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A formal justification of the Ancient Chinese Method of Computing Square Roots

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ABSTRACT: In this paper a formal justification of the ancient Chinese method for computing square roots is given. As a result, some already known properties of the square root which is computed with this method are deduced. If any other number base is used, the justification given shows that the method is applied in the same way and that the deduced properties are still being fulfilled, facts that highlight the importance of positional number systems. It also shows how to generalize the method to compute high orders roots. Although with this elementary method you can compute the square root of any real number, with the exact number of decimal places that you want, the mathematicians of ancient China were not able to generalize it for the purpose of computing irrational roots, because they did not know a positional number system. Finally, in order for high school students gain a better understanding of number systems, the examples given in this paper show how they can use the square root calculus with this method to practice elementary operations with positional number systems with different bases, and also to explore some relationships between them.

Keywords: square root, number base, positional number system

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Introduction

The method that for many years was taught in elementary education to extract the square root of a number, was created in ancient China, and was also extended to the case of the extraction of the cube root of a number. The mathematicians of ancient China also found formulas to approximate irrational roots; however, the approximations obtained were far from satisfactory (see [Jos], pp. 219-228; [YS], pp. 93-112). This method was also known by the mathematicians of Islam, who extended it to the case of extraction of square roots of fractions, isolated or joined to units (see [Sai], pp. 76-81). Perhaps the mathematicians of Islam were the ones who made it known in Europe through Spain.

Although for many years the ancient Chinese method for computing square roots was taught in elementary education, only heuristic explanations have been given to justify why it works (see [Wik]). With this method the square root of $m = 3589769.743$ is computed as shown in Figure 1:

$\sqrt{3589769.7430}$	1894.66
-1	28
258	369
-224	3784
3497	37886
-3321	378926
17669	
-15136	
253374	
-227316	
2605830	
-2273556	
332274	

Figure 1: Square root of 3589769.743.

Thus the square root of m, with an error less than 10^{-2}, is 1894.66. In order to increase the precision we simply add pairs of zeros to the right of the decimal part as many times as necessary, and continue to apply the method.

In this paper an algebraic proof is given of why the method works. This algebraic proof is not given in the mainstream literature on this subject. As a consequence, it is proved that if the square root is computed with n decimal places, $n \geq 0$, then the error committed is less than 10^{-n}, from which it is possible to show that all the digits of the computed root match with the corresponding ones of the true root. Thus, given the positive rational number m, written in positional notation, with a finite number of decimal places, with this method what is being done is to determine the greatest rational q with n decimal places such that $q^2 \leq m$.

Additionally, from the given proof it is seen that the method is applied in the same way independently of the number base that is used.

It is also shown how this method and its proof are generalized for the cases of cube roots, fourth roots, etc., although from the cube root the number of arithmetic operations that must be carried out to apply it begins to grow considerably, such that it becomes impractical.

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1 Justification of the method

1.1 Notation

(a) If u and v are real numbers, the multiplication of u and v will be denoted by $uxv, (u)(v)$, $u(v)$ o $(u)v$.

(b) If m is a positive rational such that

$$m = m_1 x 10^{k-1} + ... + m_{k-1} x 10 + m_k + m_{k+1} x 10^{-1} + ... + m_{k+s} x 10^{-s},$$

where $m_1, m_2, ..., m_{k+s}$ are no negative integers such that $1 \leq m_1 \leq 9$ and $0 \leq m_i \leq 9, i = 2, 3, ..., k+s$, m will be denoted by $m = m_1 m_2 ... m_k . m_{k+1} ... m_{k+s}$. For example,

$$9076.023 = 9x10^3 + 0x10^2 + 7x10 + 6 + 0x10^{-1} + 2x10^{-2} + 3x10^{-3}.$$

Here $m_1 = 9, m_2 = 0, m_3 = 7, m_4 = 6, m_5 = 0, m_6 = 2$ and $m_7 = 3$.

(c) For subscripts and exponents, multiplication will be denoted in the usual way.

