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The Spinning Cube: An Algebraic Excursion

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ABSTRACT: What shapes do you get when you spin a cube on one of its vertices? This article explores the algebraic aspects of a spinning cube and establishes equations for the two cones and the hyperboloid using secondary school mathematics. Both the analysis and the verification take advantage of dynamic mathematics learning technologies.

Keywords: cube, hyperboloid, algebraic analysis, dynamic learning technology

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1 Introduction

The cube is widely used in school and college mathematics classes for modeling, general references, and lively discussions of shapes (Senechal, 1990). It is also the source of numerous puzzles. While cube spinning is not often discussed in school or college mathematics, it is a classic problem with surprising geometric and aesthetic results (Hilbert & Cohn-Vossen, 1952; Steinhaus, 1969; Sobel & Maletsky, 1999). In a recent pedagogical exploration of shapes and patterns in a teacher education context, the author posed the problem to a group of teacher candidates, resorting to both physical, dynamic, and 3D-printed models (see Figs 1 and 2). The technical and pedagogical details are discussed in Bu (2019). Together, the classroom community came to appreciate the art and the mathematics of a spinning cube, which further led the author to seek an algebraic perspective on the resulting artifact.

This article takes advantage of new dynamic mathematics learning technologies (such as GeoGebra at *http://www.geogebra.org*) to set up the stage for a detailed algebraic analysis of both the cones and the one-sheeted hyperboloid, using school mathematics, in the context of a spinning cube. We further showcase the affordances of dynamic modeling and 3D design technologies in supporting mathematical problem posing, exploration, representation, solution, and communication.

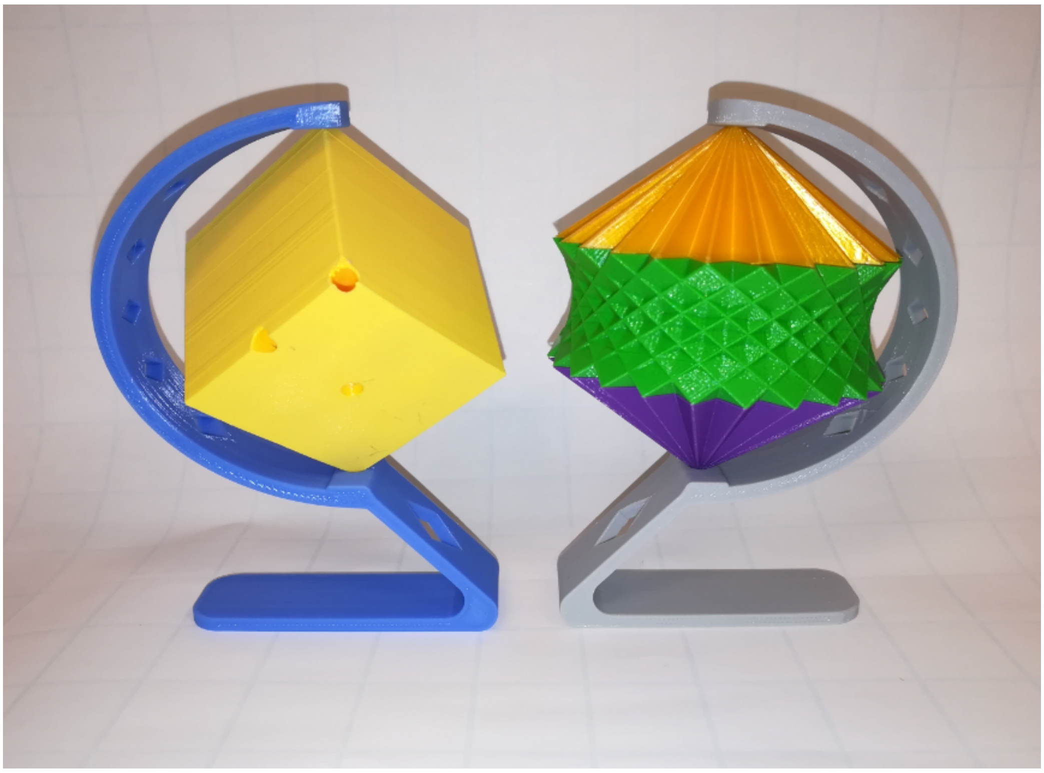

Figure 1: 3D-printed models are used for investigating a spinning cube. On the left is a cube with three rotational axes; on the right is the union of 8 copies of the cube evenly positioned.

