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Real numbers as infinite decimals

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ABSTRACT: Stemming from an idea put forward by Loo-Keng Hua in 1962, this article describes an original way to perform arithmetic directly on infinite decimals. This approach leads to a new and elementary construction of the real number system via decimal representation. Based on the least upper bound property, a definition of trigonometric functions is also included, which settles an issue that Godfrey H. Hardy called "a fatal defect" in his Course of Pure Mathematics.

Keywords: real number system, infinite decimal, trigonometric functions, rectifiable curve

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1 Introduction

By the end of the sixteenth century, Simon Stevin (1548–1620, [26], see also [15, 19, 20]) had essentially achieved the modern concept of real numbers. Promoting the systematic use of decimal notations in daily computations [27], his work marks a transition from the discrete arithmetic practiced by the Greeks to the arithmetic of the continuum widely accepted today. However, performing arithmetic directly on infinite decimals remains, to this very day, a long-standing problem ([2, p. 97], [3, p. 8], [4, p. 11], [6, p. 10], [10, p. 123], [16, p. 16], [21, p. 80], [23, p. 400], [29, pp. 105–106], [30, p. 739], [31]), that has seen the popular degeometrization [11, 19, 20, 22] of real numbers since the first constructions were published independently by Charles Méray, Eduard Heine, Georg Cantor and Richard Dedekind in around 1872. From the late 1800s onwards, and with the sudden outburst of the main classical theories, many attempts have been made to construct the real number system from various perspectives, such as the least upper bound property, the Archimedean property, equivalence classes, axiomatic approaches, the additive group of integers, continued fractions, harmonic or alternating series, and so on [5, 30].

It is well known that any element of the real number system can be identified with an infinite decimal, so why not define arithmetical operations directly on infinite decimals? There is a long list of mathematicians, including Karl Weierstrass and Otto Stolz, who prioritize the decimal system over other constructions, since we all learned at school how to perform arithmetic on terminating decimals. But many decimal approaches lack detail, and most of them are essentially not so different from the earliest theories (see [16] for a literature review). Decimal constructions of the real number system are thus rarely seen in modern Mathematical Analysis textbooks.

Our article is devoted to solving this historical problem. Only basic knowledge about elementary arithmetic on terminating decimals, or equivalently on integers, is required.

The main idea of this paper is as simple as performing arithmetic on integers, but in a slightly different way. We are usually told to add and multiply numbers from right to left. Why not do so from left to right?

Let us consider a sum $13\bullet\bullet + 45\bullet\bullet$, where each summand is of four digits and the black dots are not specified. Considering the sum of the two digits 3 and 5 in the hundreds place is 8 and the addition of the black dots is less than 199, we get

$$13\bullet\bullet + 45\bullet\bullet = 5\substack{8\\ \text{or}\\ 9}\bullet\bullet,$$

whose thousands place value 5 is independent of the values of the black dots, and hundreds place value can only be 8 or 9. A second example is

$$15\bullet\bullet + 45\bullet\bullet = 6\substack{0\\ \text{or}\\ 1}\bullet\bullet.$$

In exactly the same way, one can show that

$$0.13\bullet\bullet + 0.45\bullet\bullet = 0.5\substack{8\\ \text{or}\\ 9}\bullet\bullet,$$
$$0.15\bullet\bullet + 0.45\bullet\bullet = 0.6\substack{0\\ \text{or}\\ 1}\bullet\bullet.$$

Since none of the black dots in the latest two examples is deterministic, we can prolong them freely as long as we wish, such as

$$0.1315\bullet\bullet + 0.4545\bullet\bullet = 0.586\substack{0\\ \text{or}\\ 1}\bullet\bullet,$$

and so on. In the event that the sum of the corresponding digits is 9, we cannot make an early decision. But if the sums of digits remain to be 9 since some position, then how to define addition is much easier. The following is an illustrative example:

$$0.373737\cdots + 0.626262\cdots = 1.$$

Such observations led Loo-Keng Hua [14] to define addition on infinite decimals in 1962. Fred Richman [23] got the same idea in 1999. To summarize, the general principle (for the first scenario) is to do addition locally from right to left (that is school arithmetic) but globally from left to right[1].

To the best of the authors' knowledge, we are not aware of any works that define multiplication in a similar way, although Wen-Tsun Wu [31] believes it is doable.

Ahead of stating Hua's definition and our multiplication proposal, we fix some notations. Our ambient space is [6, 9, 17]

$$\mathbb{R} = \{a_0.a_1a_2a_3\cdots \in \mathbb{Z}\times\mathbb{Z}_{10}^{\mathbb{N}} : a_k < 9 \text{ for infinitely many } k\},$$

where $\mathbb{Z}_{10}$ denotes the set $\{0, 1, 2, \ldots, 9\}$ and $\mathbb{N}$ the set of positive integers. To be clear, the use of notation $\mathbb{Z}\times\mathbb{Z}_{10}^{\mathbb{N}}$ is convenient but casual: an arbitrary element $(b_0, b_1, b_2, b_3, \ldots)$ of $\mathbb{Z}\times\mathbb{Z}_{10}\times\mathbb{Z}_{10}\times\mathbb{Z}_{10}\times\cdots = \mathbb{Z}\times\mathbb{Z}_{10}^{\mathbb{N}}$ is always regarded the same as $b_0.b_1b_2b_3\cdots$. Note that we exclude infinite decimals ending with a string of 9s once and for all. One main reason for doing so is that every element of $\mathbb{R}$ would then represent a unique quantity that can be used to measure length, area, volume and so on. Many readers may be more familiar with the classical decimal system [29]

$$\pm z = \pm a_0.a_1a_2a_3\cdots (z \in \mathbb{R}, z \geq 0),$$

but there is no essential difference between the options as long as we primarily focus on the arithmetic of non-negative elements, then make a suitable extension. One main reason for choosing $\mathbb{R}$ is that given a real number x (or a Dedekind cut, a Cauchy sequence, a map on $\mathbb{Z}$, and so on), we generally don't need to know whether it is non-negative or negative in advance, but can find a unique $a_0 \in \mathbb{Z}$ so that $a_0 \leq x < a_0 + 1$, a unique $a_1 \in \mathbb{Z}_{10}$ so that $a_0 + \frac{a_1}{10} \leq x < a_0 + \frac{a_1+1}{10}$, and continue in this way to get $x = a_0.a_1a_2a_3\cdots$. A second reason is that the link between the model $\mathbb{R}$ and the earliest theories of real numbers can be well explained (see Section 4).

For any element $x = a_0.a_1a_2a_3\cdots$ and any non-negative integer k, denote $\theta_k(x) = a_k$, the k-th digit of x, and $x_k = a_0.a_1a_2\cdots a_k$, the terminating decimal called the truncation of x up to the k-th digit. As usual, 10^{-m} is the same as $0.00\cdots01$ whose last digit 1 is at the m-th decimal place. Adding terminating decimals is rather straightforward. For example, using the convention above,

$$(-8).765 + 5.678 = (-8) + 0.765 + 5.678 = (-8) + 6.443 = (-2).443.$$

An element of $\mathbb{R}$, $x = a_0.a_1a_2a_3\cdots$ is said to be terminating if there exists a non-negative integer m such that $a_k = 0$ for $k > m$. In this case, write $x = a_0.a_1a_2\cdots a_m$ for simplicity. We assume here that a terminating decimal is identified in the obvious way to a terminating element of $\mathbb{R}$.

