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Archimedes' Works in Conoids as a Basis for the Development of Mathematics

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Abstract: This paper explores Archimedes' works in conoids, which are three dimensional versions of conic sections, and will discuss ideas that came up in Archimedes' book *On Conoids and Spheroids*. In particular, paraboloids, or three dimensional parabolas, will be the primary focus, and a proof of one of the propositions is provided for a clearer understanding of how Archimedes proved many of his propositions. His main method is called method of exhaustion, with results justified by double contradiction. This paper will compare the ideas and problems brought up in *On Conoids and Spheroids* and how they relate to modern day calculus. This paper will also look into some basic details on the method of exhaustion and how it allowed the ancient Greek mathematicians to prove propositions without any knowledge of calculus. In addition, this paper will discuss some mathematical contributions made by Arabic mathematicians such as Ibn al-Haytham and how his work connects to mathematics in the seventeenth Century regarding sums of powers of whole numbers and the Basel Problem. Complicated forms of conoids such as hyperbolic paraboloids and other shapes that came after Archimedes will not be covered.

Keywords: Conoids; Archimedes; Paraboloids; Method of exhaustion; Double contradiction; Arabic contributions; Ibn al-Haytham; Sums of powers of whole numbers; Seventeenth Century mathematics; Basel Problem

Introduction to Conoids

Archimedes was a famous mathematician in ancient Greece. Born in 287 B.C.E., he lived during the Hellenistic period. The story about Archimedes leaping out of a bathtub and running around the streets naked shouting "Heurēka!" ("I have found it!") is known by many, and that was how he supposedly discovered a method to find the volume of any object by placing it in

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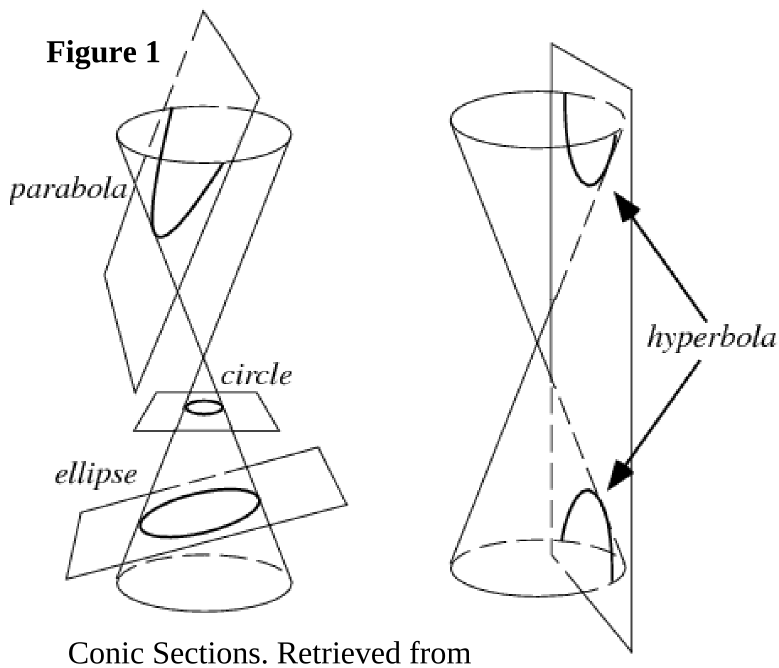
water and measuring the water's displacement. However, Archimedes is also important for his works in conoids and his discovery of the volume of a conoid (Toomer, 2018).

Throughout his life, Archimedes produced many works. One of them is *On Conoids and Spheroids,* in which Archimedes dealt with finding the volume of solids formed by conic sections (Toomer, 2018). According to Weisstein at Wolfram Research (2019), conic sections are defined as the nondegenerate curves generated by intersecting a plane with one or both pieces of a double cone. A picture depicting conic sections is shown in Figure 1:

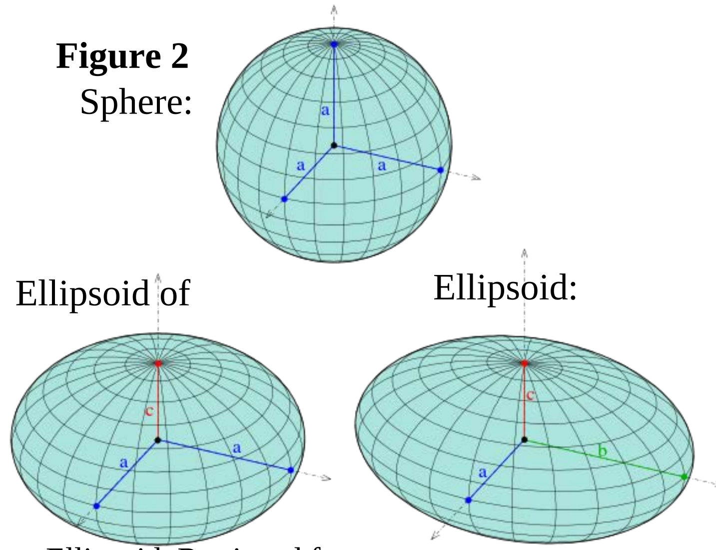
Figure 1



Conic Sections. Retrieved from
http://mathworld.wolfram.com/ConicSection.html

Conoids, however, are conic sections revolved around one of their axes (Toomer, 2018). A circle becomes a sphere, a parabola becomes a paraboloid of revolution, a hyperbola becomes a hyperboloid of revolution, and an ellipse becomes an ellipsoid of revolution, also known as a spheroid. One thing that must be noted is that not all ellipsoids are ellipsoids of revolution. An ellipsoid can have three different values for its axes, but two of an ellipsoid of revolution's axes must be equal; see Figure 2. An ellipsoid of revolution is a special case of an ellipsoid, and a sphere is a special case of an ellipsoid of revolution. In other words, Spheres ⊆ Ellipsoids of revolution ⊆ Ellipsoids.

