical activities (for examples, see the Litster, MacDonald, & Shumway (2020) and the Kuennen & Beam (2020) articles in this special issue) followed by whole-class discussion of ideas (Yackel & Cobb, 1996). Over the years, we have experimented with structures to support problem solving. What follows are strategies that worked best for us over our combined 75 years as Mathematics Teacher Educators (MTEs).

We use group-sizes of three or four; this is big enough to allow for many ideas and small enough so everyone can contribute. We choose groups at random. If we engineer groups based on ability, PTs figure it out. We change the groups every three weeks or so. This allows for group bonding, but also lets everyone work with a variety of people and helps keep PTs "on task." They become less rigorous with one another after a longer period of time together.

We pay close attention to what PTs are doing during group work but intervene little in the problem solving process. We let them make sense of activities and we let them make mistakes, and later we use their alternate conceptions and interpretations as teaching tools by making those ideas public and discussing their viability. If we know what every group did during the group work, then we know on whom to call to let interesting ideas, solutions, and alternate conceptions emerge during class discussion (Boston, Dillon, Smith, & Miller, 2017).

Fostering Perseverance

Mathematicians are not fast problem-solvers; we are slow, careful thinkers (Boaler, 2016). We tell PTs this fact explicitly, and we structure classroom experiences to foster perseverance and positive disposition (Gresalfi, 2009). We are encouraging during small-group work and we help them to manage frustration.

Bette: "I don't understand what this is even asking!"

Mathematics Teacher Educator (MTE): "Lots of people are still thinking this over. I wouldn't know what to do yet either. Remember that a problem is only a *problem* if you don't know how to solve it. There is nothing wrong with being stuck."

If the PTs need to know a definition in order to solve a problem, we help them to learn to find that information for themselves.

Cavi: "What does it mean that '*a* is a multiple of *b*?'"

MTE: [Addressing the group – and not Cavi in particular] "Perhaps you all would like to visit the glossary? That's the first place to look if you don't know what a mathematical term means." (We don't let them look anything up except definitions while in groups.)

If PTs have trouble making sense of a definition, we model how a mathematician studies a new definition. PTs have not had opportunities to make sense of mathematical terms and this is difficult for them (Seaman & Szydlik, 2007).

MTE: [Begins by reading with them from the glossary] "Let's see … *a* is a multiple of *b* if there exists whole number *c* so that $a = b \times c$." [Wait time]

Mark: "What does that mean?"

MTE: "Let's read it with some numbers in there. What do you want to test? Okay, Let's see, 20 is a multiple of 4 if there exists a whole number, *c* , so that 20 is 4 times that number."

Millie: "Oh, I get it. That's 5. So 20 is a multiple of 4 because it is 4 times 5."

MTE: "What about this: is zero a multiple of 4? Let me hear you use the definition to check."

We help them to talk to and rely on one another. When a PT asks a question during group work, we attempt to finesse it so they end up asking group members.

Amed: [Raises hand during small group work]

MTE: [Addressing the group – and not Amed in particular] "I see that your group has a question. Jesse, what are you all thinking about?"

Our goals during group work are to help PTs manage frustration, support their thinking, and teach them mathematical self-reliance. We also aim to gather ideas for the whole class conversation to come.

CCSS Mathematical Practice 2: Reason Abstractly and Quantitatively

> *Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects. (NGA Center and CCSSO, 2010)*

Contextualizing and Decontextualizing

Some "real world" problems can be solved by modeling them with mathematical representations (e.g., a system of equations or a function) and then studying or manipulating that representation before interpreting the result. The translation of a real problem to its representation is called "decontextualizing," and the act of interpreting facets of the representation in terms of the original context of the problem is termed "contextualizing." We do very little algebraic manipulation with PTs, but we do showcase this process in "balance scale" problems where each scale represents an equation (for an early study on equality as balance, see Herscovics & Kieran, 1980). For example, consider Figure 1.