When spinning a cube around its long or major diagonal, we obtain a surprisingly beautiful shape, which has a cone at both ends and a curved surface in the middle (see Figs 1 and 2). The two cones are quite expected. But the middle part is worth some deep thought. What is it algebraically? Examining the various physical and dynamic models, we recognize that the surface is swept out by the six edges in the middle of the cube, which come in two families in terms of their orientations. Furthermore, the three edges from the same family overlap in space as the cube rotates. Therefore, there are only two distinct lines contributing to the curved surface. These two lines intersect each other all around as the cube spins, which means for each point on one line, there is a corresponding point on another, constituting a circular orbit on the surface. Thus, the curved surface is essentially swept out by one skew line around the rotational axis. The surface is known as a one-sheeted ruled surface (Hilbert & Cohn-Vossen, 1952). Since it has two contributing rulings, it is also called a double-ruled one-sheeted

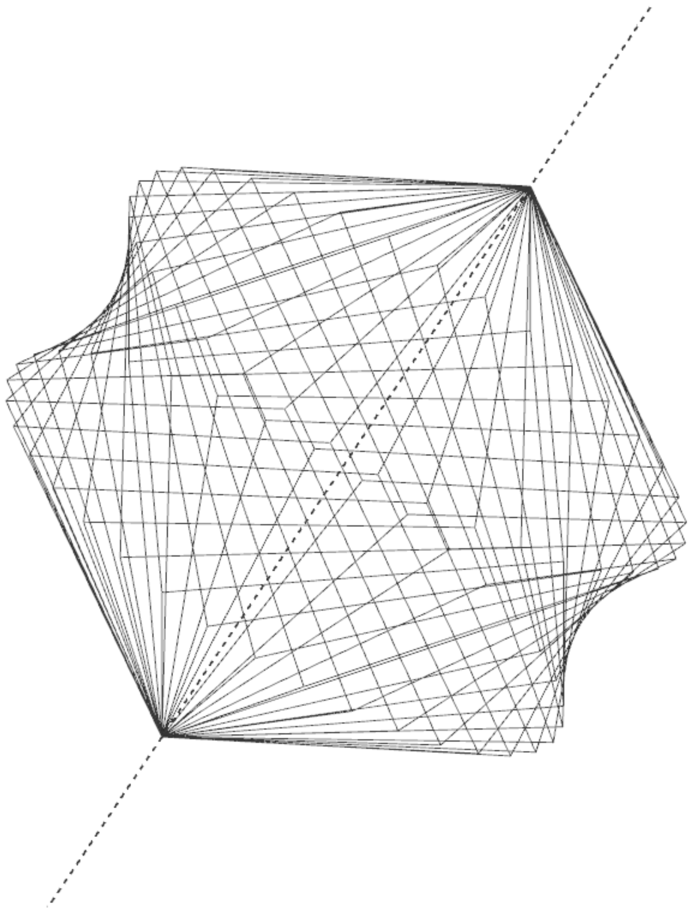

Figure 2: A dynamic model for the spinning cube can be built using GeoGebra®.

surface. In the case of a spinning cube, the surface is the locus of a quadric equation, which has an algebraic form of $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$, where $A, B, \cdots, J$ are all constants. Specifically, it is a one-sheeted hyperboloid when it is in the forms of $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ or a cone when it is in the form of $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$, where a, b, c are non-zero constants. In what follows, we first review some mathematical background and then focus on the spinning cube. We will discuss the angle of a skew line in the context of a cube and further set up a proof to show that the solid obtained when spinning a cube is a combination of two cones and a hyperboloid.

2 Mathematical Background

The big idea behind a spinning cube is that of a hyperboloid. Therefore, it is worthwhile to review existing approaches to the hyperboloid. A hyperboloid can be naturally created by revolving a hyperbola around an axis. There are two possible outcomes. If the hyperbola is revolved around the axis containing its two foci, we get a hyperboloid of two sheets. If it is revolved around the perpendicular bisector of the segment connecting the two foci, we get a hyperboloid of one sheet. The latter case has been known for its geometric and aesthetic appeals in that it is a ruled surface, which can be generated by straight lines known as rulings. To establish the algebraic equation for a one-sheeted hyperboloid from a hyperbola, let's consider a hyperbola in the xz plane (Fig. 3). We further pick an arbitrary point $A = (x_r, z)$ on the hyperbola and allow it to revolve around the z-axis. In the plane containing the orbit of point A, we consider a point $P = (x, y, z)$ on the orbit. Since point A in on the hyperbola, we have