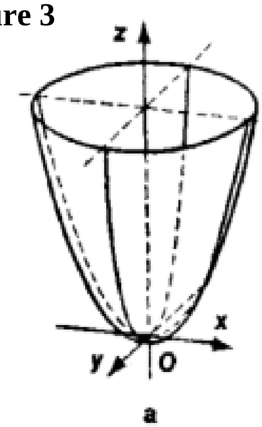
Figure 2



Ellipsoid. Retrieved from: https://commons.wikimedia.org/wiki/File:Ellipsoide.svg; Edited using Google Drawing

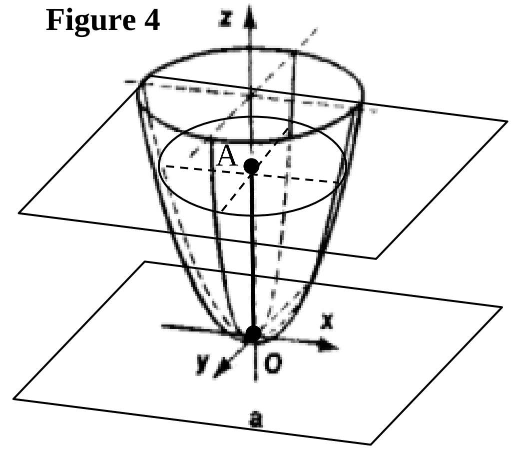
The first solid that Archimedes discusses in *On Conoids and Spheroids* is a right angled conoid. Archimedes describes a right angled conoid, also called a paraboloid of revolution, as follows: Take a right angled cone and revolve it around its diameter until returning to the position where the revolution began. A right angled cone is just a parabola, and its diameter is its axis (Archimedes, ~250 B.C.E., p. 99-100). The result is a paraboloid, as depicted in Figure 3:

Figure 3



Paraboloid. Retrieved from https://encyclopedia2.thefreedictionary.com/paraboloid

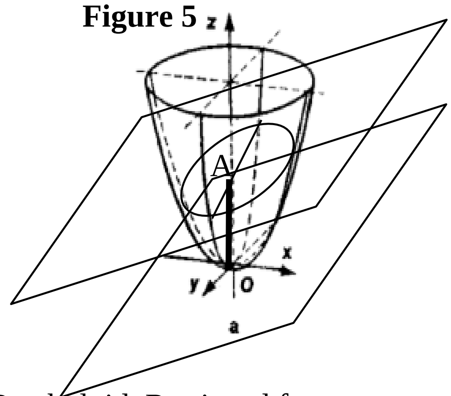
Figure 4



Paraboloid. Retrieved from https://encyclopedia2.thefreedictionary.com/paraboloid; Edited using Google Drawing

In Figure 3, the z-axis is the diameter, and the origin is the vertex. Archimedes (~250 B.C.E.) then poses a situation. Suppose a plane touches the paraboloid, that is, a plane is tangent to the paraboloid. Then draw another plane parallel to the first one and intersecting the paraboloid. If the tangent plane touches the vertex of the paraboloid, then the intersection that the second plane creates would be a circle with center A on the z-axis, as shown in Figure 4. The axis of this segment of the paraboloid is the segment connecting the points O and A. Here, Archimedes mentions two questions for consideration: 1) Why, if a segment of the right-angled conoid be cut off by a plane at right angles to the axis, will the segment so cut off be half as large again as the cylinder which has the same base as the segment and the same axis, and 2) Why, if two segments be cut off from the right-angled conoid by planes drawn in any manner, will the segments so cut off have to one another the duplicate ratio of their axes (p. 100)? He proves both of these later in *On Conoids and Spheroids*, but for the first question, instead of proving the case for planes perpendicular to the axis, he proves it for all cases. Observe: a plane tangent to the paraboloid does not always have to touch the vertex; it can touch any other point on the surface, which would mean that the plane might not be perpendicular to the paraboloid's axis. Constructing the second plane parallel to the first, the segment of the paraboloid that has been cut off would be "tilted," and the intersection formed by the second plane and the original paraboloid would be an ellipse. An example of this case is shown in Figure 5. So now the proposition that Archimedes proves is “any segment of a paraboloid of revolution is half as large again as the cylinder or segment of a cylinder which has the same base and the same axis.”(Archimedes, ~250 B.C.E., p. 131) Archimedes uses a method called “method of exhaustion,” and justifies his results using “double contradiction” or “double reductio ad absurdum.” According to Saito (2013),

Figure 5



Paraboloid. Retrieved from https://encyclopedia2.thefreedictionary.com/paraboloid; Edited using Google Drawing