Definition 1.1 (addition, Hua [14])**.** Let x, y be elements of $\mathbb{R}$.
Case 1: Suppose there exists a non-negative integer m such that $\theta_k(x_k + y_k) = 9$ for $k > m$. Then define

$$x + y = (x_m + y_m)_m + 10^{-m}.$$

Case 2: Suppose there exists a sequence of positive integers $k_1 < k_2 < k_3 < \cdots$ such that $\theta_{k_i}(x_{k_i} + y_{k_i}) \neq 9$ for $i \in \mathbb{N}$. Then $x + y$ is defined by setting

$$(x + y)_{k_i - 1} = (x_{k_i} + y_{k_i})_{k_i - 1} \quad (i \in \mathbb{N}).$$

Note that $\theta_k(x_k + y_k) = 9$ if and only if $\theta_k(x) + \theta_k(y) = 9$ and $(x_m + y_m)_m = x_m + y_m$, and both succinct substitutes were indeed used in the original definition in [14]. But to look for a reasonable multiplication rule, one should not study the analog $\theta_k(x) \times \theta_k(y)$, and the reason will be easily seen later on. An element $a_0.a_1a_2a_3\cdots$ is said to be non-negative or negative if $a_0 \geq 0$ or $a_0 < 0$. Everyone knows about performing multiplication on non-negative terminating decimals. Obviously, it suffices to consider the special case $x + y \leq 1$ as the general case is linked by $xy = 10^{2s} \times (\frac{x}{10^s} \times \frac{y}{10^s})$ via a large enough non-negative integer s. Our multiplication proposal is as follows.

Definition 1.2. Let x, y be non-negative elements of $\mathbb{R}$ such that $x + y \leq 1$.
Case 1: Suppose there exists a non-negative integer m such that $\theta_k(x_k y_k) = 9$ for $k > m$. Then define

$$xy = (x_m y_m)_m + 10^{-m}.$$

[1] To compare, for readers acquainted with the subject, addition between p-adic numbers, viewed as Hensel's expansions with p a uniformizer and $\{0; 1; \cdots; p-1\}$ the set of representatives of the classes modulo p, works both locally and globally from right to left.

Case 2: Suppose there exists a sequence of positive integers $k_1 < k_2 < k_3 < \cdots$ such that $\theta_{k_i}(x_{k_i}y_{k_i}) \neq 9$ for $i \in \mathbb{N}$. Then xy is defined by setting

$$(xy)_{k_i-1} = (x_{k_i}y_{k_i})_{k_i-1} \quad (i \in \mathbb{N}).$$

The analogy between both definitions supports the validity of the proposal, and a more convincing explanation is as follows. Let x, y be non-negative such that $x+y \leq 1$, and denote $x = x_k+\epsilon_k$, $y = y_k+\delta_k$. We naturally expect

$$xy = (x_k + \epsilon_k)y = x_k y + \epsilon_k y = x_k y_k + x_k \delta_k + \epsilon_k y,$$

which implies that

$$0 \leq xy - x_k y_k < 10^{-k} \tag{1.1}$$

as $x+y \leq 1$ and the maximum between ϵ_k and δ_k is strictly less than 10^{-k}. Consequently, if $\theta_k(x_k y_k) \leq 8$, then $(xy)_{k-1} = (x_k y_k)_{k-1}$. So the second case of the definition is feasible. To illustrate the first case, we study a special case of $m = 2$ and $(x_2y_2)_2 = 0.15$. It follows from (1.1) that

$$(x_k y_k)_k \leq xy < (x_k y_k)_k + 2 \cdot 10^{-k}. \tag{1.2}$$

Letting $k = 2$ in (1.2) gives $0.15 \leq xy < 0.17$. Note also $(x_2y_2)_2 \leq (x_3y_3)_3$. So

$$0.15 = (x_2y_2)_2 \leq (x_3y_3)_3 \leq xy < 0.17.$$

Considering the assumption $\theta_3(x_3y_3) = 9$, we get $(x_3y_3)_3 = 0.159$ or $(x_3y_3)_3 = 0.169$. The second situation actually could not happen, because if it did then

$$\begin{aligned} 0.169 = (x_3y_3)_3 &\leq (x_4y_4)_4 \leq xy < 0.17, \\ 0.1699 = (x_4y_4)_4 &\leq (x_5y_5)_5 \leq xy < 0.17, \\ 0.16999 = (x_5y_5)_5 &\leq (x_6y_6)_6 \leq xy < 0.17, \\ &\cdots\cdots\cdots, \end{aligned}$$

which implies $0.16999\cdots \leq xy < 0.17$. This is absurd, so $(x_3y_3)_3 = 0.159$. Similarly,

$$\begin{aligned} 0.159 = (x_3y_3)_3 &\leq xy < 0.161, \\ 0.1599 = (x_4y_4)_4 &\leq xy < 0.1601, \\ 0.15999 = (x_5y_5)_5 &\leq xy < 0.16001, \\ &\cdots\cdots\cdots, \end{aligned}$$

which implies $xy = 0.16 = (x_2y_2)_2 + 10^{-2}$.

To summarize, the general principle (corresponding to the second case of Definition 1.2) is to do multiplication first locally via elementary arithmetic then globally from left to right. A full definition will be given in Section 3.

We need to show that the above arithmetical operations, no matter how reasonable they may be, form a field. For whatever reason, many other decimal approaches have just stopped here [16, p. 16]. Actually, the details of proving the field structure of various models have drawn lots of negative feedback in the past. Our method is to first establish

$$|(x + y)_k - x_k - y_k| \leq M_1 \cdot 10^{-k}, \tag{1.3}$$

$$|(xy)_k - x_k y_k| \leq M_2 \cdot 10^{-k}, \tag{1.4}$$

then argue by contradiction. Here M_1 and M_2 are positive integers independent of k, and both bounds follow from the corresponding arithmetical definitions in a few lines.

We cite several classical impressions of decimal approaches to the real number system as follows. The interested reader may compare these comments with our construction.

- "It is not obvious how to perform arithmetical operations" (Brannan [3]).
- "Any solution involves more and more complications" (Bridger [4]).

- "This is not a light task" (Courant [6]).
- "Simply do not work for infinite decimals" (Gardiner [10]).
- "Even more tedious to explain multiplication" (Stolz and Gmeiner [28]).
- "Despite being the most familiar, is actually more complicated" (Tao [29]).
- "Popular approach by novices but is fraught with technical difficulties" (Weiss [30]).

This article is arranged as follows. The second section studies addition by proving that $(\mathbb{R}, +)$ is an Abelian group and admits the greatest lower bound property. The third part focuses on multiplication and concludes the proof that $(\mathbb{R}, +, \times)$ is a field. The fourth one explores the link between our construction and two classical theories of real numbers. Based on the least upper bound property, a definition of trigonometric functions is introduced in the last section, which settles an issue that Godfrey H. Hardy called "a fatal defect" in [13].

Notation. Throughout the paper, we use k and m to denote non-negative integers.

2 Totally ordered Abelian group

In the two subsequent sections, we discuss the arithmetic and ordering structure of real numbers. As announced at the end of the preceding section, we shall go into detail and progressively reach the conclusion that $\mathbb{R}$ is a field at the end of Section 3.

2.1 Addition

To begin with, we justify the addition definition given in the preceding section. To do so, we follow all the notations and assumptions in Definition 1.1.

Case 1 of Definition 1.1: For any $k > m$, one has

$$\begin{aligned} x_k + y_k + 10^{-k} &= \left(x_m + y_m + 9 \cdot 10^{-(m+1)} + 9 \cdot 10^{-(m+2)} + \cdots + 9 \cdot 10^{-k}\right) + 10^{-k} \\ &= x_m + y_m + 10^{-m}. \end{aligned}$$

So the definition is independent of the choice of m. Since terminating decimals are elements of $\mathbb{R}$, so is the sum $x + y$.