1.2 Heuristic justification

The following developments provide an intuitive justification of the method (see, for example, Joseph 2011, pp. 219-223). The appearing integers m are such that $0 < m_1 \leq 9$ and $0 \leq m_i \leq 9, i = 2, 3, 4$.

$$\begin{aligned}(10xm_1 + m_2)^2 &= 10^2 xm_1^2 + 2x10xm_1 xm_2 + m_2^2 \\ &= 10^2 xm_1^2 + (2x10xm_1 + m_2)m_2,\end{aligned}$$

$$\begin{aligned}(10^2 xm_1 + 10xm_2 + m_3)^2 &= (10(10xm_1 + m_2) + m_3)^2 \\ &= 10^2 \left(10^2 xm_1^2 + (2x10xm_1 + m_2)m_2\right) \\ &\quad +2x10(10xm_1 + m_2)m_3 + m_3^2 \\ &= 10^4 xm_1^2 + 10^2 \left((2x10xm_1 + m_2)m_2\right) \\ &\quad +2x10(m_1 m_2)m_3 + m_3^2 \\ &= 10^4 xm_1^2 + 10^2 \left((2x10xm_1 + m_2)m_2\right) \\ &\quad + \left(2x10(m_1 m_2) + m_3\right) m_3.\end{aligned}$$

In the same way it is found that

$$\begin{aligned}(10^3 xm_1 + 10^2 xm_2 + 10xm_3 + m_4)^2 &= 10^6 xm_1^2 + 10^4 (2x10xm_1 + m_2)m_2 \\ &\quad +10^2 \left(2x10(m_1 m_2) + m_3\right) m_3 \\ &\quad + \left(2x10(m_1 m_2 m_3) + m_4\right) m_4.\end{aligned}$$

It is seen that the square of any positive integer greater or equal to 10 can be written according to the previous scheme, which gives a justification of the method. Although this justification is for positive integers having exacts square root, when applied to any positive integer m the largest q positive integer is obtained such that $q^2 \leq m$.

1.3 Formal justification

(a) Consider first the case of computing the square root of a positive integer of the form $m_1 m_2 ... m_{2k}$, where $1 \leq m_1 \leq 9$ and $0 \leq m_i \leq 9, i = 2, 3, ..., 2k$. Let q_1 be the largest integer n such that $n^2 \leq m_1 m_2$. Then $0 < q_1 < 10$ because $0 < m_1 m_2 < 100$. Let

$$r_1 = m_1 m_2 - q_1^2$$

and

$$q_2 = max\{n \in \mathbf{Z} \mid (2xq_1 x10 + n)n \leq r_1 m_3 m_4\},$$

where max denotes the maximum, $\mathbf{Z}$ denotes the set of integers and $r_1 m_3 m_4$ is the integer that is formed putting m_3 y m_4, in that order, to the right of the digits of r_1. The following facts are true:

(i) $(q_1x10+q_2)^2 \leq m_1m_2m_3m_4$, because

$$\begin{array}{rcl}(2xq_1x10+q_2)q_2 & \leq & r_1m_3m_4 \\ & = & r_1x10^2+m_3m_4 \\ & = & (m_1m_2-q_1^2)x10^2+m_3m_4 \\ & = & m_1m_2m_3m_4-q_1^2x10^2.\end{array}$$

(ii) $q_2 < 10$, because if $q_2 \geq 10$, then

$$q_1q_2 = q_1x10+q_2 \geq q_1x10+10 = 10(q_1+1),$$

therefore

$$10^2(q_1+1)^2 \leq (q_1x10+q_2)^2 \leq m_1m_2m_3m_4,$$

hence

$$(q_1+1)^2 \leq m_1m_2.m_3m_4,$$

but $(q_1+1)^2$ is an integer and $0 \leq 0.m_3m_4 < 1$, from which it follows that

$$(q_1+1)^2 \leq m_1m_2,$$

which contradicts the choice of q_1.

(iii) $q_1q_2 = 10xq_1+q_2$ it is the largest integer such that $(q_1q_2)^2 \leq m_1m_2m_3m_4$, because if $(10xq_1+q_2+1)^2 \leq m_1m_2m_3m_4$, then

$$(10xq_1+q_2+1)^2 = 10^2xq_1^2+2x10xq_1(q_2+1)+(q_2+1)^2,$$

so

$$\begin{array}{rcl}2x10xq_1(q_2+1)+(q_2+1)^2 & \leq & m_1m_2m_3m_4-q_1^2x10^2 \\ & = & m_1m_2x10^2+m_3m_4-q_1^2x10^2 \\ & = & (m_1m_2-q_1^2)10^2+m_3m_4 \\ & = & r_1x10^2+m_3m_4 \\ & = & r_1m_3m_4,\end{array}$$

that is to say,

$$(2x10xq_1+(q_2+1))\,(q_2+1) \leq r_1m_3m_4,$$

which contradicts the choice of q_2.