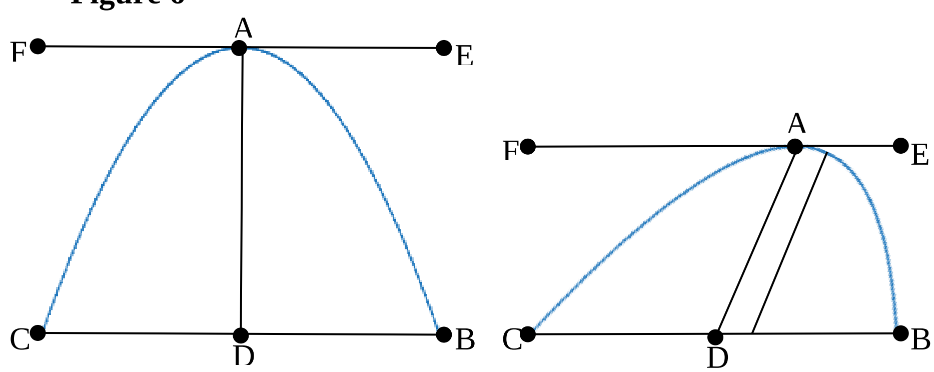
> Schematically, the model can be described in the following terms. Let P be the figure whose magnitude we wish to determine (for example, a sphere), and X be a ‘better known’ figure (for example, a cylinder) to which P is equal (in Archimedes and in Greek geometry the concepts of ‘area’ and ‘volume’ are lacking: the measurement always occurs by direct comparison of two magnitudes). Two series of figures are constructed, I and C, respectively inscribed in and circumscribed about P such that they satisfy two conditions:
>
> 1. $I < X < C$;
>
> 2. The difference $C - I$ can be made infinitely small: given a magnitude E, there can be an inscribed figure I and a circumscribed figure C such that $C - I < E$.
>
> In this case it is easy to prove that P is equal to X. In fact, if P is less than X, let $E = X - P$; by condition 2, it is possible to take C and I such that $C - I < E$. Then we would have $X - I < C - I < E = X - P$, that is, $P < I$, which is impossible because I is inscribed in P. If

P > X, let E = P − X; by condition 2, it is possible to take C and I such that C − I < E. Then X − I < C − I < E = P − X. Since I < X, P − X < P − I, and it follows that X − I < C − I < P − I. So there exists a C that would satisfy P > C > X, which contradicts the fact that C is circumscribed about P. (p. 97)

The following proof is paraphrased from *On Conoids and Spheroids*:

Proof. Set up the cross section of a segment of a paraboloid through its axis. The trace is a parabola, and label the endpoints as *B* and *C*. Let *EF* be a line tangent to the parabola and parallel to the base *BC*, and let *A* be the tangent point. Two possibilities are shown in Figure 6: The base of the segment is perpendicular to its axis, and so the tangent line intersects the parabola at its vertex *A*; and the base of the segment is not perpendicular to its axis. In the first case, the axis *AD* bisects the segment *BC*. In the second case, the axis does not bisect the segment *BC*, but the line through the tangent point *A* and parallel to the axis does. So *BD=DC*.

Figure 6



Quadratic Functions. Retrieved from https://www.shmoop.com/functions/quadratic-functions.html; Edited using Google Drawing

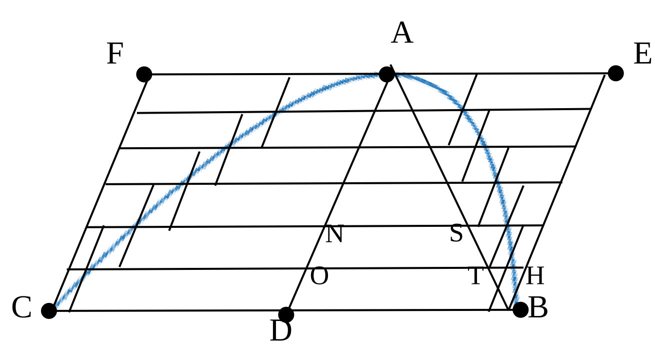
Constructing this in the third dimension, the plane through *EF* and parallel to the base touches the paraboloid at point *A*. In the first case, the base is a circle with diameter *BC*, and in the second case, the base is an ellipse with major axis *BC*. A cylinder whose surface passes through the circle or ellipse, with *AD* as its axis, can be found. Similarly, a cone whose surface passes through the circle or ellipse, with *AD* as its axis and *A* as its vertex, can be found.

Suppose *X* is a cone equal to $\frac{3}{2} \times$ (segment of cone *ABC*). It has already been proven that a cylinder is three times as large as a cone, given that they have the same base and height. So the cone *ABC* is equal to $\frac{1}{3} \times$ (cylinder bounded by *E* and *C*), and it follows that cone *X* is equal to half of cylinder *EC*. Want to show: the volume of the segment of the paraboloid is equal to *X*. If not, then the segment must either be greater than or less than *X*.

I. Suppose the volume of the segment is greater than *X*. We can then inscribe and circumscribe figures made up of cylinders of equal height as shown in Figure 7 such that

(Circumscribed figure) - (Inscribed figure) < (segment) - *X*.

Figure 7



Quadratic Functions. Retrieved from https://www.shmoop.com/functions/quadratic-functions.html; Edited using Google Drawing

Then (First section of the cylinder *EC*) : (First inscribed figure)

= *BD* : *TO*

and (Second section of the cylinder *EC*) : (Second inscribed figure)

= *HO* : *SN*

and the rest of the ratios follow.