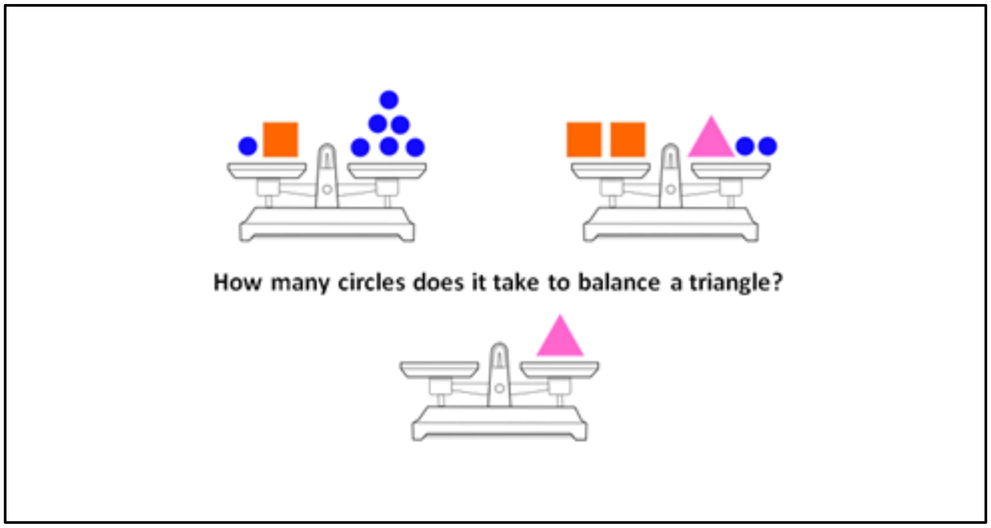

Figure 1. Balance scale problem.

We encourage PTs to solve problems like these mentally using the pictures, and then we follow their thinking in concert with a decontextualized system of equations.

Diego: "I took one circle off each side of the first balance and saw that a square weighed the same as five circles."

MTE: "Let's see, so we had that $c + s = 6c$ and you took a circle from each side. So I'll remove the weight of a circle from each side and we get that $s = 5c$. Yeah, so we can remove (or add) the same quantity to each side of an equation and it is still true. What did you do next?"

Diego: "Then I replaced each square on the second scale with five circles."

MTE: "That second equation was $2s = t + 2c$. So you substituted five circles for each square like this: $2(5c) = 10c = t + 2c$. The scale remains balanced if you substitute something of equal weight."

In this way we explicitly help PTs to give meaning (context) to equations where that context is tied to and supports all the algebraic moves.

Highlighting Properties of Operations

Humphreys & Parker (2015) describe Number Talks as a process by which students mentally solve computation problems as a way to emphasize strategies that help them to reason abstractly and quantitatively. We have found Number Talks useful in helping PTs know and use properties of operations.

The Number Talks we employ are conducted as a whole-class. After we pose a problem or series of problems, we elicit and record ideas emphasizing our *students' strategies* for solving the problems. PTs quickly create a wide variety of invented strategies (see Carpenter et. al, 2015) supported by properties of the operations. We then help to make those properties explicit.

> MTE: [Addressing the whole class] "Without paper or pencil or calculator or talking, solve this problem any way you like. If you find one way of doing the problem, then think of how you might do it another way: 12 × 6." [We give the PTs time to think without watching them because we do not want to convey that being fast is important. We do this by looking thoughtfully at the problem we've written on the board and thinking about it ourselves.]
>
> Jose: "I got 72."
>
> MTE: "How did you go about finding that solution?"
>
> Jose: "First I multiplied 10 × 6 to get 60, then I took 2 × 6 and that is 12. I added them together to get 72."
>
> MTE: [Writing $12 \times 6 = (10 + 2) \times 6 = (10 \times 6) + (2 \times 6)$] "This looks like the distributive property in action. Did anyone approach this problem a different way?"
>
> Kai: "I knew 6 × 6 is 36. Then I doubled it to get 72."
>
> MTE: "How did you know that worked?"

