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Abstract: Drawing on the Semiotic-cultural approach and the Anthropological theory of the didactic, this paper discusses how exploration of historically framed conceptualizations of mathematical objects can establish bridges between different mathematical areas such as calculus and Euclidean geometry. A classroom intervention study in two secondary mathematics classrooms involving dynamic geometry software tools to support the construction of a parabola and its tangent and aiming at the development of representational flexibility between algebraic/ functional and geometrical approaches, illustrates how students may benefit from participation in such explorations.

Keywords: dynamic geometry software; mathematics curriculum; representations; tasks with multiple solutions; parabola

1. Introduction

Mathematical objects have been conceptualized in different theoretical framings emerging over the course of the historical development of mathematics. Consequently, different approaches towards the organization of the mathematical knowledge for its teaching in school have been endorsed by different curricula. For the contemporary learner this situation naturally leads to the possibility of solving problems involving such objects in multiple ways. A comparison of these different types of solutions, in its turn, may lead to making mathematical connections between various branches of mathematics (such as algebra, geometry, theory of functions), and thus, to further meaning constructions of established results and perhaps to generating new mathematical ideas.

One object, having a long history in mathematics and commonly included in modern secondary mathematics curricula, is the parabola. As Bartolini Bussi (2005) pointed out, while the appearance of the parabola itself did not change, "the way of generating conics, the way of looking at them, and the way of studying them" (p. 40) did change over the course of history. Bergsten (2015) discussed several historical mathematical framings for the study of the parabola, including viewing the parabola as (1) a curve on a plane embedded in 3D, that is, as a section of a right angled cone in the work of Archimedes; (2) a locus of points in the plane defined by

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Apollonius, still "tied to the cone" (Fried, 2007, p. 216) drawing on the Pythagorean theory of the application of areas; (3) a locus of points in the plane equidistant from a given point and line (the focus and the directrix of the parabola, respectively) studied by Pappus; and (4) a graph, drawn in a co-ordinate system, of the quadratic function defined algebraically, which became possible due to the work of Descartes and Fermat. Bergsten (2015) also suggested that both, the algebraic representation by Descartes and the dynamic representation afforded by the development of modern computer technology including dynamic geometry software (DGS), involved *paradigmatic* historical "metamorphoses" of how to represent a parabola. Indeed, Fried (2007, pp. 216-217) writes that "Descartes relocates mathematical thought from the system of ideas of Apollonius' world to a new system of ideas in which objects like the parabola are redefined and reinterpreted in terms of abstract relations (i.e. relations lifted up and away from special subject matter)." And by computer technology the icon of a parabola has become *dynamic* by use of the drag mode function in a DGS.

The emergence of computer technology, in particular DGS, gives a learner an opportunity to review different historical approaches of looking at constructions of the parabola in an interactive and interconnecting way. Each construction in DGS draws on a defining property of the parabola, which "does not only give the shape of the icon (the curve), but also provides a basis for further explorations and analyses of the object as a support for attaching meaning to it" (Bergsten, 2015, p. 38). Bergsten also noted that the parabola is a mathematical object for which different "problems and techniques […] are ubiquitous in school mathematics" (p. 20).

While previous research has identified a wide range of didactical and learning advantages afforded by the use of DGS in mathematics teaching, including proof and justification (e.g., Laborde et al., 2006; Chan & Leung, 2014; see also DeVilliers, 2012), the aim of this paper is to illustrate, by way of an example from a secondary mathematics classroom activity, how an exploration of historically framed constructions of the parabola involving DGS can establish a bridge between different mathematical representations and branches, thus affording *explicit attention* to intra-mathematical connections seen as viable for the development of mathematical knowledge in current school curricula. The parabola has been chosen as an example for this study not only because it illustrates the multiplicity of historical treatments and their possible realizations in DGS, but also because these constructive approaches gradually have disappeared from modern mathematics curricula (e.g., Bergsten, 2015; Kondratieva, 2015). Indeed, Arzarello et al. (2012) claimed that "constructions have recently lost their centrality and almost disappeared from Geometry curriculum", and they no longer belong to "the set of problems commonly proposed in the textbooks" (p. 101). We would argue that this exclusion is detrimental to the process of meaning making in mathematics learning, and, as a consequence, to creativity and flexibility in problem solving activities (cf. Thom, 1973).