Now, let

$$\begin{array}{rcl}r_2 & = & r_1m_3m_4-(2x10xq_1+q_2)q_2 \\ & = & r_1x10^2+m_3m_4-2x10xq_1xq_2-q_2^2 \\ & = & (m_1m_2-q_1^2)10^2+m_3m_4-2x10xq_1q_2-q_1^2 \\ & = & m_1m_2m_3m_4-(10xq_1+q_2)^2 \\ & = & m_1m_2m_3m_4-(q_1q_2)^2\end{array}$$

and

$$q_3 = max\{n \in \mathbf{Z} \mid (2(q_1q_2)10+n)\,n \leq r_2m_5m_6\}.$$

For q_3 are obtained similar facts to those who were in (i), (ii) and (iii) for q_2. Continue in this way until finally get q_k, where the q_i are such that $0 < q_1 < 10$, $0 \leq q_i < 10$, for $i = 2, 3, ...k$, and $q_1q_2...q_i$ is the biggest integer such that

$$(q_1q_2...q_i)^2 \leq m_1m_2...m_{2i-1}m_{2i}.$$

(b) To compute the square root of $m_1m_2...m_{2k}m_{2k+1}$ it proceeds analogously, but starting with m_1, then it continues with m_2 and m_3, and so until finish with m_{2k} and m_{2k+1}.

(c) In order to compute the square root of $m = m_1m_2...m_{2k}.d_1d_2...d_{2s}$, where the integers m_i and d_j are such that $0 < m_1 \leq 9, 0 \leq m_i \leq 9$, for $i = 2, ...2k$ and $0 \leq d_j \leq 9$, for $j = 1, ..., 2s$, let $q = q_1...q_kq_{k+1}...q_{k+s}$ be the square root of $mx10^{2s}$ computed by the ancient Chinese method. Then

$$q^2 \leq mx10^{2s} < (q+1)^2,$$

from where

$$qx10^{-s} \leq \sqrt{m} < qx10^{-s} + 10^{-s},$$

such that

$$0 \leq \sqrt{m} - qx10^{-s} < 10^{-s}.$$

So,

$$qx10^{-s} = q_1q_2...q_k.q_{k+1}...q_{k+s}$$

is the square root of m with an error less than 10^{-s}.

Remark 1.1. As a result of what has been shown, the following facts are true when the square root is computed by the ancient Chinese method.

1. If the square root with n decimal places is computed, the committed error is less than 10^{-n}.
2. Although it is tedious to do so, it can be shown that every one digit computed of the square root match with the corresponding of the exact square root.
3. The method is applied in the same way when other number base is used.

2 Generalization of the method

In this section is shown how to generalize the method only for the case of the cube root (see [Jos], pp. 224-227), from which it is clear how to generalize it to higher-order roots.

Let m_1, m_2 and m_3 be non negative integers such that $1 \leq m_1 \leq 9$ and $0 \leq m_2, m_3 \leq 9$. Then

$$(10xm_1 + m_2)^3 = 10^3xm_1^3 + (3x10^2xm_1^2 + 3x10xm_1xm_2 + m_2^2)m_2,$$

$$\begin{aligned}
(m_1m_2m_3)^3 &= (10^2xm_1 + 10xm_2 + m_3)^3 \\
&= (10(10xm_1 + m_2) + m_3)^3 \\
&= 10^6xm_1^3 + 10^3(3x10^2xm_1^2 + 3x10xm_1xm_2 + m_2^2)m_2 \\
&\quad +3x10^2(10xm_1 + m_2)^2m_3 + 3x10(10xm_1 + m_2)m_3^2 + m_3^3 \\
&= 10^6xm_1^3 + 10^3(3x10^2xm_1^2 + 3x10xm_1xm_2 + m_2^2)m_2 \\
&\quad + \left(3x10^2(m_1m_2)^2 + 3x10(m_1m_2)m_3 + m_3^2\right) m_3.
\end{aligned}$$

It is seen that the cube of any positive integer greater or equal to 10 can be written following the previous scheme, and gives the algorithm to compute the cube root of a positive integer $m = m_1m_2m_3...m_{3k}$:

1. Separate digits of m in threes, from right to left.
2. Let q_1 be the largest integer n such that $n^3 \leq m_1m_2m_3$.
3. If $q_1, q_2, ..., q_s, s < k$, are digits already computed of the cube root, let $q = q_1q_2...q_s$ and $r_s = m_1m_2m_3...m_{3s-2}m_{3s-1}m_{3s} - q^3$. Then the next digit of the cube root, q_{s+1}, is
$$\begin{aligned} q_{s+1} &= max\{n \in \mathbf{Z} \mid (3x10^2xq^2 + 3x10xqxn + n^2)n \\ &\quad \leq r_sm_{3s+1}m_{3s+2}m_{3(s+1)}\}. \end{aligned}$$
4. Continue until $s = k$ or compute the cube root with the wanted number of decimal places.