So (Cylinder *EC*) : (Inscribed figure) = (*BD* + *HO* + …) : (*TO* + *SN* + …).

It has already been proven that $n(A_n) > 2(A_1 + A_2 + \cdots + A_{n-1})$ if the common difference is equal to A_1. In this case, *BD* = *HO* = ... = A_n and *TO* = A_{n-1}, *SN* = A_{n-2}, etc. Then (Cylinder *EC*) > 2(Inscribed figure), and it follows that

X > (Inscribed figure).

However, since (Circumscribed figure) - (Inscribed figure) < (segment) - *X*, it follows that (Inscribed figure) > [(Circumscribed figure) - (segment)] + *X*. Since the circumscribed figure is greater than the segment, we conclude that

(Inscribed figure) > *X*,

which creates a contradiction.

II. Now suppose that the segment is less than *X*. We can inscribe and circumscribe cylinders the same way as before such that

(Circumscribed figure) - (Inscribed figure) < *X* - (segment)

Then (First section of the cylinder *EC*) : (First circumscribed figure)

= *BD* : *BD*

and (Second section of the cylinder *EC*) : (Second circumscribed figure)

= *HO* : *TO*

and so on.

So (Cylinder *EC*) : (Circumscribed figure) = (*BD* + *HO* + …) : (*BD* + *TO* + …).

It has also been proven that $n(A_n) < 2(A_1 + A_2 + \cdots + A_n)$ if the common difference is equal to A_1. In this case, *BD* = *HO* = ... = A_n and *TO* = A_{n-1}, etc. Then (Cylinder *EC*) < 2(Circumscribed figure), and it follows that

X < (Circumscribed figure).

However, since (Circumscribed figure) - (Inscribed figure) < *X* - (segment), it follows that (Circumscribed figure) < [(Inscribed figure) - (segment)] + *X*. Since the inscribed figure is less than the segment, we conclude that

(Circumscribed figure) < *X*,

which creates a contradiction.

Thus, the segment is neither greater than nor less than *X*, so it must be equal to it, and therefore to $\frac{1}{2} \times$ (cylinder *EC*). (Archimedes, ~250 B.C.E., p. 131-133)

□

A Numerical Example

Let's look at an example: Suppose we have a paraboloid of revolution $z = x^2 + y^2$ and the plane $z = 2x + 3y + 1$.We want to find the volume enclosed by these two curves using the idea given in the proof above. The graphs intersect in an ellipse, and if we project the image onto the *xy*-plane, we get the circle $(x-1)^2 + (y - \frac{3}{2})^2 = \frac{17}{4}$, and the points farthest from and closest to the origin lie on the line $y = \frac{3}{2}x$.This line and the circle intersect at the points (-0.14354375, -0.21531562) and (2.14354375, 3.21531562). Plugging these two points into $z = x^2 + y^2$ or

$z = 2x + 3y + 1$, we see that (-0.14354375, -0.21531562, 0.0669656) and (2.14354375, 3.21531562,14.933034) are the endpoints of the major axes of the ellipse. The major axis *a* is half of the distance between these two points, which is 7.71362. To find the minor axis *b*, we find the intersections between $(x-1)^2 + (y-\frac{3}{2})^2 = \frac{17}{4}$ and $y = \frac{-2}{3}x + \frac{13}{6}$ because this line is perpendicular to $y = \frac{3}{2}x$ and passes through the circle's center $(1, \frac{3}{2})$. They intersect at (-0.71531562, 2.64354375) and (2.71531562, 0.35645625). Then (-0.71531562, 2.64354375, 7.5) and (2.71531562, 0.35645625, 7.5) are the endpoints of the minor axes of the ellipse. Then *b* is half the distance between these two points, which is 2.06155. Now we find the height of the segment of paraboloid, or the distance from the vertex to the plane. Using the property that the line through the vertex parallel to the axis bisects the opposite side, we can project the graphs onto the *xy*-plane; then the vertex of the paraboloid has the same (*x*, *y*) coordinates as the center of the circle, which is $\left(1, \frac{3}{2}\right)$. Plugging $\left(1, \frac{3}{2}\right)$ into $z = x^2 + y^2$, we see that the vertex of the paraboloid is $\left(1, \frac{3}{2}, \frac{13}{4}\right)$. The distance between the point $\left(1, \frac{3}{2}, \frac{13}{4}\right)$ and the plane $-2x - 3y + z - 1 = 0$ is given by $\frac{\left|(-2)(1)+(-3)\left(\frac{3}{2}\right)+(1)\left(\frac{13}{4}\right)-1\right|}{\sqrt{(-2)^2+(-3)^2+(1)^2}}$, which is $\frac{\frac{17}{4}}{\sqrt{14}} = \frac{17\sqrt{14}}{56}$. The volume of the cylinder is base area times height, and the area of the ellipse is given by *πab*. So the volume of the cylinder is *πabh* = 7.713622.06155171456 = 18.0625.The volume of the segment of paraboloid is $\frac{\pi abh}{2}$ = 9.03125.