2. Tasks with multiple solutions and mathematical instruction

Many modern curricula at all levels are moving from an acquisitionist approach to a more participationist and inquiry-based study of mathematics (cf. Sfard, 1998), focusing on developing understanding (or meaning construction) and connecting various concepts and methods. House and Coxford (1995) argued that presenting mathematics as a "woven fabric rather than a patchwork of discrete topics" is one of the most important outcomes of mathematics education research, which supports a learner in going beyond instrumental understanding, secured by knowing mathematical procedures, and achieving relational understanding of and between different mathematical concepts and methods (Skemp, 1987). In this context Hodgson (1995) argued that the "ability to establish and use a wide range of connections offers students alternative paths to the solution" (p. 19). While making connections and multiple representations of ideas are recognized among the primary processes in learning mathematics (NCTM, 2000), there is also a need for teaching strategies "for engaging students in exploring the connectedness of mathematics" (House & Coxford, 1995, p. vii).

Many examples of addressing this need by using classroom tasks which allow multiple approaches can be found in the literature. Leikin and her collaborators extensively studied "tasks that contain an explicit requirement for solving the problem in multiple ways" (Leikin & Levav-Waynberg, 2008, p. 234), particularly in the context of the development of mathematics teachers' knowledge and for an examination of mathematical creativity (Leikin & Lev, 2007). In a university mathematics context, Mingus (2002) referred to "calculation of n^{th} roots of unity" as a problem that "encourages students to see connections between geometry, vectors, group theory, algebra and long division" (p. 32) and suggested that "proving identities involving the Fibonacci numbers provide a solid connection between linear algebra, discrete mathematics, number theory and abstract algebra" (Mingus, 2002, p. 32). Winsløw (2013, p. 2481) described five ways to approach constructions of a^x for $a > 0$ and *x* real. Sun and Chan (2009) discussed nine proofs of the "Mid-Point theorem of triangles". The fact that "the sum of the interior angles in a plane triangle is 180^0" can be shown in (at least) eight different ways (Tall et al., 2012, p. 35). These and many other examples suggest that knowledge of multiple ways to treat a mathematical object may enrich students' understanding, improve their problem solving ability and strengthen the relation of a learner to the object "in the sense of providing extensions or alternatives to standard presentations" (Winsløw, 2013, p. 2483).

Kondratieva (2011) emphasized studying problems with multiple solutions, and specifically, interconnecting problems. In the latter approach students encounter an interconnecting problem several times as they progress throughout their education, each time learning a new aspect of the same problem and building their understanding on "supportive met-befores" (Tall, 2013, p. 15). This process relies on a combination of historical and didactical ways of looking at related mathematical objects. Bergsten (2015) used the term *bridging tasks* for tasks that link historically based "representations of a mathematical object" and "would have the potential to carry meanings across representations and mathematical organizations" (p. 39). Kondratieva (2013) also suggests that problems allowing a range of solutions not only can help learners to move from elementary to advanced understanding but may also be used to exemplify advanced methods in basic terms or to come up with an alternative and more elementary explanation of results found in a different way.