Case 2 of Definition 1.1: We first claim that

$$(x_n + y_n)_{k_i - 1} = (x_{k_i} + y_{k_i})_{k_i - 1}$$

for all $n > k_i$. Considering $\theta_{k_i}(x_{k_i} + y_{k_i}) \leq 8$, one has for any $n > k_i$ that

$$\begin{aligned} x_n + y_n &= x_{k_i} + y_{k_i} + (x_n - x_{k_i}) + (y_n - y_{k_i}) \\ &= (x_{k_i} + y_{k_i})_{k_i - 1} + \theta_{k_i}(x_{k_i} + y_{k_i}) \cdot 10^{-k_i} + (x_n - x_{k_i}) + (y_n - y_{k_i}) \\ &< (x_{k_i} + y_{k_i})_{k_i - 1} + 8 \cdot 10^{-k_i} + 10^{-k_i} + 10^{-k_i} \\ &= (x_{k_i} + y_{k_i})_{k_i - 1} + 10^{-(k_i - 1)}, \end{aligned}$$

where the strict inequality is due to $x_n - x_{k_i} < 10^{-k_i}$ and $y_n - y_{k_i} < 10^{-k_i}$. Note also for any $n > k_i$, $x_{k_i} + y_{k_i} \leq x_n + y_n$, which implies $(x_{k_i} + y_{k_i})_{k_i - 1} \leq (x_n + y_n)_{k_i - 1}$, and consequently

$$0 \leq (x_n + y_n)_{k_i - 1} - (x_{k_i} + y_{k_i})_{k_i - 1} \leq x_n + y_n - (x_{k_i} + y_{k_i})_{k_i - 1} < 10^{-(k_i - 1)}.$$

This suffices to prove the claim. Consequently, $x + y$ is defined as an element of $\mathbb{Z} \times \mathbb{Z}_{10}^{\mathbb{N}}$. If $x + y$ is not an element of $\mathbb{R}$, say for example $x + y = c_0.c_1c_2c_3\cdots$ with $c_k = 9$ for k bigger than or equal to some $s \in \mathbb{N}$, we then assume without loss of generality that s is of the form k_i, so $\theta_s(x_s + y_s) \leq 8$. We claim that there exists an $l > s$ such that $x_n \leq x_s + 10^{-s} - 10^{-l}$ and $y_n \leq y_s + 10^{-s} - 10^{-l}$ for all $n \geq l$. The reason is as follows. One can first pick an $l_1 > s$ so that $\theta_{l_1}(x) \leq 8$, then note for any $n \geq l_1$,

$$x_n \leq x_s + \Big(\sum_{i=s+1}^{l_1 - 1} \frac{9}{10^i}\Big) + \frac{8}{10^{l_1}} + \Big(\sum_{i=l_1+1}^{n} \frac{9}{10^i}\Big) = x_s + 10^{-s} - 10^{-l_1} - 10^{-n} \leq x_s + 10^{-s} - 10^{-l_1}.$$

Similarly, pick an $l_2 > s$ so that $\theta_{l_2}(y) \leq 8$, and finally set $l = \max\{l_1, l_2\}$ and thus prove the claim. Consequently for any $n \geq l$,

$$\begin{aligned} x_n + y_n &\leq (x_s + y_s + 2 \cdot 10^{-s}) - 2 \cdot 10^{-l} \\ &= (x_s + y_s)_{s-1} + \theta_s(x_s + y_s) \cdot 10^{-s} + 2 \cdot 10^{-s} - 2 \cdot 10^{-l} \\ &\leq (x_s + y_s)_{s-1} + 10^{-(s-1)} - 2 \cdot 10^{-l} \\ &= (x + y)_{s-1} + 10^{-(s-1)} - 2 \cdot 10^{-l} \\ &= c_0.c_1c_2 \cdots c_{s-1}999 \cdots 998, \end{aligned} \tag{2.1}$$

where the last digit 8 is in the l-th decimal place. But, from the definition of addition, there exists a (large enough) $n_0 > l$ such that $\theta_{n_0}(x_{n_0} + y_{n_0}) \leq 8$, so

$$x_{n_0} + y_{n_0} \geq (x_{n_0} + y_{n_0})_{n_0-1} = (x + y)_{n_0-1} = c_0.c_1c_2 \cdots c_{s-1}999 \cdots 999, \tag{2.2}$$

where the last digit 9 is in the $(n_0 - 1)$-th decimal place. Combining (2.1) with $n = n_0$ and (2.2) yields a contradiction. Therefore, $x + y$ must be an element of $\mathbb{R}$.

2.2 Additive inverse

Next, we justify the following definition of the additive inverse.

Definition 2.1. Let $x = a_0.a_1a_2a_3 \cdots$ be an element of $\mathbb{R}$.
Case 1: Suppose there exists a non-negative integer m such that $a_k = 0$ for $k > m$. Then define

$$-x = (-1 - a_0).(9 - a_1)(9 - a_2) \cdots (9 - a_m) + 10^{-m}.$$

Case 2: Suppose there exists a sequence of positive integers $k_1 < k_2 < k_3 < \cdots$ such that $a_{k_i} > 0$ for $i \in \mathbb{N}$. Then $-x$ is defined as the following element of $\mathbb{R}$:

$$-x = (-1 - a_0).(9 - a_1)(9 - a_2)(9 - a_3) \cdots .$$

To be precise, the integral part of $-x$ is $-1 - a_0$ and $\theta_k(-x) = 9 - \theta_k(x)$ for all $k \geq 1$.

We follow all the notations and assumptions in Definition 2.1.
Case 1 of Definition 2.1: For any $k > m$, one has

$$\begin{aligned} &(-1 - a_0).(9 - a_1)(9 - a_2) \cdots (9 - a_k) + 10^{-k} \\ &= (-1 - a_0).(9 - a_1)(9 - a_2) \cdots (9 - a_m) + 9 \cdot 10^{-(m+1)} + \cdots + 9 \cdot 10^{-k} + 10^{-k} \\ &= (-1 - a_0).(9 - a_1)(9 - a_2) \cdots (9 - a_m) + 10^{-m}. \end{aligned}$$

So the definition is independent of the choice of m. Since terminating decimals are elements of $\mathbb{R}$, so is $-x$.

Case 2 of Definition 2.1: The definition is clearly independent of the choice of positive integers $k_1 < k_2 < k_3 < \cdots$. Since $9 - a_{k_i} < 9$ for all $i \in \mathbb{N}$, we see that $-x$ is an element of $\mathbb{R}$.

2.3 Abelian group

We now proceed with the proof that $(\mathbb{R}, +)$ is an Abelian group.

Proposition 2.2. *For any two elements x, y of $\mathbb{R}$, one has $x + y = y + x$.*

This property follows immediately from the addition definition.

Proposition 2.3. *For any element x of $\mathbb{R}$, one has $x + 0 = x$.*

This property follows immediately from the second case of the addition definition.

Proposition 2.4. *For any element x of $\mathbb{R}$, one has $x + (-x) = 0$.*

Proof. Write as usual $x = a_0.a_1a_2a_3\cdots$.

Case 1 of Definition 2.1: Suppose there exists a non-negative integer m such that $a_k = 0$ for $k > m$, that is, $x = a_0.a_1a_2\cdots a_m$. According to the first case of Definition 2.1,

$$-x = (-1 - a_0).(9 - a_1)(9 - a_2)\cdots(9 - a_m) + 10^{-m}.$$

Note first

$$\begin{aligned} x_m + (-x)_m &= a_0.a_1a_2\cdots a_m + (-1 - a_0).(9 - a_1)(9 - a_2)\cdots(9 - a_m) + 10^{-m} \\ &= -1 + 9\cdot 10^{-1} + 9\cdot 10^{-2} + \cdots + 9\cdot 10^{-m} + 10^{-m} \\ &= 0. \end{aligned}$$

Since $\theta_k(x) = \theta_k(-x) = 0$ for all $k > m$, we see that $x_k = x_m$ $(k > m)$ and $(-x)_k = (-x)_m$ $(k > m)$, and consequently

$$x_k + (-x)_k = x_m + (-x)_m = 0 \quad (k > m).$$

According to the second case of the addition definition, we get $x + (-x) = 0$.

Case 2 of Definition 2.1: Suppose there exists a sequence of positive integers $k_1 < k_2 < k_3 < \cdots$ such that $a_{k_i} > 0$ for $i \in \mathbb{N}$. According to the second case of Definition 2.1,

$$-x = (-1 - a_0).(9 - a_1)(9 - a_2)(9 - a_3)\cdots.$$

So $\theta_k(x) + \theta_k(-x) = 9$ for all $k > 0$. Applying the first case of the addition definition with $m = 0$, we get $x + (-x) = x_0 + (-x)_0 + 10^{-0} = a_0 + (-1 - a_0) + 1 = 0$. □

Lemma 2.5. *Let x, y be elements of $\mathbb{R}$. Then $|(x + y)_k - x_k - y_k| \le 6\cdot 10^{-k}$ for all $k \ge 0$.*

Proof. We follow all the notations and assumptions in the addition definition.