Kai: "Because I needed 12 groups of 6 instead of only 6 groups, so I knew I had to double it or multiply by 2."

MTE: "So you did this?" [Writing $12 \times 6 = (2 \times 6) \times 6 = 2 \times (6 \times 6)$] "What property did Kai use?"

Mimi: "Oh. That's the associative property for multiplication."

Number Talks give us the opportunity to highlight structural properties of arithmetic, to embrace a wide variety of ideas, to draw connections across strategies, and to help PTs understand the *meaning* of operations and place value. Later in the course, these same strategies can be used with problems involving integers, fractions and decimal numbers.

PTs recognize the value of Number Talks; the following written reflection from a PT is typical: "I had never been introduced to a number talk. What I was familiar with is there is one way to compute. There has never been the idea of discussion of what ways work and why. I have learned that number talks allow students to learn from each other rather than being lectured at, and there are so many different ways of seeing math concepts."

CCSS Mathematical Practice 3: Construct Viable Arguments and Critique the Reasoning of Others (For more on this Practice, see the Max & Welder (2020) article in this special issue.)

> *Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others … Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. (NGA Center and CCSSO, 2010)*

Introducing the Language of Mathematics

We have found PTs need many opportunities to construct arguments in order to become familiar with the language of mathematical practice. At the start of any course, we find it helpful to define and discuss the relevant terms: conjecture, counterexample, proof and theorem. We also distinguish inductive reasoning (based on examples) from deductive reasoning (based on logic and geometric or numeric structure). Then we use this language daily in class and encourage them to do so as well.

Pressing for Justification

Many PTs are still working on making sense of mathematics. We work hard to help them see that it does make sense and to recognize when a justification is needed. An answer is not correct because an authority says it is. It is correct because it is logically consistent within the given mathematical structure. As PTs begin to understand that claims require justification, they can begin to develop insight and arguments. The instructor's role is to demand justification and to neither confirm nor deny the correctness of any conjecture. In the discussion below, PTs are working on The Locker Problem (Kamani, Olanoff, & Masingila, 2016).

> David: "Our group found a pattern! We think the answer is all the perfect squares."
>
> MTE: "Nice! But … how can you be sure that the pattern will continue? Why does it make sense? A mathematician isn't done with a problem until she knows why her solution makes sense."
>
> Eve: "Are we right?"
>
> MTE: "I'm not saying you are right or wrong. I'm just saying that you aren't *done*. Why does it make sense that open lockers are perfect squares? What is special about those numbers in the context of the problem? I know you can figure this out."

When each small group has had an opportunity to work on the problem and has something mathematical to contribute (e.g., a representation, pattern, counterexample, insight or argument), it is time for a whole-class discussion of ideas.

Fostering Public Discourse

Leading a discussion is complex (Stein, Engle, Smith, & Hughes, 2008; Lampert, 2001). The ideas belong to the PTs and we must let them debate and own them, yet we must also be sure that alternate conceptions are brought to the foreground and that there is a sense of closure in the end (for more on this balance, see Ball, 1993). In a discussion, our role is to encourage PTs to share their strategies and thinking, to insert normative language and notation, and to help them understand what counts as a mathematically *different* approach and what constitutes a viable mathematical argument. These sociomathematical norms (Yackel & Cobb, 1996) lay the foundation for mathematical discussion. In this section we provide practical ideas that support PTs in listening to and critiquing the reasoning of one another.

We have PTs move to sit in a U-shape so they can hear one another and see the board. Our intent is to invite them to explain ideas. We use and model *focusing questions* (e.g., "Can you explain what you mean?" or "Would you show us an example of that?") rather than *framing questions* designed to lead PTs through a piece of mathematics (e.g., "What do we do next?") (Herbel-Eisenmann & Breyfogle, 2005). When an important question is raised, we ask the class what *they* think. We remind PTs that they must learn to *listen* to mathematical ideas and address questions. We do not withhold key information, but we do insist that they begin to develop their own judgment. We refrain from rephrasing everything. If we do that, PTs need only listen to us and not one another. We have the PT repeat what he or she has said if others have not heard it or if we need to hear it again to think it through. All of this supports the class in making sense of student work and helps them to see how they might do the same in their own teaching.