3. Representational flexibility

The issues of using different approaches to a mathematical object and tasks rich in affording multiple solutions strongly link to the role of representations in mathematical learning and problem solving. In their review of research on "representational flexibility" in these contexts, Acevedo Nistal et al. (2009) report that while many studies find that using and switching between multiple representations "facilitates mathematical problem solving and learning", some studies indicate that such use also could have a hindering "opposite effect" (p. 627). Thus, on the one hand, by developing representational flexibility of mathematical objects students become better prepared for solving non-routine problems by reconstructing their knowledge to "fit the demands of the task" in new and complex situations (Spiro & Jehng, 1990; cited in Acevedo Nistal et al., 2009, pp. 627-628). By using not only one but multiple representations students can gain a more broad, deep, robust and flexible understanding (for references, see Acevedo Nistal et al., 2009, p. 628). On the other hand, it is argued, in order to benefit from the use of a representation, students need to be familiar with how it is normally used, its relations to reality and to other representations of the same mathematical concept. While having such familiarity is necessary for being able "to select the most appropriate representation for each situation", not having it may impact negatively on mathematical performance (Acevedo Nistal et al., 2009, p. 628). With the notion of representational flexibility, the authors refer to "students' disposition to make appropriate representational choices taking into account the task, student and context characteristics that come into play in the resolution of the mathematical task at hand" (p. 629). Here a wide spectrum of *student* characteristics is considered, including their conceptual and procedural as well as abstract conditional knowledge about representations, domain-specific knowledge, representational preference and affective factors. The *context* characteristics include curriculum factors, which could be observed also in the present study. The *task* in our study was selected with reference to its historical development and curricular variance involving multiple approaches and representations regarding the parabola.

4. Organizing Mathematical Knowledge

In our discussion of meaning construction, we will draw on the *Semiotic-cultural approach* (Radford, 2006), where the meaning of the objects of study constructed by a learner is both a subjective and cultural construct: it depends on the learner's background as well as on "semiotic means of objectification – e.g. objects, artifacts, linguistic devices and signs that are intentionally used by the individuals in social processes of meaning production" (Radford, 2002, p. 14). Consequently, meaning construction of the mathematical object *parabola* draws on the historical mathematical framing used. Different framings may employ different semiotic registers, such as figural or algebraic. Duval (2006) proposed that the use of different representations and registers might lead to adjustments of meaning. Certain flexibility of thinking is needed "to accommodate different views at the same time, so that various semiotic means of objectification can be integrated in the act of knowing" (Radford, 2002, p. 21).

As a language of description (e.g., Dowling, 1994) for discussing the issue of multiple solutions to mathematical problems, we will use the notion of *praxeology* (or *mathematical organization*) from the *Anthropological theory of the didactic* (ATD; e.g., Bosch & Gascon, 2006). With a focus on mathematical activity within an institution, a (mathematical) praxeology is characterized on the one hand by the types of tasks and techniques endorsed for solving these tasks (its *praxis*), and on the other hand how these are justified by the technologies employed and, at a deeper level, embedded in a theory (its *logos*) (e.g., Bosch & Gascon, 2006). As different solutions to one type of problem (task) involves the use of different techniques that may draw on different technologies and/or theories within mathematics for its justification, an analysis of the praxeologies coming into play can clarify the character of the arguments made and the relations established between mathematical ideas that are being evoked. To illustrate the use of this language of description, we will discuss three different constructions of the parabola (and its tangent) and their inter-relationships (connections). This will be done in some detail to highlight the types of mathematical arguments that could be required from students when they encounter bridging tasks in geometry involving both treatment and conversion of semiotic registers (in the sense of Duval, 2006).

Construction 1: A functional approach

A parabola can be constructed by plotting a graph of a function defined in algebraic terms. Let the parabola be given by the equation $f(x) = ax^2$, $a \neq 0$. Its tangent can be found using the derivative of the function, that is $f'(x) = 2ax$, the equation of the tangent line at $x = x_0$ being $y - ax_0^2 = 2ax_0(x - x_0)$. The plotting of the parabola and a tangent line can be supported by a software tool for graphing functions given in the form $y = f(x)$ for some non-zero values of a and x_0. Figure 1 illustrates this approach for $a = 1$ and $x_0 = 1$.