According to Toomer (2018), this is a problem of integration if put into modern terms. To find the volume of a region bounded by these two curves in the third dimension, we would use multiple integrals: $\iiint 1\, dzdydx$. The variable *z* is bounded by $x^2 + y^2$ and $2x + 3y + 1$; setting $x^2 + y^2 = 2x + 3y + 1$, we get a circle $(x-1)^2 + (y - \frac{3}{2})^2 = \frac{17}{4}$. Solving for *y*, we get that *y* is bounded by $\frac{3}{2} \pm \sqrt{\frac{17}{4} - (x-1)^2}$; and *x* is bounded by $1 \pm \frac{\sqrt{17}}{2}$. So now we have $\int_{1-\frac{\sqrt{17}}{2}}^{1+\frac{\sqrt{17}}{2}} \int_{\frac{3}{2}-\sqrt{\frac{17}{4}-(x-1)^2}}^{\frac{3}{2}+\sqrt{\frac{17}{4}-(x-1)^2}} \int_{x^2+y^2}^{2x+3y+1} dzdydx$.By using the transformations $x = 1 + rcos(\theta)$ and $y = \frac{3}{2} + rsin(\theta)$, we get $\int_0^{2\pi} \int_0^{\frac{\sqrt{17}}{2}} \left(\frac{17}{4} - r^2\right) r\, drd\theta$. This evaluates to $\frac{289\pi}{32} = 9.03125$, which is the exact same answer we got using Archimedes' proposition.

Many teachers today teach integration by drawing rectangles of equal width under a curve and then explaining how the sum of the areas of the rectangles would approximate the area under the curve. This is the same idea that Archimedes used when he inscribed and circumscribed figures in/around the segment of paraboloid. But when Archimedes proved this problem, there was no calculus. Instead, he used a common method that proved many propositions involving areas or volumes - method of exhaustion, which is mostly based on logic. In fact, there wasn't a particular outline on how to prove something using this method. The outline that Saito (2013) provided, which is quoted before Archimedes' proof, was derived from the various proofs undertaken by Archimedes (p. 97).

Archimedes' method for proving volumes of conoids reveals the beauty of ancient Greek mathematics. Archimedes may not have been the first person to come up with the idea of a conoid, but he was one of the first people who made significant advancements in this area.

However, credit must also be given to Antiphon for coming up with the idea of method of exhaustion, and to Eudoxus and Euclid for advancing this method (Sack, 2014). As the concept of calculus did not exist during Archimedes' time, it is quite profound to realise how a logical idea like double contradiction could prove many abstract ideas and propositions. Ultimately, Archimedes' 32 propositions and their proofs in his book *On Conoids and Spheroids* served as a stepping stone towards the development of the conoids that we know today.

Advances a Millennium after Archimedes

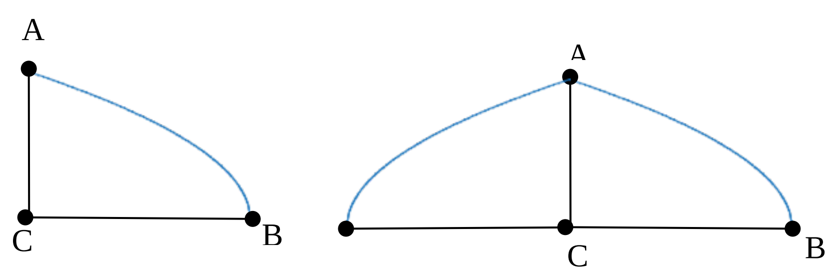
Approximately a thousand years after Archimedes, Hasan Thabit ibn Qurra Marwan al-Harrani was born in Northern Mesopotamia in 836 CE. With his proficiency in Arabic and Greek, and a passion for mathematics, Thabit studied and translated many works from Greek mathematicians. Several of Archimedes' works were translated, and Thabit built on his works on conic sections and the measurement of parabolas and paraboloids. He was able to find the volume of a paraboloid, and some people consider him as the essential link between Archimedes and later European mathematicians such as Cavalieri, Kepler, and Wallis (Joseph, 2011, p.459). In 940 CE, another Arabic mathematician, Abū Sahl al-Qūhī, was born. In his book *Risala ft istikhraj misahat al-mujassam al-mukafi* ("Measuring the Parabolic Body"), Al-Qūhī provided a clearer and simpler solution to finding the volume of a paraboloid than the proof written by Archimedes in *On Conoids and Spheroids* ("Al-Qūh.", 2019).

According to Rashed (2017, p. 143), the works of Thabit ibn Qurra and Al-Qūhī heavily influenced Ibn al-Haytham, an Arabic mathematician born in 965 C.E. His interest in optics led him to prove sums of powers of whole numbers, which then became an important part of his measurement of the volume of a certain kind of paraboloid (Joseph, 2011, p. 494). As Rashed

(2017, p. 143) states, Ibn al-Haytham's work was the last done in Arabic: no further contributions using the exhaustion method are seen after this, nor indeed was any further research undertaken.

Al-Haytham used a method slightly different from Archimedes' to show that the volume of a paraboloid of revolution around a diameter is equal to half the volume of the circumscribed cylinder, but he also proved volumes of different types of paraboloids. One of the conjectures that he shows is "Let *ABC* be a semi-parabola, *BC* its diameter, *AC* its ordinate, *v* the volume of a paraboloid generated by the rotation of *ABC* around *AC*; *V* the volume of the circumscribed cylinder, then $v = \frac{8}{15}V$ " (Rashed, 2017, p. 160). Figure 8 shows an example of what a semi-parabola is and what it looks like after rotating around *AC*.