$$\frac{x_r^2}{a^2} - \frac{z^2}{c^2} = 1. \tag{2.1}$$

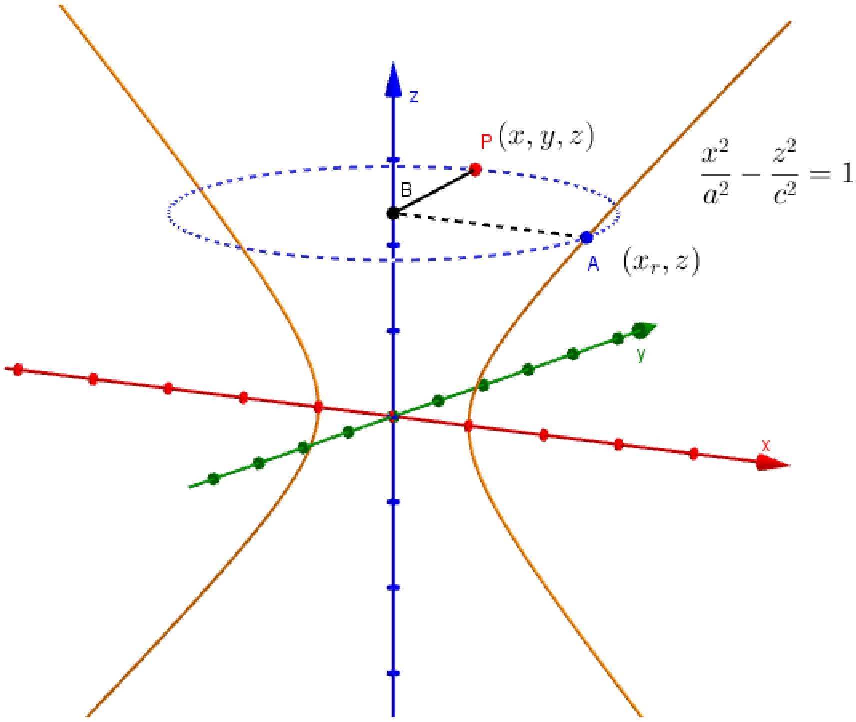

Figure 3: A hyperboloid of one sheet is created by revolving a hyperbola in the xz plane around the z-axis (created with GeoGebra®).

Further, since AB and BP are both radii of the circular orbit of A and P is on the circle, then we have $BP = x_r$ and $x^2 + y^2 = x_r^2$, which leads to an equation for the hyperboloid created by the hyperboloa as it revolves around the z-axis,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{c^2} = 1. \tag{2.2}$$

Next we consider the hyperboloid of one sheet as a ruled surface (Hilbert & Cohn-Vossen, 1952). Hilbert and Cohn-Vossen discussed in great detail the geometric processes of a variety of ruled surfaces, including the hyperboloid of one sheet, indicating that a proof would require some analytic methods. Steinhaus (1969) approached the hyperboloid using overlapping photographs of a spinning cube, highlighting its aesthetic outcome. Analytically, both Wilson (1911) and Ogilvy (1966) proposed resourceful ideas for constructing an algebraic proof in the general case of generating a hyperboloid from a skew line. In particular, Wilson (1911) considered a special location of the revolving skew line and extended his observation to an arbitrary point on the hyperboloid. As shown in Fig. 4, line g is the generating skew line revolving around the z-axis. We consider the special case where g is parallel to the yz plane at its initial location and intersects the x-axis at $D = (a, 0, 0)$. Line g' is the projection of g in the yz plane and $\angle\phi$ is the skew angle. For simplicity, let $m = \tan(\phi)$, representing the slope of g' as $y = mz$ in the yz plane. For a point $C = (x_r, y_r, z)$ on line g, we know that $x_r = a$ and $y_r = mz$. In the plane containing the orbit of C as it revolves around the z-axis, we focus on right triangle $\triangle ABC$, where $AB = a$, $BC = y_r = mz$, and thus $AC^2 = a^2 + m^2z^2$. Now, for an arbitrary point $P = (x, y, z)$ on the orbit of C, we have $x^2 + y^2 = AC^2$, because AC is the radius of the circle. Therefore, we have established an algebraic equation for the hyperboloid generated by line g,

$$x^2 + y^2 = a^2 + m^2z^2, \tag{2.3}$$

which can be further manipulated as necessary. Later on, we will revisit (2.3) in the context of a cube. By contrast, Ogilvy (1966) approached the same problem from a general perspective. He used the angle of rotation as a parameter to obtain the equation of a hyperbola in the xy plane, which was further used to argue for the hyperboloid generated by the ruling. Having briefly reviewed the mathematical background of the hyperboloid, we move on to the spinning cube and look into the geometric components in an analytic manner, using methods that complement those of Wilson (1911) and Ogilvy (1966).