Case 1 of Definition 1.1 (to be continued): One has

$$x + y = x_k + y_k + 10^{-k} \quad (k > m),$$

which implies $x + y = (x + y)_k$ $(k > m)$, and consequently

$$|(x + y)_k - x_k - y_k| = |(x + y) - x_k - y_k| = 10^{-k} \quad (k > m).$$

Case 2 of Definition 1.1 (to be continued): $|(x+y)_{k_i-1} - x_{k_i-1} - y_{k_i-1}|$ is bounded from above by

$$|(x_{k_i} + y_{k_i})_{k_i-1} - (x_{k_i} + y_{k_i})| + |x_{k_i} - x_{k_i-1}| + |y_{k_i} - y_{k_i-1}|,$$

which is less than $3\cdot 10^{-(k_i-1)}$.

Cases 1 and 2 (continued): No matter which case happens, there exist infinitely many positive integers k so that

$$|(x + y)_k - x_k - y_k| \le 3\cdot 10^{-k}. \tag{2.3}$$

Let q be an arbitrary non-negative integer. Fixing a $k > q$ such that (2.3) holds, one gets

$$|(x + y)_q - x_q - y_q| \le |(x + y)_k - x_k - y_k| + 3\cdot 10^{-q} \le 6\cdot 10^{-q},$$

which proves Lemma 2.5. □

Lemma 2.6. *Let x and y be two distinct elements of $\mathbb{R}$. Then there exists an $l \in \mathbb{N}$ such that $|x_k - y_k| \ge 10^{-l}$ for $k > l$.*

Proof. Since x and y are distinct elements, there exists a non-negative integer m such that $x_m \ne y_m$. We may assume without loss of generality that $x_m < y_m$. Fix a positive integer $l > m$ so that $\theta_l(x) \le 8$. For any $k > l$, we have $x_k \le x_l + 10^{-l}$, and consequently

$$\begin{aligned} y_k - x_k &\ge y_m - \Big(x_m + \Big(\sum_{i=m+1}^{l} \frac{\theta_i(x)}{10^i}\Big) + 10^{-l}\Big) \ge y_m - \Big(x_m + \sum_{i=m+1}^{l} \frac{9}{10^i}\Big) \\ &= (y_m - x_m - 10^{-m}) + 10^{-l} \ge 10^{-l}. \end{aligned}$$

This finishes the proof. □

Proposition 2.7. *For any three elements x, y, z of $\mathbb{R}$, one has $(x + y) + z = x + (y + z)$.*

Proof. It follows from Lemma 2.5 that

$$|((x+y)+z)_k - x_k - y_k - z_k| = |((x+y)+z)_k - (x+y)_k - z_k + (x+y)_k - x_k - y_k| \leq 12 \cdot 10^{-k}$$

for all $k \geq 0$. Similarly,

$$|(x+(y+z))_k - x_k - y_k - z_k| \leq 12 \cdot 10^{-k} \quad (k \geq 0).$$

Combining the above two inequalities gives

$$|((x+y)+z)_k - (x+(y+z))_k| \leq 24 \cdot 10^{-k} \quad (k \geq 0).$$

If $(x + y) + z$ and $x + (y + z)$ are not the same, then according to Lemma 2.6, there exists an $l \in \mathbb{N}$ such that

$$|((x+y)+z)_k - (x+(y+z))_k| \geq 10^{-l} \quad (k > l).$$

Hence $10^{-l} \leq 24 \cdot 10^{-k}$ for $k > l$, which is absurd when $k = l + 2$. This proves the proposition. □

To conclude, we have proved that $(\mathbb{R}, +)$ is an Abelian group.

Before defining multiplication and studying the ring structure, we state the greatest lower bound property. This property implies that the set of real numbers is complete. In general, completeness means the real axis has no gaps. There are several equivalent ways to characterize the completeness of $\mathbb{R}$, depending on whether it is regarded as a metric space or a totally ordered set. If $\mathbb{R}$ is viewed as a metric space, then Cauchy's criterion for convergence is a completeness property; if it is treated as a totally ordered set, then the greatest lower bound property plays the same role.

We draw the reader's attention to the fact that, although at this stage we are conveniently supplying a proof of the greatest lower bound property, we won't need it to study multiplication and establish the field structure. In this regard, our construction of the real number system differentiates itself from most other suggested decimal approaches [12, 16, 18, 25] that rely on completeness as an essential to deal with the arithmetic structure.

2.4 Greatest lower bound property

We include a proof of the greatest lower bound property of $(\mathbb{R}, \preceq)$, where $\preceq$ denotes the lexicographical order of $\mathbb{R}$. To be precise, $x \preceq y$ means $x_k \leq y_k$ for all $k \geq 0$. Clearly, $x \preceq x$, $x \preceq y$ and $y \preceq x$ imply $x = y$, $x \preceq y$ and $y \preceq z$ imply $x \preceq z$, so $(\mathbb{R}, \preceq)$ is a partially ordered set. Suppose x and y are distinct elements of $\mathbb{R}$, and let m be the smallest non-negative integer such that $x_m \neq y_m$. If $x_m < y_m$, then for any $k > m$,

$$y_k - x_k \geq y_m - x_k \geq y_m - x_m - \sum_{i=m+1}^{k} \frac{9}{10^i} = (y_m - x_m - 10^{-m}) + 10^{-k} \geq 10^{-k} > 0.$$

So, in this case, one has $x \preceq y$. Similarly, if $y_m < x_m$, then one can get $y \preceq x$. This shows that $(\mathbb{R}, \preceq)$ is a totally ordered set. As usual, $x \prec y$ means $x \preceq y$ but $x \neq y$, and $y \succeq x$ ($y \succ x$) means exactly the same as $x \preceq y$ ($x \prec y$). A real number x is said to be non-negative (positive, or negative) if $x \succeq 0$ ($x \succ 0$, or $x \prec 0$). Hereafter, we shall simply write $\preceq$ ($\prec$, $\succeq$, or $\succ$) as $\leq$ ($<$, $\geq$, or $>$). The following greatest lower bound property is essentially contained in [1, 12, 16]. The key ingredient of the proof is the fact that every non-empty bounded below subset of $\mathbb{Z}$ has a (unique) smallest element.

Theorem 2.8. *Every non-empty bounded below subset of $\mathbb{R}$ admits a greatest lower bound in $\mathbb{R}$.*

Proof. Step 1 (ideal candidate): Let A be a non-empty bounded below subset of $\mathbb{R}$. Since $\{\theta_0(x) : x \in A\}$ is a non-empty bounded below subset of $\mathbb{Z}$, it has a smallest element a_0. In other words, $x_0 \geq a_0$ for all $x \in A$. Similarly, $\{\theta_1(x) : x \in A, \theta_0(x) = a_0\}$ has a smallest element, a digit a_1. Let x be an arbitrary element of A. If $x_0 = a_0$, we then deduce from the definition of a_1 that $x_1 \geq a_0.a_1$. This latter inequality clearly holds when $x_0 > a_0$. Thus, for all $x \in A$, $x_1 \geq a_0.a_1$. Continuing in this way, we see that for any $k \geq 2$, the non-empty subset

$$\{\theta_k(x) : x \in A, \theta_0(x) = a_0, \theta_1(x) = a_1, \ldots, \theta_{k-1}(x) = a_{k-1}\}$$

of $\mathbb{Z}$ has a smallest element a_k, and

$$x_k \geq a_0.a_1a_2\cdots a_k \tag{2.4}$$

for all $x \in A$. Now define $y = a_0.a_1a_2a_3\cdots$ as an element of $\mathbb{Z}\times\mathbb{Z}_{10}^{\mathbb{N}}$.