Figure 8



Quadratic Functions. Retrieved from https://www.shmoop.com/functions/quadratic-functions.html; Edited using Google Drawing

To prove this, Al-Haytham uses a Lemma:

$$\sum_{k=1}^{n-1}(n^2 - k^2)^2 \leq \frac{8}{15}n \times n^4 \leq \sum_{k=0}^{n-1}(n^2 - k^2)^2$$

The following proof is paraphrased from Rashed's (2017) book *Ibn al-Haytham and Analytical Mathematics*:

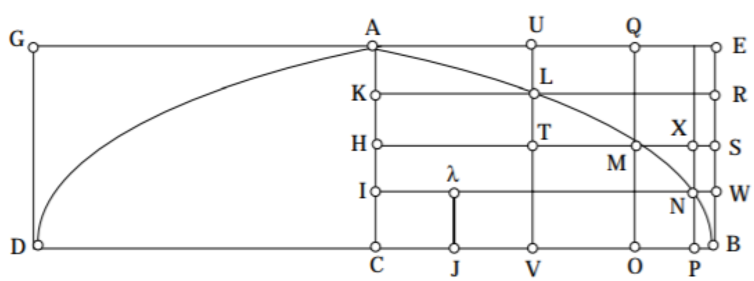
Proof. Let *ABC* be a semi-parabola, *BC* its diameter, *AC* its ordinate, *v* the volume of a paraboloid generated by the rotation of *ABC* around *AC*, and *V* the volume of the circumscribed cylinder. Notice that the angle *ACB* is not always a right angle, but we will only show the case when $\angle ACB = \frac{\pi}{2}$.

Case 1: Assume that $\angle ACB = \frac{\pi}{2}$.

I. First assume that $v > \frac{8}{15}V$; then $v - \frac{8}{15}V = \varepsilon$.

Let *H* be the midpoint of *AC* and construct *HS* ∥ *BC*. *HS* intersects the semi-parabola at *M*, and construct *QO* ∥ *AC* and passing through point *M*. Let *K* be the midpoint of *AH* and *I* be the midpoint of *HC* and construct *KR* ∥ *IW* ∥ *BC*. Let *L* be the intersecting point of the semi-parabola and *KR*, and construct *UV* ∥ *AC* and passing through *L*. Let *N* be the intersecting point of the semi-parabola and *IW*, and construct *XP* ∥ *AC* and passing through *N*, with *X* lying on *HS*.

Figure 9



Ibn al-Haytham and Analytical Mathematics. (Rashed, 2017, p. 160)

Let [*U*] be the volume generated by the rotation of the surface (*U*).

From Figure 9, [*EM*] = [*MB*] and [*AM*] = [*MC*], hence [*EM*] + [*MC*] = [*AM*] + [*MB*] $= \frac{1}{2}V$.

In a similar manner, [*QL*] + [*LH*] = $\frac{1}{2}$ [*AM*] and [*SN*] + [*NO*] = $\frac{1}{2}$ [*MB*], and it follows that [*QL*] + [*LH*] + [*SN*] + [*NO*] = $\frac{1}{2}$ [*AM*]+$\frac{1}{2}$[*MB*] = $\frac{1}{2}\left(\frac{1}{2}V\right) = \frac{1}{4}V$.

By continuing to take subdivisions and successively subtract $\frac{1}{2}V$, $\frac{1}{4}V$, etc, we will inevitably reach a point where the remainder is less than ε.

Assume that we have reached that step in Figure 9, that is,

$$[BN] + [NM] + [ML] + [LA] < \varepsilon.$$

Let V_n = [*BN*] + [*NM*] + [*ML*] + [*LA*], and let v_n be the volume of V_n inside the paraboloid.

With our assumption that $v - \frac{8}{15}V = \varepsilon$, we know that $v - v_n > \frac{8}{15}V$.

Because of the properties of a parabola, $\frac{AC^2}{LV^2} = \frac{BC}{BV}$, $\frac{LV^2}{MO^2} = \frac{BV}{BO}$, $\frac{MO^2}{NP^2} = \frac{BO}{BP}$.

MO = 2*NP*, *LV* = 3*NP*, *AC* = 4*NP*, so if we set *NP* = 1, then *NP* : *MO* : *LV* : *AC* = $1{:}\,2{:}\,3{:}\,4$, which is the first four natural numbers. Then *BP* : *BO* : *BV* : *BC* = $1^2{:}\,2^2{:}\,3^2{:}\,4^2$, which is the square of the first four natural numbers. It follows that *WN* : *SM* : *RL* : *EA* = $1^2{:}\,2^2{:}\,3^2{:}\,4^2$.But

$$WI=SH=RK=EA$$

and

$$\frac{WN}{SM} = \frac{1^2}{2^2}, \ldots, \frac{RL}{EA} = \frac{3^2}{4^2} = \frac{(n-1)^2}{n^2}.$$