The formal justification of the method to compute cube roots is similar to that given to compute square roots. Also are obtained the same facts seen in the remark 1.1.

3 Examples

3.1 Example 1

Through the algorithm described above, the cube root of 47698.75987 is computed:

(a) Digits are separated into three from the decimal point, rightward and leftward, so that the separation is as 47,698.759,870. Here $m_1 = 4, m_2 = 7, m_3 = 6, m_4 = 9, m_5 = 8, m_6 = 7, m_7 = 5, m_8 = 9, m_9 = 8, m_{10} = 7$ and $m_{11} = 0$

(b) The largest integer n such that $n^3 \leq 47$ is 3, so that $q_1 = 3$ and $r_1 = 47 - 27 = 20$.

(c) $r_1m_3m_4m_5 = 20698$,

$$6 = max\{n \in \mathbf{Z} \mid (3x10^2x3^2 + 3x10x3xn + n^2)n \leq 20698\},$$

which means $q_2 = 6$, and

$$r_2 = 20698 - (3x3^2x10^2 + 3x3x10x6 + 6^2)6 = 20698 - 3276x6 = 1042.$$

(d) $r_2m_6m_7m_8 = 1042759$,

$$2 = max\{n \in \mathbf{Z} \mid (3x10^2x(36)^2 + 3x10x36xn + n^2)n \leq 1042759\},$$

that is $q_3 = 2$, and

$$\begin{aligned} r_3 &= 1042759 - \left(3x(36)^2x10^2 + 3x36x10x2 + 2^2\right)2 \\ &= 1042759 - 390964x2 \\ &= 260831. \end{aligned}$$

(e) $r_3m_9m_{10}m_{11} = 260831870$,

$$6 = max\{n \in \mathbf{Z} \mid (3x10^2x(362)^2 + 3x10x362xn + n^2)n \leq 260831870\},$$

that is $q_4 = 6$, and

$$\begin{aligned} r_4 &= 260831870 - \left(3x(362)^2x10^2 + 3x362x10x6 + 6^2\right)6 \\ &= 260831870 - 39378396x6 \\ &= 24561494. \end{aligned}$$

Thus, the cube root of 47698.75987 is $(q_1q_2q_3q_4)x10^{-2} = 36.26$ with an error less than 10^{-2}.

The previous procedure is shown in Figure 2.

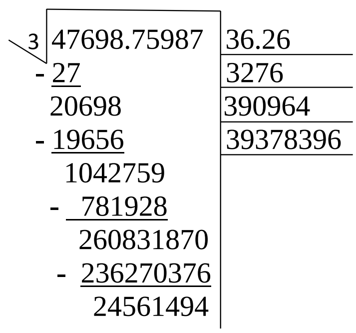

Figure 2: Cube root of 47698.75987.

$\sqrt{2}$	1.414213
-1	24
100	281
-96	2824
400	28282
-281	282841
11900	2828423
-11296	
60400	
-56564	
383600	
-282841	
10075900	
- 8485269	
1590631	

Figure 3: Square root of 2 in decimal system.

3.2 Example 2

In this example the square root of 2 is computed in the decimal, binary and quaternary number systems.

(a) Decimal system: From Figure 3 it is seen that the square root of 2 with six decimal places is 1.414213, with an error less than 10^{-6}.

(b) Binary system: Figure 4 shows the procedure for computing the square root of 2 with thirteen "binary places". With an error less than 10^{-1101}, the square root of 2 in the binary system is 1.0110101000001. So, in decimal system the square root of 2 is

$$1 + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \frac{1}{2^{13}} = 1.41418457,$$

with an error less than $2^{-13} = 0.00012207$.

(c) Quaternary system: From Figure 5 it is seen that with seven "quaternary places" the square root of 2 is 1.1222002, with an error less than 10^{-13}. Then, in decimal system the square root of 2 is

$$1 + \frac{1}{4} + \frac{2}{4^2} + \frac{2}{4^3} + \frac{2}{4^4} + \frac{2}{4^7} = 1.41418457,$$

with an error less than $4^{-7} = 0.0000610352$.

3.3 Example 3

In this example the cube root of 2 is computed, also in the decimal, binary and quaternary systems.

(a) Decimal system: Figure 6 shows that the cube root of 2, with four decimal places, is 1.2599. The error is less than $10^{-4} = 0.0001$.