Step 2 (element of $\mathbb{R}$): We claim that y is an element of $\mathbb{R}$. If this is not the case, then y is of the form $a_0.a_1a_2\cdots a_m999\cdots$ for some $m \in \mathbb{N}$. The first step guarantees that there exists a $z \in A$ such that $z_m = y_m$. Applying (2.4) to $x = z$ and all $k > m$ yields $z = a_0.a_1a_2\cdots a_m999\cdots$, which is absurd. Hence y is an element of $\mathbb{R}$.

Step 3 (greatest lower bound): It follows from (2.4) that $x \succeq y$ for all $x \in A$. In other words, y is a lower bound for A. Let $w \in \mathbb{R}$ be an arbitrary lower bound for A. For any fixed $k \geq 0$, the first step guarantees that there exists a $z \in A$ (depending on k) such that $z_k = y_k$ (see also Step 2). Since w is a lower bound for A, one has $w_k \leq z_k = y_k$. This means $w \preceq y$. In other words, y is the greatest lower bound for A. □

We remark that the least upper bound property can be derived in much the same way, except that one more identification procedure needs to be included. Alternatively, since we have shown that $(\mathbb{R},+)$ is a totally ordered Abelian group, this property follows immediately from Theorem 2.8 and the fact that

$$\inf A = -\sup(-A),$$

where A denotes any bounded below subset of $\mathbb{R}$, $-A \doteq \{-x : x \in A\}$, $\inf(\cdot)$ denotes the greatest lower bound (also called the infimum), and $\sup(\cdot)$ denotes the least upper bound (also called the supremum).

2.5 Addition and ordering

The purpose of this part is to establish some basic relations between addition and ordering.

Proposition 2.9. *Let x, y, z, w be elements of $\mathbb{R}$.*
(1) If $x \geq 0$ and $y \geq 0$, then $x + y \geq 0$.
(2) $x < y$ if and only if $y - x > 0$.
(3) $x \leq y$ if and only if $y - x \geq 0$.
(4) If $x \leq y$, then $x + z \leq y + z$.
(5) If $x \leq z$ and $y \leq w$, then $x + y \leq z + w$.

Proof. It follows from the addition definition that property (1) holds. If $y - x > 0$, then the proof of Lemma 2.6 implies that there exists an $l \in \mathbb{N}$ such that

$$(y-x)_k \geq 10^{-l} \quad (k > l).$$

According to Lemma 2.5,

$$y_k + (-x)_k + 6\cdot 10^{-k} \geq (y-x)_k \quad (k \geq 0).$$

Applying Lemma 2.5 with $y = -x$ gives

$$6\cdot 10^{-k} \geq x_k + (-x)_k \quad (k \geq 0).$$

Combining the above three inequalities gives

$$y_k - x_k \geq 10^{-l} - 12\cdot 10^{-k} \quad (k > l),$$

which implies $x < y$. In much the same way, one can show that $y - x < 0$ implies $y < x$. If $y - x = 0$, then $x = 0 + x = (y + (-x)) + x = y + ((-x) + x) = y + 0 = y$. To conclude, we have established properties (2) and (3). If $x \leq y$, then $(y+z) - (x+z) = y - x \geq 0$, which implies $x + z \leq y + z$. This proves property (4). Finally, if $x \leq z$ and $y \leq w$, then $(z+w) - (x+y) = (z-x) + (w-y) \geq 0$, which implies $x + y \leq z + w$. This proves property (5). □

3 Field structure

3.1 Multiplication

In this part, we state our definition of multiplication which was not given in Hua's work [14]. We then justify that the proposed two-case definition of multiplication is consistent.

Definition 3.1. Let x, y be elements of $\mathbb{R}$.
(1) Suppose x, y are non-negative. Fix a non-negative integer s such that $x + y \le 10^s$.
Case 1: Suppose there exists a non-negative integer m such that $\theta_k(x_{k+s}y_{k+s}) = 9$ for $k > m$. Then define $xy = (x_{m+s}y_{m+s})_m + 10^{-m}$.
Case 2: Suppose there exists a sequence of positive integers $k_1 < k_2 < k_3 < \cdots$ such that $\theta_{k_i}(x_{k_i+s}y_{k_i+s}) \neq 9$ for $i \in \mathbb{N}$. Then xy is defined by setting

$$(xy)_{k_i-1} = (x_{k_i+s}y_{k_i+s})_{k_i-1} \quad (i \in \mathbb{N}).$$

(2) Suppose x, y are negative. Then define $xy = (-x)(-y)$.
(3) Suppose only one of x and y is negative. Then define $xy = -(x(-y))$.

We follow all the notations and assumptions in Definition 3.1 (1). Regarding the other cases (2) and (3), there is nothing to do. The justification is as follows.

Case 1 of Definition 3.1 (1): First, we claim that for any $n > m$,

$$(x_{n+s}y_{n+s})_m = (x_{m+s}y_{m+s})_m. \tag{3.1}$$

To verify (3.1), it suffices to consider $n = m + 1$, and suppose this is the case. Then

$$\begin{aligned} x_{n+s}y_{n+s} &= x_{m+s}y_{m+s} + (x_{n+s} - x_{m+s})y_{m+s} + x_{n+s}(y_{n+s} - y_{m+s}) \\ &\le x_{m+s}y_{m+s} + (x_{n+s} + y_{n+s}) \cdot \frac{9}{10^{n+s}} \le x_{m+s}y_{m+s} + \frac{9}{10^n}, \end{aligned}$$

where the last inequality is due to $x_{n+s} + y_{n+s} \le 10^s$. Thus

$$(x_{n+s}y_{n+s})_n \le (x_{m+s}y_{m+s} + \frac{9}{10^n})_n \le (x_{m+s}y_{m+s})_m + \frac{9}{10^n} + \frac{9}{10^n}. \tag{3.2}$$

Considering the assumption $\theta_n(x_{n+s}y_{n+s}) = 9$, one gets

$$(x_{n+s}y_{n+s})_n = (x_{n+s}y_{n+s})_m + \frac{9}{10^n}. \tag{3.3}$$

Combining (3.2) and (3.3) yields

$$0 \le (x_{n+s}y_{n+s})_m - (x_{m+s}y_{m+s})_m \le \frac{9}{10^n} < \frac{1}{10^m},$$

which proves claim (3.1). Next, we claim that for any $n > m$,

$$(x_{n+s}y_{n+s})_n + 10^{-n} = (x_{m+s}y_{m+s})_m + 10^{-m}. \tag{3.4}$$

To verify (3.4), it suffices to consider $n = m + 1$, and suppose this is the case. Recall $\theta_n(x_{n+s}y_{n+s}) = 9$, so (3.4) is equivalent to (3.1). Therefore, the definition is independent of the choice of m. On the other hand, it follows from (3.1) that

$$(x_{n+s}y_{n+s})_m + 10^{-m} = (x_{m+s}y_{m+s})_m + 10^{-m}, \tag{3.5}$$

so the definition is also independent of the choice of s. Since terminating decimals are elements of $\mathbb{R}$, so is the product xy.

Case 2 of Definition 3.1 (1): We claim that

$$(x_n y_n)_{k_i-1} = (x_{k_i+s}y_{k_i+s})_{k_i-1} \tag{3.6}$$

for $n > k_i + s$. Similar to the verification of the previous case, one gets

$$x_n y_n < x_{k_i+s} y_{k_i+s} + \frac{1}{10^{k_i}}. \tag{3.7}$$

Considering the assumption $\theta_{k_i}(x_{k_i+s} y_{k_i+s}) \leq 8$, one has

$$x_{k_i+s} y_{k_i+s} < (x_{k_i+s} y_{k_i+s})_{k_i-1} + \frac{9}{10^{k_i}}. \tag{3.8}$$