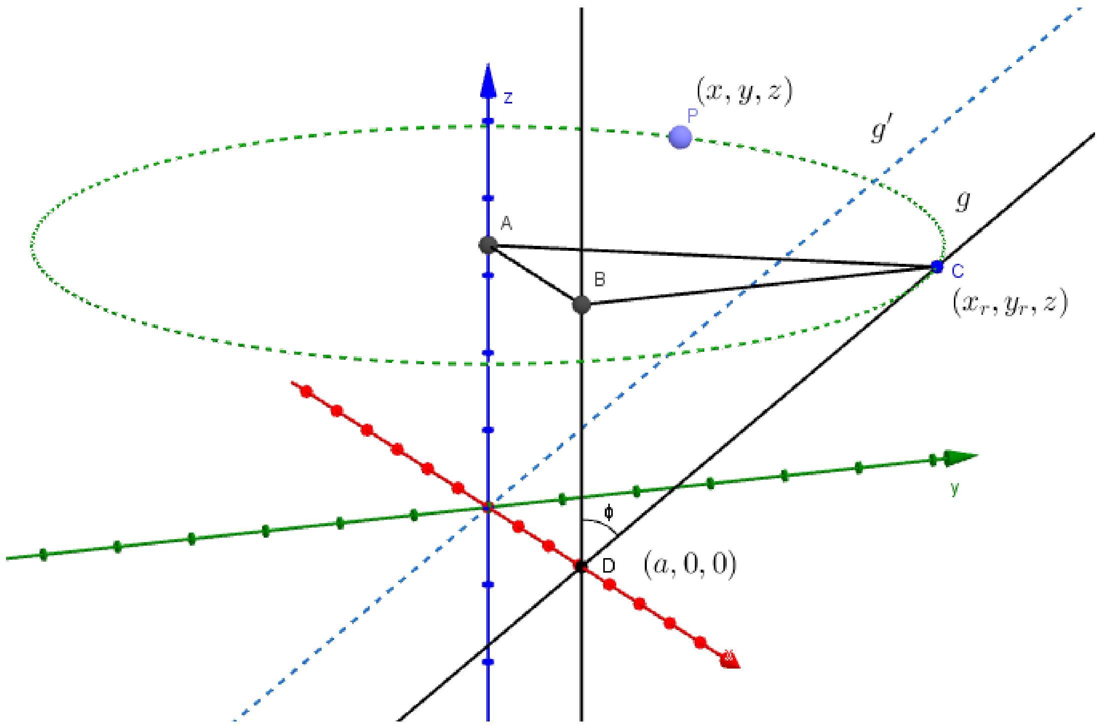

Figure 4: Wilson's (1911) method for establishing an equation for a hyperboloid of one sheet generated by a skew line (created with GeoGebra®).

3 The Skew Angle

In spinning a cube, the *skew angle* of the middle edges is of critical importance; otherwise, we would get a cylinder in the middle. Let's consider a unit cube shown in Fig. 5, where BH is the rotational axis and FG is a middle edge to be examined. During the rotation, B and H stay fixed, while D, E, and G constitute one orbit, and A, F, and C the other. To find the skew angle of FG with respect to BH, we construct a line FJ perpendicular, at point F, to the plane containing points A, C, and F. Angle $\angle GFJ$ or $\angle\alpha$ is the skew angle of edge FG with respect to line FJ. As illustrated in Fig. 5, $\angle GFJ$ is simply the translation of $\angle CBH$ or $\angle\beta$ along edge BF. Thus, the skew angle $\angle GFJ$ or $\angle\alpha$ is congruent to $\angle CBH$ or $\angle\beta$, which is $\tan^{-1}(\sqrt{2})$ with reference to the right triangle $\triangle BCH$, where $BC = 1$ and $CH = \sqrt{2}$. Specifically, the skew angle of a middle edge of the cube is approximately $54.74°$.

4 The Cones

To conduct an algebraic analysis of the cones resulting from a spinning cube, we continue to consider a unit cube, whose long diagonal is $\sqrt{3}$ and is divided into three equal parts by the two planes defined respectively by the two groups of three vertices in the middle (Fig. 6). Since our primary focus is on the middle hyperboloid, we position the unit cube such that a long diagonal is on the z-axis and the corresponding vertices are at $B = (0, 0, -\frac{\sqrt{3}}{3})$ and $H = (0, 0, \frac{2\sqrt{3}}{3})$ for ease of presentation (Fig. 6). This setup could also be pedagogically more accessible to students in the beginning. The resulting equation can be easily adjusted when necessary.

In Fig. 6, $\angle CBH$ or $\angle\beta$ is $\tan^{-1}(\sqrt{2})$, evident with respect to $\triangle BCH$. To establish an algebraic equation for the cone swept out by BC as it revolves around the z-axis, let's pick an arbitrary point $P = (x, y, z)$ on BC and further construct a plane through P that is perpendicular to the z-axis at Q. In $\triangle BPQ$, $PQ = BQ \cdot \tan(\beta)$, where $\tan(\beta) = \sqrt{2}$. Thus,we have

$$PQ^2 = 2BQ^2 \tag{4.1}$$

We know that $PQ^2 = x^2 + y^2$ and $BQ^2 = (z + \frac{\sqrt{3}}{3})^2 = z^2 + \frac{2\sqrt{3}}{3}z + \frac{1}{3}$. Therefore, from (4.1), the

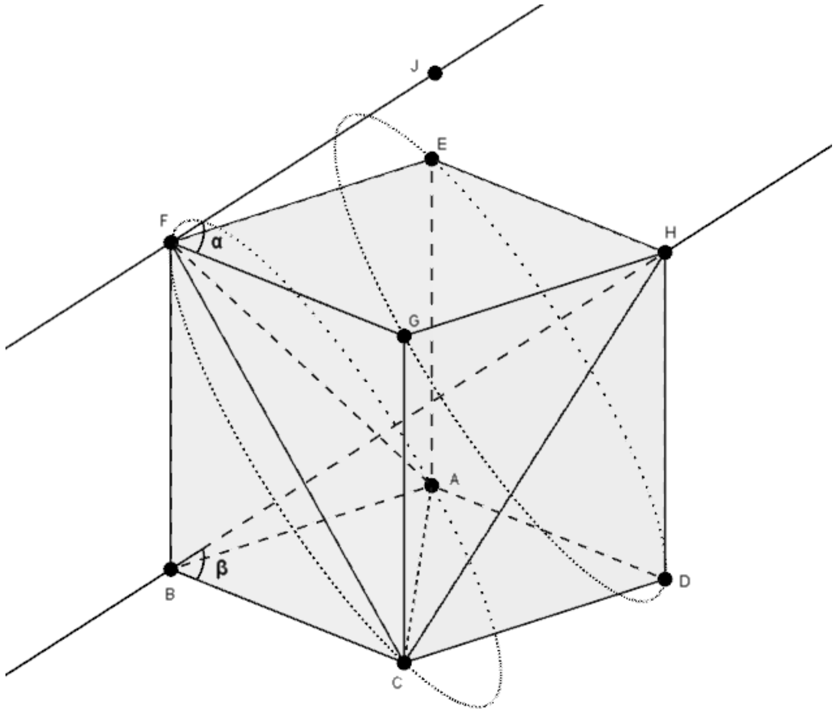

Figure 5: The skew angle of the middle edge FG with respect to axis BH is $\tan^{-1}(\sqrt{2})$ (created with GeoGebra®).