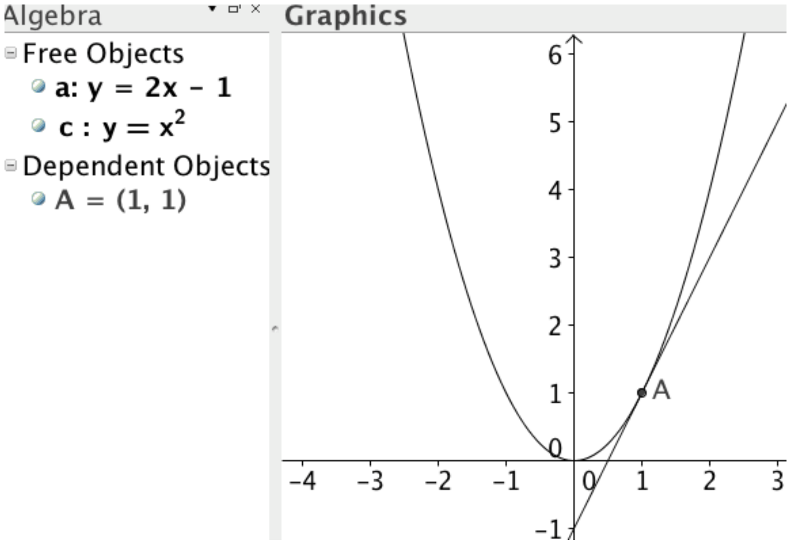

Figure 1. Plotting (in *Geogebra*) of the parabola $y = x^2$ and its tangent line $y = 2x - 1$ at $x_0 = 1$.

This construction can be viewed as embedded in a *reference mathematical organization* (or praxeology) (Barbé et al., 2005, p. 241) for the study of real functions of one real variable, where the graph of the function f is the set of points (x, y) in the Cartesian coordinate system with $y = f(x)$. For a differentiable function, its tangent line through the point $(x_0, f(x_0))$ has the equation $y - f(x_0) = f'(x_0)(x - x_0)$. This technology provides a technique for constructing the tangent (when $f'(x_0)$ can be computed), which can be justified within the theory of differentiable functions (a part of calculus). Points on the graph can be determined by the use of a table of function values, or with an instrumented technique via a computer software. Within this praxeology the parabola is defined as the graph of the quadratic function (i.e., $f(x) = ax^2 + bx + c$, where $a \neq 0, b$ and c are constants).

Construction 2: A geometrical approach involving distances

A parabola can also be geometrically defined as the locus of points equidistant from a given point F (*focus*) and a given line *l* (*directrix*) not passing through F. This definition enables the following (standard) construction (see Figure 2): Drop the perpendicular from F to A on *l*; pick any point B on *l*; join F and B and draw the perpendicular bisector *l'* to *FB* at M; draw *l''* perpendicular to *l* at B; let C be the intersection point of *l'* and *l''*. As B moves along *l*, the point C traces a parabola. The line *l'* is tangent to the parabola with C being the point of tangency. The midpoint V of the segment *FA* is the *vertex* of the parabola.

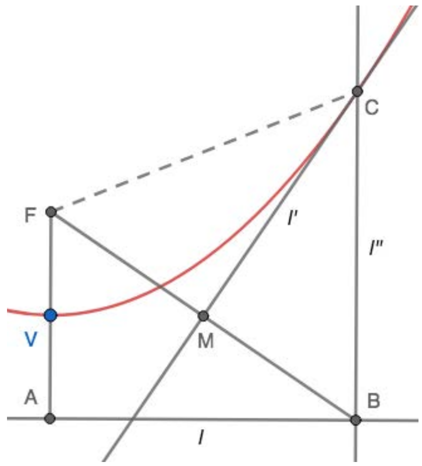

Figure 2: Construction of a parabola given its focus F and directrix *l*.

A reference praxeology for this construction is provided by Euclidean geometry, where the technique used in Construction 2 can be justified by theorems on congruent triangles. That *l'* is tangent to the parabola at C can be justified within this praxeology by, for example, a proof by contradiction (see a student's proof below).