Combining (3.7) and (3.8) yields

$$0 \leq x_n y_n - (x_{k_i+s} y_{k_i+s})_{k_i-1} < \frac{1}{10^{k_i-1}},$$

which proves claim (3.6). Consequently, xy is defined as an element of $\mathbb{Z} \times \mathbb{Z}_{10}^{\mathbb{N}}$. If xy is not an element of $\mathbb{R}$, say for example $xy = c_0.c_1c_2c_3\cdots$ with $c_k = 9$ for k bigger than or equal to some $q \in \mathbb{N}$, we then assume without loss of generality that q is of the form k_i, so $\theta_q(x_{q+s}y_{q+s}) \leq 8$. Similar to the justification of the second case of the addition definition, one can fix an $l > q + s$ so that $x_n \leq x_{q+s} + 10^{-(q+s)} - 10^{-l}$ and $y_n \leq y_{q+s} + 10^{-(q+s)} - 10^{-l}$ for all $n \geq l$. So for any $n \geq l$,

$$x_n y_n \leq x_{q+s} y_{q+s} + (x_{q+s} + y_{q+s})(10^{-(q+s)} - 10^{-l}) + (10^{-(q+s)} - 10^{-l})^2.$$

The three summands of the right-hand side of the above inequality will be studied in detail as follows. First,

$$\begin{aligned} x_{q+s} y_{q+s} &\leq (x_{q+s} y_{q+s})_{q-1} + 8 \cdot 10^{-q} + (10^{-q} - 10^{-(2q+2s)}) \\ &= (xy)_{q-1} + 9 \cdot 10^{-q} - 10^{-(2q+2s)}. \end{aligned}$$

Second,

$$\begin{aligned} (x_{q+s} + y_{q+s})(10^{-(q+s)} - 10^{-l}) &\leq 10^s \cdot (10^{-(q+s)} - 10^{-l}) \\ &= 10^{-q} - 10^{-(l-s)}. \end{aligned}$$

Finally,

$$(10^{-(q+s)} - 10^{-l})^2 \leq 10^{-(2q+2s)}.$$

Combining the above three inequalities gives

$$x_n y_n \leq (xy)_{q-1} + 10^{-(q-1)} - 10^{-(l-s)} \quad (n \geq l). \tag{3.9}$$

On the other hand, fixing a (large enough) $n_0 > l + 1$ with $\theta_{n_0-s}(x_{n_0} y_{n_0}) \leq 8$, one gets

$$x_{n_0} y_{n_0} \geq (x_{n_0} y_{n_0})_{n_0-s-1} = (xy)_{n_0-s-1} = (xy)_{q-1} + 10^{-(q-1)} - 10^{-(n_0-s-1)}. \tag{3.10}$$

Combining (3.9) with $n = n_0$ and (3.10) yields a contradiction. Therefore, xy must be an element of $\mathbb{R}$.

3.2 Ring structure

As usual, we also denote xy by $x \times y$. Our present purpose is to show that $(\mathbb{R}, +, \times)$ is a ring.

Proposition 3.2. *For any two elements x, y of $\mathbb{R}$, one has $xy = yx$.*

Proof. We have three cases to consider.

Case 1: Suppose both x and y are non-negative. Then it is clear that $xy = yx$.

Case 2: Suppose both x and y are negative. Then both $-x$ and $-y$ are non-negative, and consequently $xy = (-x)(-y) = (-y)(-x) = yx$.

Case 3: Suppose only one of x and y is negative. Then the proof of the second case implies $(-y)x = y(-x)$, hence $xy = -(x(-y)) = -((-y)x) = -(y(-x)) = yx$. □

Proposition 3.3. *For any element x of $\mathbb{R}$, one has $x \times 1 = x$.*

Proof. We have two cases to consider.

Case 1: Suppose $x \geq 0$. Then it is clear that $x \times 1 = x$.

Case 2: Suppose $x < 0$. Then $x \times 1 = -(x \times (-1)) = -((-x) \times 1) = -(-x) = x$. □

Lemma 3.4. *Let x, y be non-negative elements of $\mathbb{R}$. Then $|(xy)_k - x_k y_k| \leq M \cdot 10^{-k}$ for all $k \geq 0$, where M is a positive integer depending only on x and y.*

Proof. We follow all the notations and assumptions in Definition 3.1 (1).

Case 1 of Definition 3.1 (1) (to be continued): Let $k > m$ be arbitrary. One has (see also (3.4))

$$xy = (xy)_k = (x_{k+s} y_{k+s})_k + 10^{-k}. \tag{3.11}$$

Note

$$(x_{k+s} y_{k+s})_k \leq x_{k+s} y_{k+s} \leq (x_k + 10^{-k})(y_k + 10^{-k}) \leq x_k y_k + (10^s + 1) \cdot 10^{-k}. \tag{3.12}$$

On the other hand,

$$(x_{k+s} y_{k+s})_k \geq (x_k y_k)_k \geq x_k y_k - 10^{-k}. \tag{3.13}$$

Combining (3.11) $\sim$ (3.13) yields

$$|(xy)_k - x_k y_k| \leq (10^s + 2) \cdot 10^{-k} \quad (k > m).$$

Case 2 of Definition 3.1 (1) (to be continued): Let $i \in \mathbb{N}$ be arbitrary. One has

$$(xy)_{k_i - 1} = (x_{k_i + s} y_{k_i + s})_{k_i - 1}.$$

Similar to the previous case, one can establish

$$|(xy)_{k_i - 1} - x_{k_i - 1} y_{k_i - 1}| \leq (10^s + 1) \cdot 10^{-(k_i - 1)}.$$

Cases 1 and 2 (continued): No matter which case happens, there exist infinitely many positive integers k so that

$$|(xy)_k - x_k y_k| \leq (10^s + 2) \cdot 10^{-k}. \tag{3.14}$$

Let q be an arbitrary non-negative integer. Fixing a $k > q$ such that (3.14) holds, one gets

$$\begin{aligned} |(xy)_q - x_q y_q| &\leq |(xy)_k - (xy)_q| + |x_k y_k - x_q y_q| + (10^s + 2) \cdot 10^{-k} \\ &\leq 10^{-q} + |x_k - x_q| y_k + x_q |y_k - y_q| + (10^s + 2) \cdot 10^{-q} \\ &\leq (2 \cdot 10^s + 3) \cdot 10^{-q}, \end{aligned}$$

which proves Lemma 3.4. □

In the proofs of the next two propositions, we need a notion of sign. By sign, we mean that an element of $\mathbb{R}$ is either non-negative or negative. Elements of the same sign are thus either all non-negative or all negative.

Proposition 3.5. *For any three elements x, y, z of $\mathbb{R}$, one has $x(y + z) = xy + xz$.*

Proof. We first prove the proposition when x, y, z are non-negative, then discuss how to establish it in general. Now suppose x, y, z are non-negative. It follows from Lemma 2.5 and Lemma 3.4 that

$$\begin{aligned} |(x(y+z))_k - x_k(y_k + z_k)| &= |(x(y+z))_k - x_k(y+z)_k + x_k(y+z)_k - x_k(y_k + z_k)| \\ &\leq M_1 \cdot 10^{-k} \quad (k \geq 0), \end{aligned}$$

where M_1 is a positive integer depending only on x, y and z. Similarly,

$$\begin{aligned} |(xy + xz)_k - x_k(y_k + z_k)| &= |(xy + xz)_k - (xy)_k - (xz)_k + (xy)_k + (xz)_k - x_k y_k - x_k z_k| \\ &\leq M_2 \cdot 10^{-k} \quad (k \geq 0), \end{aligned}$$

where M_2 is a positive integer depending only on x, y and z. Combining the above two inequalities gives

$$|(x(y+z))_k - (xz + yz)_k| \leq (M_1 + M_2) \cdot 10^{-k} \quad (k \geq 0).$$

If $x(y+z)$ and $xy+xz$ are not the same, then according to Lemma 2.6, there exists an $l \in \mathbb{N}$ such that

$$|(x(y+z))_k - (xy+xz)_k| \geq 10^{-l} \quad (k > l).$$

Hence $10^{-l} \leq (M_1 + M_2) \cdot 10^{-k}$ for $k > l$, which is absurd when k is large enough. So we must have $x(y+z) = xy + xz$. In general, Definition 3.1 and the proof of Proposition 3.2 imply that $xw = -(-x)w$ for all $x, w \in \mathbb{R}$, so to establish the proposition it suffices to assume from now on that $x \geq 0$. Similarly, we have $xw = -x(-w)$ for all $x, w \in \mathbb{R}$. If both y and z are negative, then

$$x(y+z) = -x(-(y+z)) = -x(-y-z) = -(x(-y)+x(-z)) = -x(-y) - x(-z) = xy + xz.$$

Finally, suppose y and z are not of the same sign. Since $y+z$ and only one of y and z must be of the same sign, we may assume without loss of generality that $y+z$, y and $-z$ are of the same sign. Then

$$x(y+z) + x(-z) = xy,$$

which combining $x(-z) = -xz$ gives $x(y+z) = xy + xz$. This finishes the proof of the proposition. □