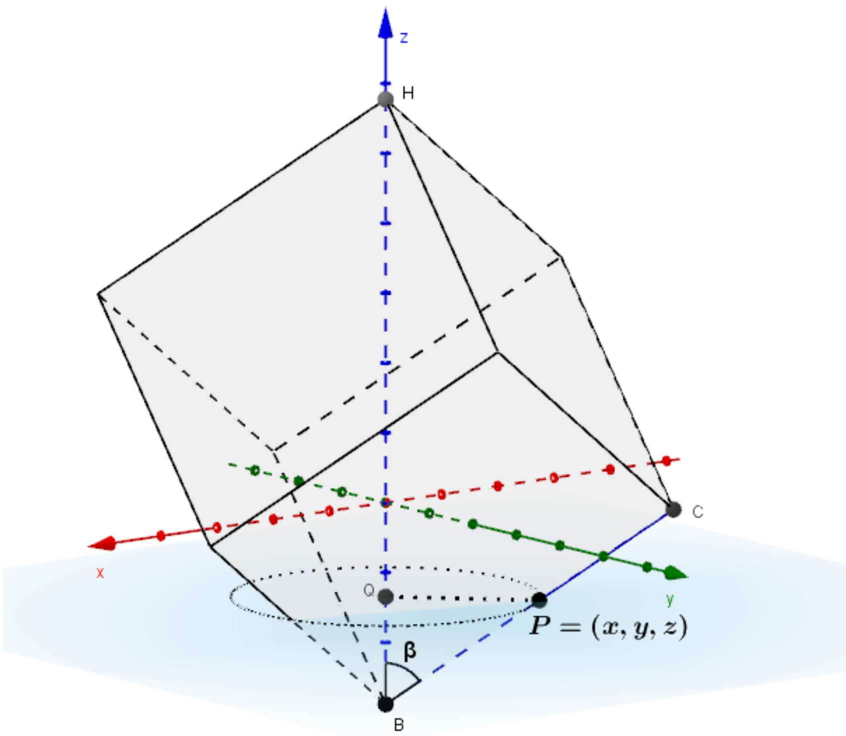

Figure 6: Finding an algebraic equation for the lower cone of a spinning cube (created with GeoGebra®).

equation for the lower cone is:

$$x^2 + y^2 - 2\left(z + \frac{\sqrt{3}}{3}\right)^2 = 0, \tag{4.2}$$

or

$$x^2 + y^2 - 2z^2 - \frac{4\sqrt{3}}{3}z - \frac{2}{3} = 0. \tag{4.3}$$

The equation for the upper cone can be established in a similar way. Alternatively, we can take advantage of the symmetry of the structure, recognizing that the upper cone is a reflection of the lower one about the plane $z = \frac{\sqrt{3}}{6}$ in our setup, which cuts through the waist of the spinning cube. Thus, substituting $(\frac{\sqrt{3}}{3} - z)$ for z in (4.3), we get

$$x^2 + y^2 - 2\left(\frac{\sqrt{3}}{3} - z\right)^2 - \frac{4\sqrt{3}}{3}\left(\frac{\sqrt{3}}{3} - z\right) - \frac{2}{3} = 0, \tag{4.4}$$

which can be simplified to

$$x^2 + y^2 - 2z^2 + \frac{8\sqrt{3}}{3}z - \frac{8}{3} = 0. \tag{4.5}$$

Both the equations above can be visually verified in GeoGebra in its 3D graphics view (see Fig. 8). If the two vertices of the long diagonal are positioned at $(0, 0, 0)$ and $(0, 0, \sqrt{3})$, respectively, we could substitute $(z - \frac{\sqrt{3}}{3})$ for z in (4.3) and (4.5) to obtain their equations as follows:

$$x^2 + y^2 - 2z^2 = 0. \tag{4.6}$$

$$x^2 + y^2 - 2z^2 + 4\sqrt{3}z - 6 = 0. \tag{4.7}$$

5 The Middle Hyperboloid

The problem, spinning the cube, is a special case of the traditional problem of ruled surfaces, which can be constructed by rotating a line in space around another. In the case of a one-sheeted hyperboloid, the axis and the ruling are skew lines, meaning that they do not intersect and are not parallel to each other in space. The surface is visually a hyperboloid; but a proof requires analytic methods (Hilbert & Cohn-Vossen, 1952; Ogilvy, 1966). To show that the surface swept out by the middle edges of a cube is indeed a hyperboloid, we set up a slightly more general case shown in Fig. 7, where segment AB rotates around the z-axis for a one-sheeted hyperboloid. Point A orbits on a circle of radius r in the xy-plane, while point B rotates correspondingly on its circle of radius r that is h units above the xy-plane (h is one third of the long diagonal for a cube). The orbit plane of B intersects with the z-axis at S. We further construct line g perpendicular to the xy-plane at point A, which intersects B's orbit plane at T. We note that $ST = OA = r$. Angle $\angle BAT$ or $\angle\phi$ between line g and line AB is thus the skew angle of AB, which is $\tan^{-1}(\sqrt{2})$ for a cube, as discussed earlier.