Construction 3: A geometrical approach involving areas

The third construction draws on Bergsten (2015), with reference to the definition of the parabola by Apollonius (see Apollonius of Perga, 1952). Given the segment *VL* (see Figure 3), pick an arbitrary point N on *VL* (or its extension) and draw lines perpendicular to *VL* at V, N, and L. Points H, E and K are placed on the three lines such that *VH* = *NE* = *LK* = *VN*. Let *VK* and *NE* meet at C. When N moves along *VL* (or its extension beyond L), C traces a parabola (see Bergsten, 2015). The line connecting C and the midpoint M of the segment *VN*, is tangent to the parabola at C.

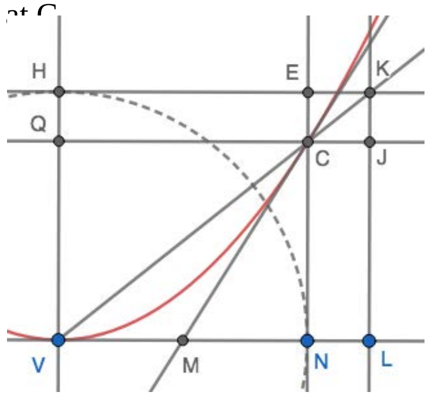

Figure 3. Construction of a parabola by the application of areas.

Embedded in the reference praxeology of Euclidean geometry, the defining property of the parabola so constructed can be set to the requirement of the equality of areas of the square VNEH and the rectangle VLJQ in Figure 3, which follows from the (obvious) equality of the areas of the rectangles QCEH and NLJC (Euclid I.43). The construction technique is an example of the Pythagorean method of application of areas. In Bergsten (2015) also the more general case with VLKH being any parallelogram is discussed. The tangency at C of the line through M and C (Figure 3) can be established (within this praxeology) by drawing the perpendicular to *MC* at M and relating to the construction of the tangent line in Figure 2 (see the next section). Alternatively, a proof involving the use of algebra can be employed (see Appendix).

Bridging different constructions of the parabola

The constructions above draw on different defining properties of a parabola. The question then arises whether the curves so constructed are identical. To answer this question it is necessary to establish connections between the different constructions. A first observation is that the definition of the parabola as a curve in all three cases employs the use of a parameter. In Construction 1 it is the coefficient a, in Construction 2 the distance between the focus and the directrix (in the following denoted by $2p$), and in Construction 3 the length of the segment *VL* (in the following denoted by q). A change in the value of the parameter entails a change in the shape of the parabola as more or less “open” (as can be experimentally observed when employing a DGS for the constructions). Between these parameters the relations $q = 4p = 1/a$ hold (see below).

Connecting Construction 1 and Construction 2

First, to link Construction 1 with Construction 2, the graph of the function $y = ax^2$ (with the constant $a > 0$) is shown in Figure 4 (Cartesian coordinates). Let $p = 1/4a$ and F and A have coordinates $(0, p)$ and $(0, -p)$, respectively. Let C with coordinates (x, y) be an arbitrary point on the graph. By Pythagoras’ Theorem the difference between the (squared) lengths of segments *FC* and *BC* is $(ax^2 - p)^2 + x^2 - (ax^2 + p)^2 = x^2 - 4apx^2 = 0$ for all x. This means that the equidistance property used in Construction 2 is fulfilled. In addition, using differentiation, the parabola tangent at the point (x_0, y_0) with $x_0 \neq 0$, has the equation $y - y_0 = 2ax_0(x - x_0)$, which (for $x_0 \neq 0$) meets the x-axis when $-ax_0{}^2 = 2ax_0(x - x_0)$, that is when $x = x_0/2$. The tangent is therefore the midpoint perpendicular to the segment *FB*.

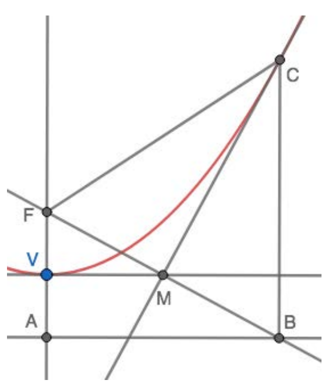

Figure 4. Connecting Construction 1 and Construction 2.