We are relentless in our press for justification. We insert "why" and "how" questions to indicate justification is wanted ("*Why* will that pattern continue?", "*Why* do perfect squares have an odd number of factors?", "*How* can we see the doubling?", and "*How* do you know that is true for all numbers?") and we encourage PTs to make arguments using examples that reveal all the structure of the general case. Carpenter, Franke, & Levi (2003) term such arguments e*xamples that are more than examples*. For example, a PT might argue the sum of 7 and 5 is even in such a way that we see why *any* two odd numbers must have an even sum (see Figure 2). The tally marks reveal that each odd number of objects can be paired with one "leftover," and so when the two odds are combined, the "leftover" from each makes a pair.

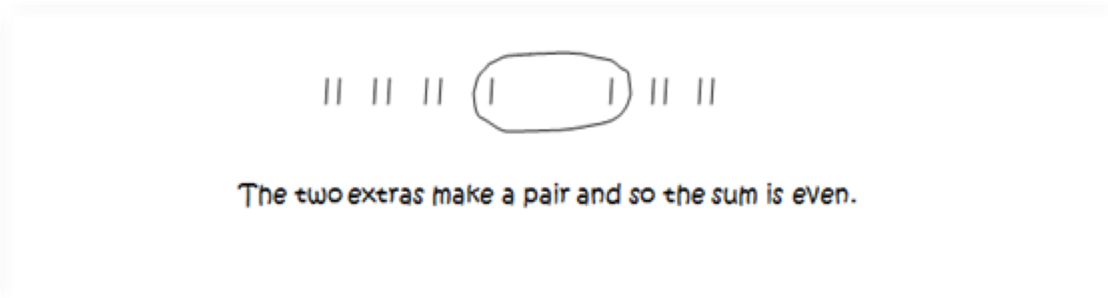

Figure 2. An example that reveals the structure of a general argument.

While the argument centers on a specific example, the *idea* holds for all pairs of odd numbers. This type of justification is particularly useful for PTs because their future students can make and understand such arguments (Carpenter, 2003).

CCSS Mathematical Practice 4: Model with Mathematics

> *Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. ... Mathematically proficient students who can apply what they know are comfortable making assumptions*

and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose. (NGA Center and CCSSO, 2010)

Modeling with "3 ACT Math" Tasks

The mathematics that PTs interact with outside the classroom -- and the mathematics the majority of *their students* will interact with outside the classroom -- requires them to apply knowledge in new and flexible ways. PTs need experiences in mathematical modeling in order to develop their own understanding of the modeling process (Lesh, Doerr, Carmona & Hajlmarson, 2003). As a way to bring the full messiness of modeling to the fore, we recommend "3 ACT Math" tasks (Meyer, 2011) because they allow PTs to both model with mathematics and learn to teach their students to model with mathematics.

ACT 1 – The MTE poses a visual representation of a conflict with few to no words. Consider the example in the Girl Scout Cookie problem posed by a 20 second time-lapse video clip of Girl Scout cookies being piled into the back of an SUV (http://www.101qs.com/3675). In ACT 1, encourage PTs to pose questions.

MTE: "After watching that video, what questions come to mind?"

Zulema: "How many boxes of cookies will fit into the back end of that SUV?"

Mark: "How many cookies did they sell? How long will it take them to deliver them all?"

MTE: "Let's begin with Zulema's question, how many cookies will fit into the back end of that SUV? Now your turn, think for a bit and make your very best guess." [MTE allows PTs time to talk and think and then takes some estimates, highlighting the range.