The surface under consideration consists of a family of infinitely many segments AB as it rotates around the z-axis. Let's pick an arbitrary point $P = (x, y, z)$ on AB and look into a few relationships in Fig. 7. The orbit plane of P intersects with line g at R and the z-axis at Q.

First, we note that $PQ = \sqrt{x^2 + y^2}$, because we could project point P to P' in the xy-plane and apply the Pythagorean Theorem.

Second, the skew angle $\angle\phi$ is a constant. Within right triangle $\triangle ABT$, we know that $BT = AT \cdot \tan(\phi)$. Since $AT = h$ by our setup, then $BT = h \cdot \tan(\phi)$. Furthermore, within $\triangle SBT$, we know that $SB = ST = r$, because both SB and ST are the radius of the circular orbit of point B. Therefore, $m(\angle BTS) = \cos^{-1}(h \cdot \tan(\phi)/(2r))$, which can be found by considering the midpoint of BT. In the case of the cube, $m(\angle BTS) = \cos^{-1}(\frac{1}{2}) = 60°$.

Third, within $\triangle PQR$, $QR = r$ because of our setup. Also, $\angle PRQ$ or $\angle\alpha$ is congruent to $\angle BTS$, because they are both the angle between plane $APBTRA$ and plane $ARTSQOA$. Therefore, $\angle PRQ$ or $\angle\alpha$ is a constant, with $m(\angle\alpha) = \cos^{-1}(h \cdot \tan(\phi)/(2r))$ and accordingly $\cos(\alpha) = h \cdot \tan(\phi)/(2r)$.

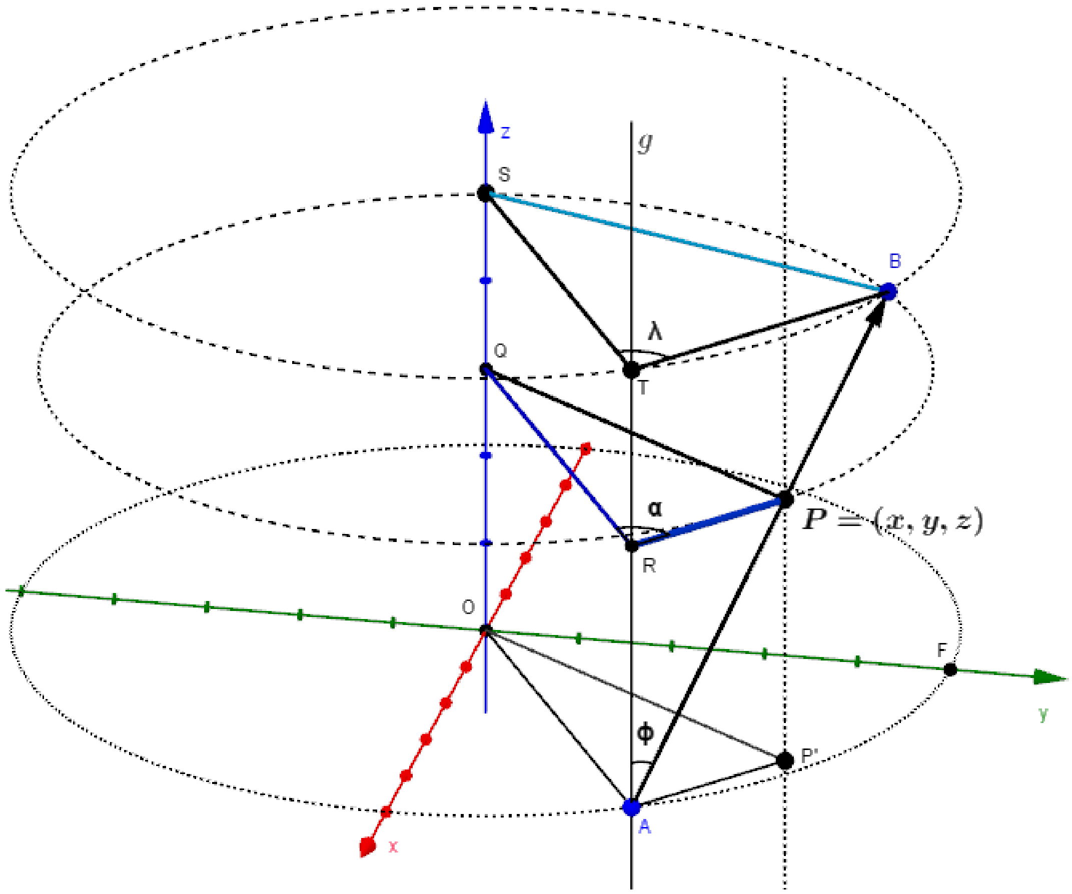

Figure 7: An arbitrary point $P = (x, y, z)$ on the ruling AB is on the surface of a hyperboloid (created with GeoGebra®).