Second, after producing a parabola using Construction 2, introduce a Cartesian coordinate system on the curve so that (with notations in Figure 5) the coordinates of V, F, A and C are (0,0), (0,p), (0, –*p*), and (*x*,*y*), respectively. By the equalities of the distances *FC* and *BC* and Pythagoras Theorem, we have $x^2 + (y-p)^2 = (y+p)^2$, which simplifies to $x^2 = 4py$, that is $y = ax^2$ (with $a = 1/4p$). As M has coordinates $\frac{1}{2}\left((0,p) + (x,-p)\right) = (x/2,0)$, the slope of the tangent *MC* then is $\frac{y-0}{x-x/2} = \frac{2y}{x} = 2ax$, that is the derivative of the function $y = ax^2$.

Connecting Construction 1 and Construction 3

First, after producing a parabola $y = ax^2$, using Construction 1, referring to the notations in Figure 3 the points V and C have coordinates (0,0) and (*x*,*y*), respectively. Choose L so that it has coordinates $(1/a\,, 0)$. Let the *y*-coordinate of point K (that is the intersection of the line through V and C and the perpendicular to at L) be *z*. By similarity of the triangles VLK and VNC we have $\frac{z}{y} = \frac{1/a}{x}$. As $y = ax^2$, this shows that $z = x$ and that $x^2 = \frac{1}{a} \cdot y$, that is the square VNEH is being applied on the rectangle VLJQ. That the tangent at C defined by the derivative passes the point M (Figure 3) follows from the definition of the tangent as the line through C (with given coordinates (x_0, ax_0^2)) with slope $y' = 2ax_0$, which has the equation $y = 2ax_0x - ax_0^2$. Setting $y = 0$, it is clear that this line meets the *x*-axis at $x = x_0/2$, that is at M.

Second, in order to establish a connection between Construction 3 and Construction 1, we introduce Cartesian coordinates with the origin at V, associating *VL* with the *x*-axis and *VH* with the *y*-axis (Figure 3). Then assuming that point C has coordinates (*x*,*y*) and VL = *q*, from the equality of the areas of VNEH and VLJQ we immediately obtain $x^2 = qy$, thus $a = 1/q$. As M is the midpoint of *VN* it has coordinates $(x/2,0)$. The slope of the tangent *MC* is then $\frac{y-0}{x-x/2} = \frac{2y}{x} = 2ax$, that is the derivative of the function $y = ax^2$.

Connecting Construction 2 and Construction 3

The construction of a point C on a parabola by Construction 2 is displayed in Figure 5, where F is a given point (the focus), and the line through A (perpendicular to *AF*) is the given directrix (cf. Figure 2). The parabola has been indicated in the figure.

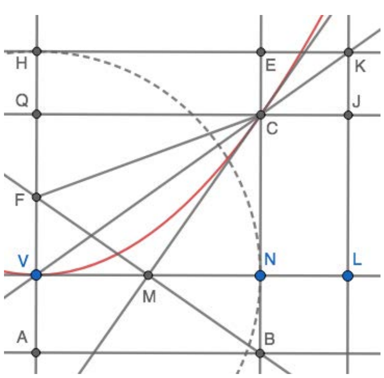

Figure 5. Connecting Construction 2 and Construction 3

First, the line through M perpendicular to *AF* meets *AF* at V and *BC* at N. As the triangles MVF and MBN are congruent, so are *VF* and *NB* as well as *MV* and *MN*. The line through V and C meets the perpendicular to (the extension of) *AF* at H (where *VH* equals *VN*, congruent to *AB*) at the point K. *KL* is perpendicular to *VN*. The square on *VN* then equals the rectangle on *VL* and *NC* (Euclid I:43), which means that the defining property of Construction 3 is fulfilled. As the midpoint perpendicular *MC* is tangent to the parabola, this tangent passes the midpoint M of *VN* since *MV* and *MN* are congruent. Finally, we note that *MN* is the mean proportional to *BN* and *NC*, showing that the square on *MN* equals the rectangle on *BN* and *NC*. As the square on *VN*, equalling the rectangle on *VL* and *NC*, is four times the square on *MN*, the segment *VL* is four times that of *BN* and thus constant, not depending on the position of B (on the directrix). So, *VL* is four times the segment *VF*.