Proposition 3.6. *For any three elements x, y, z of $\mathbb{R}$, one has $(xy)z = x(yz)$.*

Proof. We first prove the proposition when x, y, z are non-negative, then discuss how to establish it in general. Now suppose x, y, z are non-negative. Similar to the proofs of Proposition 2.7 and the corresponding part of Proposition 3.5, one can establish $(xy)z = x(yz)$. Next, we study the general situation of the proposition. If x, y, z are negative, then

$$(xy)z = -(((-x)(-y))(-z)) = -(-x)((-y)(-z)) = x(yz).$$

Finally, suppose only two of x, y and z are of the same sign. Note that

$$-(xy)z = ((-x)y)z = (x(-y))z = (xy)(-z)$$
$$-x(yz) = (-x)(yz) = x((-y)z) = x(y(-z)).$$

We now have three cases to consider. If $-x$, y and z are of the same sign, then $-(xy)z = ((-x)y)z = (-x)(yz) = -x(yz)$, which implies $(xy)z = x(yz)$. The other two cases can be studied in exactly the same way. This finishes the proof of the proposition. □

To conclude, we have proved that $(\mathbb{R}, +, \times)$ is a ring.

3.3 Multiplication and ordering

Proposition 3.7. *Let x, y, z be elements of $\mathbb{R}$.*
(1) If x and y are non-negative, then xy is non-negative.
(2) If x is non-negative and if $y \leq z$, then $xy \leq xz$.

Proof. Property (1) follows from the multiplication definition. As an application, if x is non-negative and if $y \leq z$, then $xz - xy = x(z-y) \geq 0$, which implies $xy \leq xz$ by Proposition 2.9 (3). □

3.4 Field structure

We prove next that $(\mathbb{R}, +, \times)$ is a field. To this end, we define the reciprocal of a non-zero element of $\mathbb{R}$.

Definition 3.8. Let x be a non-zero element of $\mathbb{R}$.
(1) Suppose x is positive. There exists a unique non-negative integer a_0 so that $x \times a_0 \leq 1 < x \times (a_0+1)$, a unique $a_1 \in \mathbb{Z}_{10}$ so that $x \times (a_0 + \frac{a_1}{10}) \leq 1 < x \times (a_0 + \frac{a_1+1}{10})$, and continuing in this way yields an element $a_0.a_1a_2a_3\cdots$ of $\mathbb{Z} \times \mathbb{Z}_{10}^{\mathbb{N}}$. Then define $x^{-1} = a_0.a_1a_2a_3\cdots$.
(2) Suppose x is negative. Then define $x^{-1} = -((-x)^{-1})$.

To justify the first case of the reciprocal definition, we need to show that the element $a_0.a_1a_2a_3\cdots$ uniquely generated by the procedure

$$x \times a_0.a_1a_2\cdots a_k \leq 1 < x \times (a_0.a_1a_2\cdots a_k + 10^{-k}) \quad (k = 0, 1, 2, \ldots) \tag{3.15}$$

must be an element of $\mathbb{R}$. If this is not true, then x^{-1} is of the form $a_0.a_1a_2\cdots a_m999\cdots$ instead for some $m \in \mathbb{N}$. For any $k > m$, one has

$$x \times a_0.a_1a_2\cdots a_k \leq 1 < x \times (a_0.a_1a_2\cdots a_k + 10^{-k}) = x \times (a_0.a_1a_2\cdots a_m + 10^{-m}),$$

which yields (see also Proposition 2.9)

$$0 < x \times (a_0.a_1a_2\cdots a_m + 10^{-m}) - 1 \leq x \times 10^{-k}, \tag{3.16}$$

where Proposition 3.5 has been used to get

$$x \times 10^{-k} = x \times (a_0.a_1a_2\cdots a_k + 10^{-k}) - x \times a_0.a_1a_2\cdots a_k.$$

Formula (3.16) implies that the set $\{x \times 10^{-k} : k > m\}$ has a positive lower bound, which is absurd. This finishes the justification of the first case of the reciprocal definition. Regarding the second case, we have nothing to do.

Proposition 3.9. *For any non-zero element x of $\mathbb{R}$, one has $xx^{-1} = 1$.*

Proof. We have two cases to consider.

Case 1: Suppose x is positive. Then $x^{-1} = a_0.a_1a_2a_3\cdots$ is generated by the procedure (3.15), which following Proposition 3.7 gives

$$x(x^{-1} - 10^{-k}) \leq 1 \leq x(x^{-1} + 10^{-k}) \quad (k = 0, 1, 2, \ldots),$$

where we have used the simple fact $x^{-1} - 10^{-k} \leq (x^{-1})_k$. Equivalently (see also Proposition 2.9),

$$-x \cdot 10^{-k} \leq 1 - xx^{-1} \leq x \cdot 10^{-k} \quad (k = 0, 1, 2, \ldots).$$

Clearly, the set $\{-x \cdot 10^{-k} : k > m\}$ cannot have a negative upper bound, and the set $\{x \cdot 10^{-k} : k > m\}$ cannot have a positive lower bound, so we must have $1 - xx^{-1} = 0$, or equivalently, $xx^{-1} = 1$.

Case 2: Suppose x is negative. Then $xx^{-1} = (-x)(-(x^{-1})) = (-x)(-x)^{-1} = 1$. □

To conclude, we have proved that $(\mathbb{R}, +, \times)$ is a field.

4 Classical theories

In this section, we discuss Dedekind and Cantor's theories from Stevin's viewpoint of infinite decimal expansions. Many introductory analysis books [9, 13, 17, 24] choose Dedekind's approach, some [29] prefer Cantor's theory, but most avoid a detailed construction. Note that Cantor's work is essentially the same as those of Méray and Heine [20]. An object could have various characterizations or disguises; so do real numbers.

4.1 Dedekind's theory

A Dedekind cut $(A|B)$ is formed of two subsets A, B of $\mathbb{Q}$ such that [23] $A \cup B = \mathbb{Q}$, $a < b$ for any $a \in A$ and $b \in B$, and B has no smallest element[2]. Given a Dedekind cut $(A|B)$, there exists a unique integer a_0 such that $a_0 \in A$, $a_0 + 1 \in B$. Similarly, there exists a unique integer $a_1 \in \mathbb{Z}_{10}$ such that $a_0 + \frac{a_1}{10} \in A$, $a_0 + \frac{a_1+1}{10} \in B$. Continuing in this way yields an element $a_0.a_1a_2a_3\cdots$ of $\mathbb{R}$, which is naturally identified with the cut $(A|B)$. One can easily show such an identification is a bijection.

A disadvantage of Dedekind's approach may be that this language is rarely used in advanced courses and research activities.

[2]Many authors [8, 13, 17] replace this uniqueness condition with A having no greatest element. If so, then the identification of $(\{x \in \mathbb{Q} : x < 1\}|\{x \in \mathbb{Q} : x \geq 1\})$ is $0.999\cdots$, which does not belong to $\mathbb{R}$.

4.2 Cantor's theory

A sequence of rational numbers $\{q_n\}_{n=1}^{\infty}$ is said to be Cauchy if for any $\epsilon > 0$ there exists a natural number N (depending on ϵ) such that $|q_m - q_n| < \epsilon$ for all $m, n > N$. First, show that the given sequence is bounded. By the pigeonhole principle, we then pick an $a_0 \in \mathbb{Z}$ so that infinitely many elements of the sequence lie in $[a_0, a_0 + 1)$, and $a_1 \in \mathbb{Z}_{10}$ so that infinitely many elements of the sequence lie in $[a_0 + \frac{a_1}{10}, a_0 + \frac{a_1+1}{10})$. Continuing in this way yields an element $a_0.a_1a_2a_3\cdots$ of $\mathbb{Z}\times\mathbb{Z}_{10}^{\mathbb{N}}$. In most cases this procedure outputs a unique identification element of $\mathbb{R}$; sometimes it could also provide two "*different*" elements such as $0.999\cdots$ and $1.000\cdots$, so we need to identify them; but three or more identification elements could never all exist together. To verify this claim, one needs to explore what Cauchy sequence really means. Once again, it is not difficult to show that the above identification is a bijection.