(b) Binary system: Figure 7 shows that the cube root of 2 with seven "binary places" is 1.0100001, with an error less than 10^{-111}. In decimal system the cube root of 2 is

$$1 + \frac{1}{2^2} + \frac{1}{2^7} = 1.2599,$$

with an error less than $2^{-7} = 0.0078125$.

```
√ 10                  | 1.0110101000001
 - 1                  | 100
   100                | 1001
   - 0                | 10101
   10000              | 101100
   - 1001             | 1011001
    0011100           | 10110100
   - 10101            | 101101001
     00011100         | 1011010100
    - 0               | 10110101000
         1110000      | 101101010000
       - 1011001      | 1011010100000
         001011100    | 10110101000000
       - 0            | 101101010000001
           101110000
         - 101101001
            00000011100
          - 0
                1110000
              - 0
                111000000
              - 0
                11100000000
              - 0
                1110000000000
              - 0
                111000000000000
              - 101101010000001
                001010101111111
```

Figure 4: Square root of 2 in binary system.

```
√ 2             | 1.1222002
 -1             | 21
  100           | 222
  - 21          | 2302
  01300         | 23102
  - 1110        | 231100
    013000      | 2311000
   - 11210      | 23110002
      0113000
    - 112210
       00013000
      - 0
          1300000
        - 0
          130000000
        - 112220010
          011113330
```

Figure 5: Square root of 2 in quaternary system.

$\sqrt[3]{2}$	1.2599
- 1	364
1000	45025
- 728	4721331
0272000	475864311
- 225125	47621196244
046875000	
- 42491979	
04383021000	
- 4282778799	
0100242201000	
- 95242392488	
4999808512	

Figure 6: Cube root of 2 in decimal system.

$\sqrt[3]{10}$	1.0100001
- 1	1100
1000	111101
- 0	100101100
1000000	10010110000
- 111101	1001011000000
0000011000	100101100000000
- 0	10010110111100001
11000000	
- 0	
11000000000	
- 0	
11000000000000	
- 0	
11000000000000000	
- 10010110111100001	
00101001000011111	

Figure 7: Cube root of 2 in binary system.

$\sqrt[3]{2}$	1.10022
- 1	331
1000	102300
- 331	10230000
0003000	1023202010
- 0	103001023310
3000000	
- 0	
3000000000	
- 2113010020	
0220323320000	
- 212002103220	
002321210120	

Figure 8: Cube root of 2 in quaternary system.

(c) Quaternary system: From Figure 8 we have that with five "quaternary places" the cube root of 2 is 1.10022, with an error less than 10^{-11}. Therefore, in decimal system the cube root of 2 is

$$1 + \frac{1}{4} + \frac{2}{4^4} + \frac{2}{4^5} = 1.25976563,$$

with an error less than $4^{-5} = 0.00097656$.

Remark 3.1. From examples 3 (b) and 3 (c), the cube root of 2 with nine "binary places" is 1.010000101 in binary system, with an error of less than 10^{-1001}. Hence, in decimal system the cube root of 2 is

$$1 + \frac{1}{2^2} + \frac{1}{2^7} + \frac{1}{2^9} = 1.25976563.$$

Remark 3.2. From the examples 2 (b), 2 (c), 3 (b) and 3 (c) an interesting relation between the binary and quaternary number systems is observed. That is, we can see how we can pass from the binary system to the quaternary system and, reciprocally, from the quaternary system to the binary system.

4 Conclusions

When the square root is computed by the ancient Chinese method we have the following advantages:

1. Every one digit computed of the square root match with the corresponding of the exact square root.
2. If the square root with n decimal places is computed, then the error committed is less than 10^{-n}.
3. The procedure is applied in the same manner, regardless of the number base used.

However, although the method can be generalized to compute roots of order n greater than two, the procedure is impractical to implement because it is based on the expansion of a binomial power of order n, $(a + b)^n$.

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References

[Jos] Joseph, G. C. (2011). *The Crest of the Peacock: Non-European Roots of Mathematics.* USA: Princeton Univ. Press.

[Sai] Saidan, A. S. (1978). *The Arithmetic of Al-Uqlīdisī.* Dordrecht, Holland: D. Reidel Publishing Company.

[Wik] Wikipedia contributors, Methods of computing square roots, *Wikipedia, The Free Encyclopedia,* http://en.wikipedia.org/wiki/Methods of computing square roots.

[YS] Yong, L. L. & Se, A. T. (2004). *Fleeting Footsteps: Tracing the Concepcion of Arithmetic and Algebra in Ancient China.* Revised edition. Singapore: World Scientific Publishing Co. Pte. Ltd.

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