This is followed by an opportunity for PTs to also make a guess that they know is too low and too high to refine their estimation skills.]

ACT 2 – The class discusses ways to solve the problem. In ACT 2, PTs need to decide what mathematical concepts may be needed in order to build their own model.

MTE: "What information would we need in order to solve this problem? How would we approach answering the question how many boxes of cookies will fit in the back of that SUV?"

Grace: "We would need to know the size of the cookie boxes."

John: "We also need to know the size of the back of the SUV" [The MTE then helps the PTs to find the information requested (e.g., the size of cookie boxes or dimensions of the pictured vehicle) and also may explore other questions such as, are all boxes the same size? Do they pack using all the space in the SUV?]

ACT 3 – The class resolves the conflict. PTs solve the problem using their understanding of the underlying mathematics in order to go through the mathematical modeling process. In the example given, an understanding of volume and unit conversion in multiple dimensions may be needed. This often requires digging deeper into the particular mathematical concepts needed to solve the problem. We then have the opportunity to compare results to initial estimations and discuss the ways in which different PTs solved the problem.

CCSS Mathematical Practice 5: Use Appropriate Tools Strategically

> *Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitation. (NGA Center and CCSSO, 2010)*

Making Tools Available for Thinking

"Because of the abstract nature of mathematics, people have access to mathematical ideas only through the representation of those ideas" (National Research Council, 2004, p. 94). Some representations are mental constructs, but others are physical models. This is true for both children and adults, and so we incorporate a variety of manipulatives in our courses: base-ten blocks, fraction squares, scales, chips, cubes, nets, dot paper, graph paper, dice and spinners, to name a few. We require that PTs know how to use them to illustrate mathematical ideas because we would like them to use similar tools with their own students.

> Mia: "I don't want to use fraction squares. Why can't I just do $\frac{3}{4} \div \frac{1}{8}$ using 'invert and multiply'? There. I got 6."
>
> MTE: "Because it is not just about *you* anymore and it is not just about getting the correct answer. I want you to be able to make sense of *why* 6 is the answer, and I want you to be able to model this for your future students also."

We encourage PTs to consider a variety of tools and to discuss their appropriateness both for the problem situation and the developmental level of their future students. For example, when teaching children strategies for subtraction, tools such as counters, base-ten blocks, hundreds-charts or number lines might be helpful depending on the size of the numbers involved and the age of the children. (For more discussion on the use of manipulatives, see the Kuennen & Beam (2020) article in this special issue.) When modeling linear functions in middle school, the use of an online graphing calculator can aid in conceptual understanding of transformations of the parent function without students having to plot many different functions by hand.

CCSS Mathematical Practice 6: Attend to Precision

> *Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the*

> *meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions. (NGA Center and CCSSO, 2010)*

Attending to Language

According to the CCSS, precision means much more than calculating accurately or expressing numerical answers with an appropriate degree of precision. It also means communicating precisely with others in speaking and writing (Devlin, 1998). PTs must practice using normative vocabulary and mathematical definitions, and learn to use careful language in discussions and in making arguments. For example, a triangle has three vertices, not "three points." Using the word "point" for vertex is ambiguous since each side of a triangle contains an infinite number of points.

Amanda: "This tip is more than 90 degrees and so the triangle is obtuse."

MTE: "So *the angle at that vertex measures* more than 90 degrees?"

Amanda: "Oh, yeah."

We try to polish PT language without being disruptive to the conversation at hand.

Attending to Notation

Another way in which we attend to precision is in our use of mathematical symbols. For example, we have found that it is important to discuss appropriate use of the equals sign early in any class for PTs.

MTE: "Suppose that you presented this problem to your second-graders:" [Writes on the board: 6 + 4 = ____ + 3 = ____] "Suppose that you asked them, 'What numbers go in the blanks to make this number sentence true and why?' What do you think they'd say?"

After letting our PTs discuss this, we invite them to consider each of the following scenarios:

MTE: "One child says that 10 and 13 go in the blanks. Why might he say this, and what would you say to him as his teacher?"