By Lemma, which states,

$$\sum_{k=1}^{n-1}(n^2-k^2)^2 \leq \frac{8}{15}n\times n^4 \leq \sum_{k=0}^{n-1}(n^2-k^2)^2,$$

we have

$$NI^2+MH^2+LK^2 \leq \frac{8}{15}(WI^2+SH^2+RK^2+AE^2)$$

and

$$\frac{8}{15}(WI^2+SH^2+RK^2+AE^2) \leq BC^2+NI^2+MH^2+LK^2$$

Areas of discs with respective radii are marked by S_i, so $S_k = \pi(n^2-k^2)$, and $S_0 = \pi n^4$. Therefore,

$$\sum_{k=1}^{n-1} S_k \leq \frac{8}{15}n\times S_0 \leq \sum_{k=0}^{n-1} S_k.$$

For cylinders with base S_k and height $h = \frac{AC}{n}$, we mark as W_k. Then we get

$$\sum_{k=1}^{n-1} W_k \leq \frac{8}{15}V.$$

But by construction,

$$\sum_{k=1}^{n-1} W_k = v - v_n.$$

Therefore,

$$v - v_n < \frac{8}{15}V,$$

which is absurd, so we conclude that

$$v \leq \frac{8}{15}V.$$

II. Now assume that $v < \frac{8}{15}V$; then $\frac{8}{15}V - v = \varepsilon$.

Consider the same subdivision as used at the step where the total of surfaces which surround the parabola is smaller than ε. Let u_n be the volume of V_n, outside the paraboloid; therefore $u_n < \varepsilon$, so $v + u_n < \frac{8}{15}V$. The solid $v + u_n$ is nothing more than a solid whose base is the disc of radius *BC* and whose vertex is the disc of radius *AU*, but we have shown that

$$\frac{8}{15} n \times S_0 \leq \sum_{k=0}^{n-1} S_k\ ;$$

therefore,

$$\frac{8}{15} V \leq \sum_{k=0}^{n-1} W_k,$$

which is absurd, since

$$\sum_{k=0}^{n-1} W_k = v + u_n < \frac{8}{15} V.$$

It follows that

$$v \geq \frac{8}{15} V.$$

So from I and II, we conclude that $v = \frac{8}{15}V$.(Rashed, 2017, p. 160-163)

□

For cases 2 and 3, Al-Haytham assumes that $\angle ACB < \frac{\pi}{2}$ and $\angle ACB > \frac{\pi}{2}$ and, using the same method, shows that $v = \frac{8}{15}V$.

While there are some minor differences, it can be seen that Archimedes' and Al-Haytham's proofs are quite similar - Both used the method of exhaustion, both involved cutting

the segment into discs or cylinders, and both had ideas pertaining to summation. Al-Haytham's proof came around thirteen centuries after Archimedes, and according to Rashed (2017, p. 143), "no further contributions using the exhaustion method are seen after
this, nor indeed was any further research undertaken. This is an area that no historian can fail to investigate, as we now witness a second halt, just as brutal as the first had been, thirteen centuries before."

At the same time, Al-Haytham's works regarding the sums of powers of whole numbers was highly influential on calculus. According to Rashed (2017, p. 144), Al-Haytham derived the formulas for $\sum_{k=1}^{n} k^i$ for $i \in \{1, 2, 3, 4\}$.

Let's examine his proof for $\sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$.

Proof. Let $S_n = \sum_{k=1}^{n} k^1$ and $S_n^{(2)} = \sum_{k=1}^{n} k^2$.

Let $P_k = (k+1)S_k = S_k^{(2)} + S_k + S_{k-1} + \cdots + S_1$.

1. $P_1 = 1(1+1) = 1^2 + 1 = S_1^{(2)} + S_1$;
2. $P_2 = (1+2)(2+1) = 2^2 + 1^2 + (1+2) + 1 = S_2^{(2)} + S_2 + S_1$;
3. $P_3 = (1+2+3)(3+1) = 3^2 + 2^2 + 1^2 + (1+2+3) + (1+2) + 1 = S_3^{(2)} + S_3 + S_2 + S_1$;
4. $P_4 = (1+2+3+4)(4+1) = 4^2 + 3^2 + 2^2 + 1^2 + (1+2+3+4) + (1+2+3) + (1+2) + 1 = S_4^{(2)} + S_4 + S_3 + S_2 + S_1$

Now we assume $P_k = (k+1)S_k = S_k^{(2)} + S_k + S_{k-1} + \cdots + S_1$ is true for some k and we prove that this property holds for $k+1$.

$P_{k+1} = [(k+1)+1]S_{k+1} = (k+1)S_{k+1} + S_{k+1}$

$= (S_k + (k+1))(k+1) + S_{k+1} = P_k + (k+1)^2 + S_{k+1}$

$= S_{k+1}^{(2)} + S_{k+1} + S_k + \cdots + S_1.$

Observe: $S_n + S_{n-1} + \cdots + S_1 = \frac{1}{2}\big(1 \times (1+1) + 2 \times (2+1) + \cdots + n \times (n+1)\big) =$

$\frac{1}{2}(1^2 + 2^2 + \cdots + n^2 + 1 + 2 + \cdots + n) = \frac{1}{2}(S_n^{(2)} + S_n).$

Then, $(n+1)S_n = S_n^{(2)} + \frac{1}{2}S_n^{(2)} + \frac{1}{2}S_n.$

But $(n+1)S_n = \left(n + \frac{1}{2}\right)S_n + \frac{1}{2}S_n$, so we can set $S_n^{(2)} + \frac{1}{2}S_n^{(2)} + \frac{1}{2}S_n = \left(n + \frac{1}{2}\right)S_n +$

$\frac{1}{2}S_n$. So $S_n^{(2)} + \frac{1}{2}S_n^{(2)} = \frac{3}{2}S_n^{(2)} = \left(n + \frac{1}{2}\right)S_n.$