Cantor's approach is the first example of completing metric spaces in functional analysis. Although he calls Cauchy sequences real numbers, his greatest mathematical contributions were inspired by decimal expansions. For example, Cantor's idea of proving $\mathbb{R}^2$ is the same size (or cardinality) as $\mathbb{R}$, which he called the continuum, can be described as follows. Given any two real numbers $x = a_0.a_1a_2a_3\cdots$ and $y = b_0.b_1b_2b_3\cdots$, define

$$\Psi(x, y) = 0.a_1b_1\Box a_2b_2\Box a_3b_3\Box\cdots,$$

where the sequence of empty boxes is left to encode the integer parts of x and y, and there are plenty of ways to do so. Obviously, Ψ is injective, so the size of $\mathbb{R}^2$ is not greater than that of $\mathbb{R}$. Clearly, the continuum is not greater than the size of $\mathbb{R}^2$ either, hence by the Schröder-Bernstein theorem [7, p. 4], both sets are of the same cardinality.

5 Definition of trigonometric functions

One of the main upshots of any decimal approach to the real number system is to offer a rather straightforward and tangible proof of the least upper bound property (see Section 2.4). We are going to take advantage of this feature and expand, in this section, into a natural and rigorous way of defining trigonometric functions.

Historically, trigonometric functions were defined in the popular geometric style, assuming that the length of arc is a known mathematical or physical concept, which it is unlikely anyone would doubt. But without a complete construction of the real number system, how can we rigorously talk about lengths, areas, volumes and so on? Some readers may claim that the length of smooth curves has been given in calculus books as a standard application of Riemann's integration theory. The trouble is that these books have already used $\sin x$ or $\cos x$ profusely as basic examples before integration. This issue was clearly underlined by Godfrey H. Hardy, who called it "a fatal defect" in his Course of Pure Mathematics [13, p. 316]. Therefore, ideally, it still remains to define both functions as early as possible.

A classical idea of defining the length of arcs is as follows. Let γ be an arbitrary arc on the plane. Let us divide the arc γ equally into 2^n parts for each $n \in \mathbb{N}$ and consider the sequence of the lengths of the piecewise linear functions with all ending points of the 2^n subarcs as its vertices. This sequence can be shown to be increasing and have an upper bound. One then defines the length of γ as the limit of this sequence. However, this method does not automatically imply that the length of the curve concatenating two concentric arcs is the sum of those two pieces. We will remedy it and ground this classical geometric approach in what follows.

Let $[a, b] \subset \mathbb{R}$ be a bounded interval. A curve $F = (f_1, f_2) : [a, b] \to \mathbb{R}^2$ is said to be monotone if its coordinate functions f_1 and f_2 are monotone (increasing or decreasing). Since the "*shortest*" curve connecting two points is a straight line, we expect the "*length*" of F, denoted by $\mathscr{L}(F)$, to be an upper bound for the "*length*" of the line segment $\overline{F(a)F(b)}$ with ending points $F(a)$ and $F(b)$. Suppose we divide $[a, b]$ into two pieces $[a, b] = [a, c] \cup [c, b]$. Clearly,

$$\mathscr{L}(F) = \mathscr{L}(F|_{[a,c]}) + \mathscr{L}(F|_{[c,b]}) \geq \mathscr{L}(\overline{F(a)F(c)}) + \mathscr{L}(\overline{F(c)F(b)}) \geq \mathscr{L}(\overline{F(a)F(b)}),$$

where the last inequality is due to the fact that the sum of two sides of a triangle is not smaller than the third. Continuing this procedure, the closest quantity to $\mathscr{L}(F)$ we can find is

$$\sup\Big\{\sum_{i=1}^{n} \mathscr{L}(\overline{F(c_{i-1})F(c_i)}) : a = c_0 < c_1 < c_2 < \cdots < c_n = b, \quad n \in \mathbb{N}\Big\}, \tag{5.1}$$

where $\sup(\cdot)$ denotes the supremum of a given bounded above subset of $\mathbb{R}$ (see Section 2.4). Note that, considering the "*length*" of a line segment to be its Euclidean length,

$$\sum_{i=1}^{n} \mathscr{L}(\overline{F(c_{i-1})F(c_i)}) = \sum_{i=1}^{n} \sqrt{(f_1(c_{i-1}) - f_1(c_i))^2 + (f_2(c_{i-1}) - f_2(c_i))^2}$$
$$\leq \sum_{i=1}^{n} \big(|f_1(c_{i-1}) - f_1(c_i)| + |f_2(c_{i-1}) - f_2(c_i)|\big)$$
$$= |f_1(a) - f_1(b)| + |f_2(a) - f_2(b)|, \tag{5.2}$$

where the inequality is due to $\sqrt{x^2+y^2} \leq |x| + |y|$, and the last equality owes to the monotonicity of f_1 and f_2. So the existence of (5.1) is guaranteed by the least upper bound property of the real number system.

Definition 5.1. Given a monotone curve $F : [a,b] \to \mathbb{R}^2$, define its length $\mathscr{L}(F)$ to be the supremum (5.1).

More generally, a curve $F : [a,b] \to \mathbb{R}^2$ is said to be rectifiable [6, 24] if the supremum (5.1) exists (as an element of $\mathbb{R}$), or equivalently if the coordinate functions of F are of bounded variation. Jordan's decomposition theorem [7, p. 173] states that any function of bounded variation can be written as the difference between two monotone functions, hence Definition 5.1 is very close to the ultimate scenario of rectifiable curves.

One can easily show now, applying Definition 5.1, that the length of the curve concatenating two concentric arcs is the sum of those two pieces.

Next, let us rigorously define the "*well-known*" sine function and establish its continuity. Let $y \in [0,1]$ be a parameter and introduce the map

$$\gamma_y : t \mapsto (t, \sqrt{1-t^2})$$

on the interval $[\sqrt{1-y^2}, 1]$. It is clear that γ_y is a monotone curve, so its length $\mathscr{L}(\gamma_y)$ is well defined by Definition 5.1.

Definition 5.2. Define a real number $\pi^\bullet$ by $\frac{\pi^\bullet}{2} = \mathscr{L}(\gamma_1)$.

To avoid misunderstanding, we create the new symbol $\pi^\bullet$ at this stage. Nevertheless, the real number $\pi^\bullet$ should be regarded as the same as π later on.

Definition 5.3. Define a function f on $[0,1]$ by $f(y) = \mathscr{L}(\gamma_y)$.

Proposition 5.4. *f is a strictly increasing continuous function.*

Proof. Let $0 \leq y_1 < y_2 \leq 1$ be arbitrary. Similar to the upper bound (5.2), one can obtain

$$0 \leq \mathscr{L}(\gamma_{y_2}) - \mathscr{L}(\gamma_{y_1}) \leq (y_2 - y_1) + (\sqrt{1-y_1^2} - \sqrt{1-y_2^2}),$$

which implies that f is a continuous function on $[0,1]$. Obviously, f is strictly increasing. This finishes the proof of Proposition 5.4. □

A well-known result in calculus says that a strictly increasing continuous real-valued function on an interval has an inverse that is also strictly increasing and continuous. So according to Proposition 5.4, we have the following result.

Theorem 5.5. *The sine function on $[0, \frac{\pi^\bullet}{2}]$, defined as the inverse function of f, is strictly increasing and continuous.*

The interested reader may now extend the above definition to the full line $\mathbb{R}$ in the popular geometric style, define the cosine function, and establish the differentiability and other analytic properties of these trigonometric functions.

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