MTE: "Another child, Mari, argues that 7 and then 10 go in the blanks because 6 and 4 is 10. And you have to add 7 to three to get 10."

MTE: "A third child, Peter, argues that 7 and then 10 go in the blanks because 3 is one less than 4 and so you have to fill in the blank with one more than 6."

Many children (and many PTs) interpret '=' as 'write the answer.' A vast majority will write '10' in the first blank and then add 3 to ten and write 13 in the second blank (Carpenter, Franke, & Levi, 2003).

We want to make the point to PTs that a mathematician thinks of an equation as a relationship between two *numbers*: the two quantities on either side of the equal-sign are exactly the same. This way of thinking about the equals sign is powerful and provides a foundation for algebraic thinking (Empson, Levi, & Carpenter, 2011; Carpenter, Leof Franke, & Levi, 2003).

Furthermore, we use the problem to help PTs distinguish between procedural thinking and relational thinking. Mari's argument is procedurally focused, whereas Peter's relies on the relationship between the numbers on each side of the balance. While both types of thinking are important, the second type shows a deep understanding of equality. Finally, we want to remind PTs that *their* use of the '=' sign is important; their students will be watching and learning from the way *they* write and talk about mathematics. We always point it out when they write strings of non-equal expressions using the equals sign.

Max: [Writing on the board as he explain his thinking] "12 + 8 = 20 ÷ 5 = 4"

MTE: "Hmm. To me that says 12 + 8 = 4 … is that how you wanted to write it?"

Max: "I'm just showing my steps."

MTE: "I like that you're explaining the steps, but I worry that your notation will confuse children about the meaning of the equals sign. Might you instead write 12 + 8 = 20, and then 20 ÷ 5 = 4?"

CCSS Mathematical Practice 7: Look for and Make Use of Structure

Mathematically proficient students look closely to discern a pattern or structure … They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 × 8 equals the well-remembered 7 × 5 + 7 × 3, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$*, older students can see the 14 as 2 × 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see* $5 - 3(x - y)^2$ *as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y. (NGA Center and CCSSO, 2010)*

Using Representations to Reveal Structure

In order for PTs to use this practice in their problem-solving and in their teaching, they need experiences making arguments based on mathematical structure. Allowing PTs to discuss problems represented in different ways deepens their learning and allows them to link symbols back to contexts (Greeno & Hall, 1997). We use a variety of problems to reveal structure; here is an example from geometry: *Consider the maximum number of regions into which a plane can be partitioned by 1 line, 2 lines, 3 lines, and 4 lines as shown below. If there are n lines, what is the maximum number of regions?*

In small groups, we allow PTs to explore this problem and we discuss why the word "maximum" is important in the problem statement. After 30 minutes or so, most groups will have drawn a series of pictures and collected data on number of lines and number of regions. We encourage PTs to organize the information so that we can all look for patterns. Eventually we'll have some pictures (Figure 3) and some data (Figure 4) on the board or the document camera.

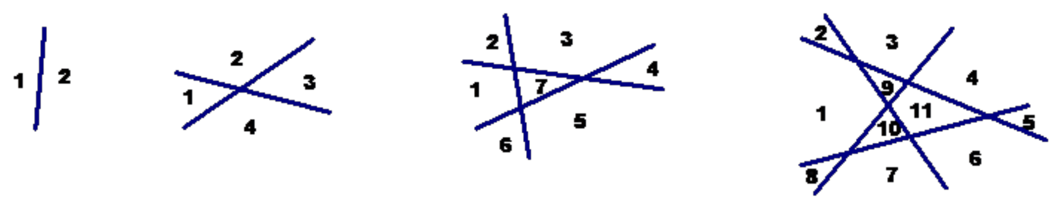

Figure 3: Regions pictures.

Number of lines (n):	Maximum Number of Regions:
1	2
2	4
3	7
4	11
5	16

Figure 4: Regions data.