Fourth, within $\triangle APR$, we know that $AR = z$, where z is the z-coordinate of point P and $PR = z \cdot \tan(\phi)$.

Finally, we focus on $\triangle PQR$, about which we know that $PQ = \sqrt{x^2 + y^2}$, $QR = r$, and $PR = z \cdot \tan(\phi)$. By the law of cosines, therefore, we have

$$PQ^2 = QR^2 + PR^2 - 2 \cdot QR \cdot PR \cdot \cos(\alpha), \tag{5.1}$$

which yields

$$x^2 + y^2 - (\tan(\phi))^2 \cdot z^2 + 2 \cdot r \cdot \tan(\phi) \cdot \cos(\alpha) \cdot z - r^2 = 0. \tag{5.2}$$

For simplicity, let $c = (\tan(\phi))^2$ and note that $2 \cdot r \cdot \tan(\phi) \cdot \cos(\alpha) = ch$, where we use the fact that $\cos(\alpha) = h \cdot \tan(\phi)/(2r)$. Then, equation (5.2) becomes

$$x^2 + y^2 - cz^2 + chz - r^2 = 0, \tag{5.3}$$

which matches the general form of a hyperboloid. Specifically, in the case of the unit cube, $c = 2$ because $\phi = \tan^{-1}(\sqrt{2})$, $h = \frac{\sqrt{3}}{3}$, and $r = \frac{\sqrt{6}}{3}$. Therefore, if the unit cube is positioned with a long diagonal on the z-axis and the corresponding vertices at $(0, 0, -\frac{\sqrt{3}}{3})$ and $(0, 0, \frac{2\sqrt{3}}{3})$, respectively, then its algebraic equation is

$$x^2 + y^2 - 2z^2 + \frac{2\sqrt{3}}{3}z - \frac{2}{3} = 0, \tag{5.4}$$

which is verified in Fig. 8 using GeoGebra. Further, if the two vertices of the long diagonal are positioned at $(0, 0, 0)$ and $(0, 0, \sqrt{3})$, respectively, we could substitute $(z - \frac{\sqrt{3}}{3})$ for z in (5.4) to obtain its equation:

$$x^2 + y^2 - 2z^2 + 2\sqrt{3}z - 2 = 0. \tag{5.5}$$

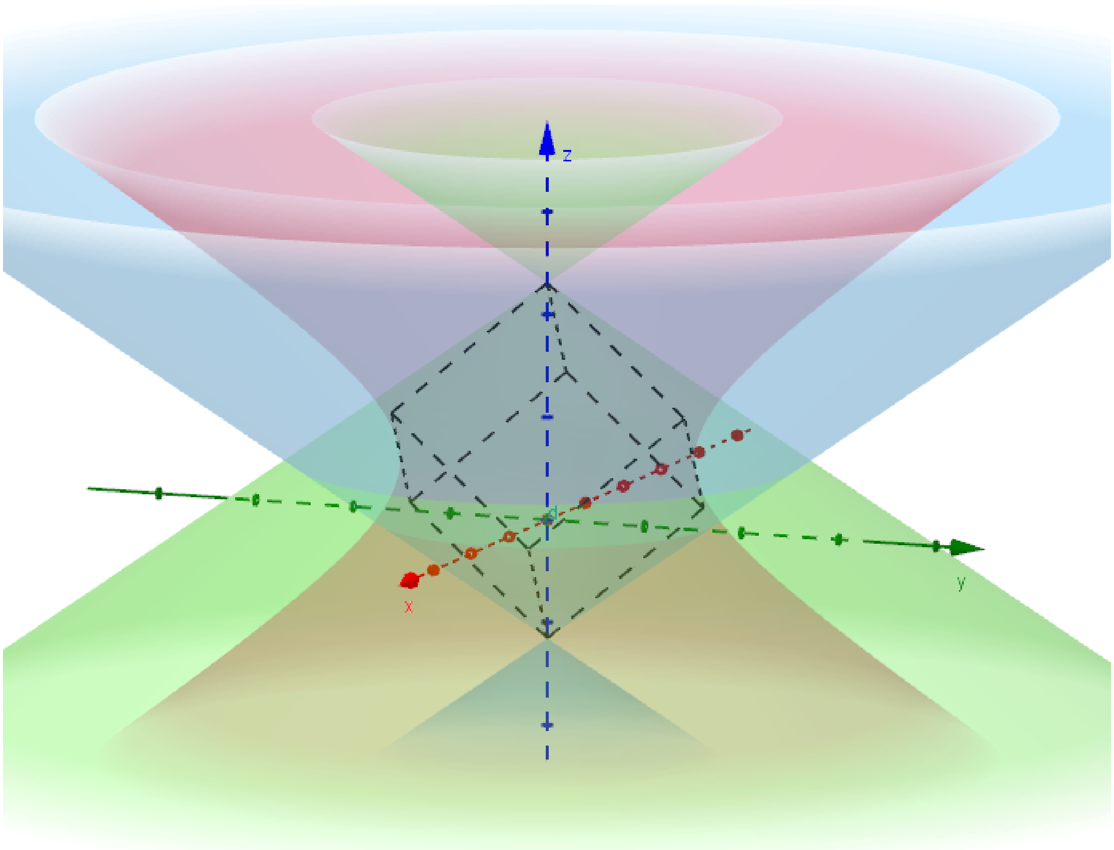

Figure 8: The two cones and the middle hyperboloid of a spinning cube are verified using GeoGebra®.