Then $S_n^{(2)} = \frac{2}{3}\left(n + \frac{1}{2}\right)S_n = \frac{2}{3}\left(n + \frac{1}{2}\right)\left(\frac{n(n+1)}{2}\right) = \frac{1}{3}(n+1)n\left(n + \frac{1}{2}\right) = \frac{1}{3}n^3 + \frac{1}{2}n^2 +$

$\frac{1}{6}n$. (Rashed, 2017, p. 144-146)

□

In this proof, Al-Haytham used an archaic form of finite induction, which was still being used in the seventeenth century. He used the same method to prove the formulas for $\sum_{k=1}^{n} k^3$ and $\sum_{k=1}^{n} k^4$, and then he identified a general rule: $(n+1)\sum_{k=1}^{n} k^i = \sum_{k=1}^{n} k^{i+1} + \sum_{p=1}^{n}[\sum_{k=1}^{p} k^i]$ (Rashed, 2017, p. 144-148). According to Sebah and Gourdon (2002, p. 2), many mathematicians in the early years of calculus, such as Pierre de Fermat, Gilles de Roberval, Johann Faulhaber, and Jakob Bernoulli, had taken an interest in sums of powers of whole numbers. For example, on September 22, 1636, Fermat wrote to Roberval that he could “square infinitely many figures composed of curved lines,” and Roberval replied that he could do that too with the inequalities $\frac{n^{k+1}}{k+1} < \sum_{i=1}^{n} i^k < \frac{(n+1)^{k+1}}{k+1}$ (Knoebel, Lodder, Laubenbacher, & Pengelley, 2007, p. 9). As we learn in Calculus II today, summations are strongly related to integrals, and these inequalities may look somewhat familiar as we see that $\int n^k \, dn = \frac{(n+1)^{k+1}}{k+1} + C$ from the power rule.

A Basel Problem

Regarding these sums, Jakob Bernoulli mainly looked into the following:$S_m(n) = 0^m + 1^m + \cdots + (n-1)^m = \sum_{k=0}^{n-1} k^m$. Bernoulli listed out the sequence of formulas:

$$S_0(n) = n$$

$$S_1(n) = \frac{1}{2}n^2 - \frac{1}{2}n$$

$$S_2(n) = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$$

$$S_3(n) = \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2$$

$$S_4(n) = \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

$$S_5(n) = \frac{1}{6}n^6 - \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$

$$S_6(n) = \frac{1}{7}n^7 - \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$$

$$S_7(n) = \frac{1}{8}n^8 - \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2$$

$$S_8(n) = \frac{1}{9}n^9 - \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n$$

$$S_9(n) = \frac{1}{10}n^{10} - \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{12}n^4 - \frac{3}{20}n^2$$

$$S_{10}(n) = \frac{1}{11}n^{11} - \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$$

(Graham, Knuth, Patashnik, & .., 2017, p. 283)

He then empirically noticed that the polynomials $S_m(n)$ have the form $S_m(n) = \frac{1}{m+1}n^{m+1} - \frac{1}{2}n^m + \frac{n}{12}n^{m-1} + 0(n^{m-2}) - + \cdots$ (Sebah & Gourdon, 2002). This form can also be written as $S_m(n) = \frac{1}{m+1}\left(B_0 n^{m+1} + \binom{m+1}{1}B_1 n^m + \cdots + \binom{m+1}{m}B_m n\right) = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k n^{m+1-k}$, where B_k is the k^{th} Bernoulli number defined by a recurring relation $\sum_{j=0}^{m}\binom{m+1}{j}B_j = 0\ \forall m \geq 0$. (Graham, Knuth, Patashnik, & .., 2017, p.283-284). The first few Bernoulli numbers are:

k	0	1	2	3	4	5	6	7	8	9	10	11	12
B_k	1	$\frac{-1}{2}$	$\frac{1}{6}$	0	$\frac{-1}{30}$	0	$\frac{1}{42}$	0	$\frac{-1}{30}$	0	$\frac{5}{66}$	0	$\frac{-691}{2730}$

From the surface, Bernoulli numbers may seem quite odd and somewhat unimportant. However, Euler was able to utilize Bernoulli numbers to find a summation formula for a continuous function: $\sum_{i=1}^{n} f(i) \approx C + \square\ ^{n}f(x)\,dx + \frac{f(n)}{2} + B_2\frac{f'(n)}{2!} + B_3\frac{f''(n)}{3!} + B_4\frac{f'''(n)}{4!} + \cdots$ (Knoebel, Lodder, Laubenbacher, & Pengelley, 2007, p. 14). One of the first problems that Euler tackles using this formula is finding the value of $\sum_{k=1}^{\infty}\frac{1}{k^2}$, also known as the Basel Problem. He used this formula to calculate this sum correctly to twenty decimals $\left(\approx \frac{\pi^2}{6}\right)$ (Knoebel, Lodder, Laubenbacher, & Pengelley, 2007, p. 14).

The above account demonstrates the significance of Greek contributions to mathematics. From Archimedes' propositions on finding the volume of a segment of paraboloid, Al-Haytham devised his own proof and built on the idea of summing powers of whole numbers. This later became an area of interest for many mathematicians in the 17th Century, which led to the development of Bernoulli numbers and a solution to the Basel problem. Overall, Greek contributions became the basis for calculus, and also indirectly led to developments in number theory through the works of Ibn Al-Haytham and Jakob Bernoulli. It is fair to say that mathematics would not be the way it is today without the Greeks.

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