PTs may observe that the differences between the numbers of regions in each successive row are 2, then 3, then 4, and then 5. We might record that observation like this (Figure 5) so that the structure of these numbers is more obvious.

Max Number of Regions:	
2	= 1 + 1
4	= 1 + 1 + 2
7	= 1 + 1 + 2 + 3
11	= 1 + 1 + 2 + 3 + 4

Figure 5: Regions data written to reveal structure.

They will conjecture that for *n* lines, there will be 1 + 1 + 2 + 3 + …. + *n* regions. We remind them that this is still a *conjecture* because thus far, we have based our thinking on patterns in the table. This is *inductive reasoning*. Next we help them to draw a geometric picture of these numbers (Figure 6).

```
X  X       X  X       X  X       X  X
              X X        X X        X X
                         X X X      X X X
                                    X X X X
```

Figure 6: Geometric pictures to reveal structure.

These pictures help PTs further see mathematical structure and may even help them to create a closed form for the number of regions: $1 + \frac{n\,(n+1)}{2}$.

We remind our PTs again that a mathematician isn't done with a problem until he or she has made a deductive argument based on mathematical structure to justify *why* the pattern (or answer) makes sense. Then we help PTs construct a viable deductive argument: In the case of no lines we have one region. The first line divides that region into two regions. In order to obtain the maximum number of regions the second line must intersect the first line. This means that the second line divides each of the two existing regions into two more regions. In order to obtain the maximum number of regions the third line must intersect both of the first two lines. This means that the third line divides three of the existing regions into two regions each, creating three more regions. Likewise, the nth line will need to intersect each of the previous $n – 1$ lines, dividing n of the existing regions into two regions each, creating n more regions. Thus the maximum number of regions created by n lines is the sum of $1 + 1 + 2 + 3 + … + n$.

CCSS Mathematical Practice 8: Look for and Express Regularity and Repeated Reasoning

> *Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation* $(y - 2)/(x - 1) = 3$*. Noticing the regularity in the way terms cancel when expanding* $(x - 1)(x + 1)$*,* $(x - 1)(x^2 + x + 1)$*, and* $(x - 1)(x^3 + x^2 + x + 1)$ *might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results. (NGA Center and CCSSO, 2010)*

Creating Systematic Lists

We have found that many PTs benefit from opportunities to make lists or to organize data in ways that help them to see regularity. For example, when asked to list all the groups of size three that can be made from five total people, many PTs will make groups arbitrarily: ABC, ABE, CDE, BED, etc. The difficulty with this type of data recording is that PTs do not know when they are done. When a situation like this arises in class, we stop and discuss ideas for making an organized list before we let them return to their data collection. If we want PTs to see regularity, repeated reasoning, and structure, we must help them to value systemic thinking and data recording.

Leading Pattern Talks

Earlier in this article, we discussed Number Talks. Here we advocate another type of Number Talk that we will call Pattern Talks. The format for class discussion is the same as in a Number Talk, but the discussion is focused on making sense of a geometric pattern. Here's an example:

MTE: "Without talking or drawing anything, I'd like you to consider the following pattern of stars (Figure 7). How do you see the pattern?"

```
* *       * * * *       * * * * * *
*         * * * *       * * * * * *
          *             * * * * * *
          *             *
                        *
                        *
```

Figure 7: Pattern talk problem.

Terry: "I see two squares and a stem."

MTE: [Drawing on the picture to show Terry's thinking.] Okay. Did anyone see it differently?"

Amy: "I see a rectangle twice as long as it is wide and a stem."

Owen: "I see a 2*n* stem and a rectangle that is almost twice as long as it is wide."

After we have heard all of the PT ideas, we can use their thinking to address the question: assuming the pattern continues, how many stars are in the Figure *n*? This way they learn to use regularity to move flexibly between algebraic and geometric representations.