As a final extension, the middle hyperboloid can be approached using Wilson's (1911) strategy as illustrated in Fig. 4 and equation (2.3). For convenience, the unit cube can be positioned such that its center is at the system origin $(0, 0, 0)$ and a long diagonal coincides with the z-axis. In such a position, the two vertices of the rotational long diagonal will be at $(0, 0, -\frac{\sqrt{3}}{2})$ and $(0, 0, \frac{\sqrt{3}}{2})$. Using Wilson's approach, the slope $m = \tan(\phi) = \sqrt{2}$ and the x-coordinate $a = \frac{\sqrt{2}}{2}$. Therefore, equation (2.3) becomes

$$x^2 + y^2 = \frac{1}{2} + 2z^2, \tag{5.6}$$

which yields equation (5.5) when we move the surface $\frac{\sqrt{3}}{2}$ units up by substituting $(z - \frac{\sqrt{3}}{2})$ for z.

6 Conclusion

The spinning cube provides an appealing problem for rich discussions about the nature of mathematical exploration and sense-making (Tall, 2013) as well as the fascinating world of shapes, symmetries, spatial relations, and their algebraic manifestations (Senechal, 1990). The spinning cube can be readily created in a modern 3D design environment for mathematical exhibits and pedagogical discussions (Bu, 2019). However, the resulting 3D solid is not easy to print on a 3D printer without extensive support. We can take advantage of its symmetries to slice it using a cone's base plane, the waist plane, or a plane containing the axis of rotation (see Fig. 9). While physical manipulations, dynamic modeling, and 3D printing all help us reach out to the world of mathematics, an algebraic analysis integrates a host of secondary and college mathematics and unveils the engaging mathematical structure behind the problem. There are certainly other approaches to generating a hyperboloid and establishing its equations (Wilson, 1911; Ogilvy, 1966). The present discussion is situated in the context of cube spinning, secondary mathematics, and mathematics teacher education. Similar questions can certainly be posed for other solids in the three-dimensional space.

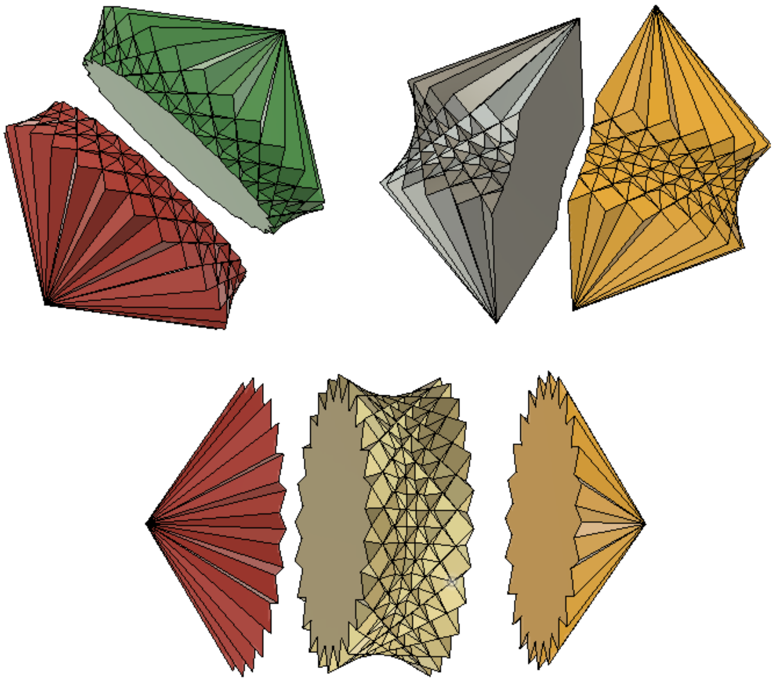

Figure 9: The spinning cube can be sliced in three ways for easy 3D printing using its symmetries (created with Autodesk Fusion 360®).

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References

Bu, L. (2019). Spinning the cube with technologies. *Mathematics Teacher,112*, 551-554.

Hilbert, D. & Cohn-Vossen, S. (1952). *Geometry and the imagination* (P. Nemenyi, Trans.). New York, NY: Chelsea Publishing Company.

Ogilvy, C. S. (1966). Generating a hyperboloid. *Mathematics Magazine*, *39*, 276-277.

Senechal, M. (1990). Shape. In L. A. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 139-181). Washington, DC: National Academy Press.

Sobel, M. A. & Maletsky, E. M. (1999). *Teaching mathematics: A sourcebook of aids, activities, and strategies*. Boston, MA: Allyn and Bacon.

Steinhaus, H. (1969). *Mathematical snapshots* (3rd American Edition). New York, NY: Oxford University Press.

Tall, D. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics*. New York, NY: Cambridge University Press.

Wilson, J. P. (1911). The hyperboloid as a ruled surface. *American Mathematical Monthly*, *18*, 158-159.

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