Trouble-Shooting: Some Advice for Dealing with Common Difficulties

During the past twenty-five years of leading classes that focus on mathematical practices, we have dealt with a few frustrated PTs (and not always well). What follows is our advice based on experience.

What if prospective teachers say "you aren't *teaching* me anything"?

This is often an initial reaction to a practice-based course and not a lasting complaint. However, if this attitude persists, then it may be that we are not providing enough closure to the class discussions. We might try leaving the last minutes of class to summarize big ideas or to tell PTs what they should reflect on for next time. Sometimes we tell them it is okay if they are still confused, and that the homework will give them opportunities to think about it some more. We also tell them that they must come see us if, after the next class, an idea does not make sense to them.

What if a prospective teacher complains that "my group moves too fast for me"?

There will be a variety of group-mate complaints during the course of the term, and we work hard during the first weeks to encourage slower or more careful thinkers to slow their groups by asking for explanations. We tell them to give their groups a chance to be teachers, and we tell them that quicker students need to learn to listen to and address questions. We do this publicly – not around any particular person, just as a way to have them think about their behavior as learners and as potential teachers. If there seems to be an irresolvable group conflict, then we do a random group switch earlier than usual with no particular reason given.

What if a certain prospective teacher dominates the class discussion?

We recommend having a private word with that PT to get them on board. We tell them that we are going to call on them less and we tell them why.

What if the prospective teachers will not talk to one another during class discussion?

Only the savviest of instructors can get PTs to address one another directly and seamlessly without having to intervene between every utterance. To encourage student-to-student discourse, we refrain from rephrasing everything PTs say. They will not learn to listen to one another if we always say it better, and they *must* learn to listen and evaluate the imperfect mathematical utterances of others. We also try simply looking confused (or interested or inquisitive) as appropriate (often another PT will jump in to clarify). Or we may try sitting down with the class rather than standing in front – sometimes that helps them attend to one another rather than to us.

What if *no one* wants to talk during class discussion?

This *is* likely our fault, and that means there are things we can do to improve the situation. First, we think about the questions we are posing. Are they too easy? No one wants to answer questions that are too easy; it is humiliating to get them wrong and no one gets excited or inspired if you get them "right" (i.e., poor expected return). We aim for moderately-tough questions and we try to ask questions that can be answered in more than one way. Second, we sometimes have PTs discuss with a partner before giving public responses. That provides PTs a sounding board and confidence. Third, we wait longer before stepping in. We watch the clock and try to wait a full 30 seconds. If we've asked a good question, then it should take a bit of thinking to address it. We encourage them to talk to one another in general. A buzzing room will be a productive room, and no one likes to break a silence. Fourth, we don't summarize everything that is said; it's boring and shuts down dialogue. We move along to another idea, another PT's approach, or another question.

Here are some additional techniques we have found particularly useful to encourage discussion: Individual work with hand-raising: “Everyone find the LCM of 1200 and 325 and raise your hand when you are done. I’m not going to call on you; I just want to see when everyone is done. You can talk to each other.” Partner work: “Would each of you make up a partitive division problem for your partner to solve?” Forced interaction: “Amy, what do you think John means by that?” “Carol, I saw that you did something like that during group work. Could you show us?” Thanking PTs for their participation: Write notes on homework or thank them verbally after class. This encourages people to speak.

Conclusion

Whether we have them for one content course or three, it is impossible to teach PTs all the mathematics they will need to be good elementary grades teachers. Therefore the most important knowledge we can help them to construct is an understanding of how to *learn* mathematics: to make sense of problems and persevere; to reason abstractly and quantitatively; to construct and evaluate arguments; to model with mathematics; to use tools appropriately; to attend to and value precision in language; to make use of structure; and to look for regularity and repeated reasoning (CCSSI, 2010). We do this by doing a few good problems and activities well, rather than many problems poorly. We do this by fostering a culture of doing mathematics in our classrooms. We do this by helping PTs to construct the powerful tools that mathematicians use to learn and do mathematics: the mathematical practices.

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