Second, the construction of a point C on a parabola by Construction 3 is also displayed in Figure 5, where *VL* is a given segment, and N is an arbitrary point on *VL* or its extension (cf. Figure 3). C is found as the intersection of *NE* and *VK*. Select M as the midpoint on *VN*, draw *MC* and its perpendicular at M meeting *VH* at F and (the extension of) *NE* at B. Compared to Construction 2, F will be the focus of the parabola and its directrix will pass B parallel to *VL*.

We first note that the triangles MFV and MBN are congruent (as two angles and one side are congruent). This means that *MF* and *MB* are congruent, as well as *VF* and *NB*, and that *MC* is the midpoint perpendicular to *FB*. It follows that also the triangles MCF and MCB are congruent (as two sides and one angle are congruent). This means that *FC* and *BC* are congruent (i.e., C is equidistant to F and B). Finally, we note that *MN* is the mean proportional to *BN* and *NC*, showing that the square on *MN* equals the rectangle on *BN* and *NC*. As the square on *MN* is a fourth of the square on *VN*, equalling the rectangle on *VL* and *NC*, the segment *BN* is a fourth of *VL*. Thus, *VF* is a fourth of *VL*; F is thus invariant as N moves along *VL* (and its extension).

5. The classroom study

In this section the work of the students participating in our study as they encounter the multiplicity of constructions of the parabola in DGS is discussed.

Context and setting

The empirical study comprised a teaching intervention organized at two different occasions, with twenty students participating at the first occasion and twelve (other) students at the second occasion. The students were all enrolled in grade 11 of an upper secondary school in Moscow (Russia) specializing in mathematics. Admission to the school is based on an entrance examination in mathematics, and once admitted, students follow an accelerated curriculum in the subject. The level of preparation of the students participating in this study was thus higher than in a regular upper secondary school. Some of these students also attended mathematical circles or camps and participated in other outreach mathematical activities in their spare time.

The study involved sessions with each group that took place in the students' normal classroom, led by the first author who is fluent in Russian. However, at the students' request, the sessions were conducted in English. The students were given the task to provide a definition of a parabola, a method for drawing a parabola and a tangent, and a justification of why the suggested method works. For these students, the concept of a parabola had not been a specific topic of study during the regular class work other than in the context of graphing elementary functions (including second degree polynomials).

Before meeting with the students, the first author shared with their mathematics teacher three different approaches that could be used for the construction of a parabola in a school context as presented above: one functional approach (Construction 1); and two geometrical approaches (Construction 2 and Construction 3). The teacher explained that while the students might have learned about other constructions from other sources, the construction drawing on a functional approach is consistent with the current (Russian) curriculum.

Each meeting took place in a regular class time for two lesson-periods (50 minutes each) with a short intermission. Participation of students was voluntary and no marks were assigned to this activity. All students present in class chose to participate in the activity. During the meeting a DGS was available in the classroom for whole class demonstrations and discussions, as well as on students' personal devices. The DGS was capable of drawing points, segments, parallel and perpendicular lines, circles etc., as well as graphs of functions given by their equations in the form $y = f(x)$, and manipulating with these objects, including the drag mode. Tools for drawing a parabola by its focus and directrix, and a tangent line to a given curve, were not amongst the available menu options.

At the beginning of the first lesson the students were asked to write a definition of the parabola on a piece of paper. They were given five minutes for this task and were encouraged to include more than one definition if possible. After the students' responses had been shared with the whole class, they were asked to propose a way of drawing a parabola and its tangent line at a chosen point. The students again were working individually for 10-20 minutes. Then again they were asked to share their ideas with the whole group. Based on the discussion of these ideas, the teacher suggested alternative constructions and asked students to explain why they work.

In the next section observations, as well as some dialogues that took place between the students and the teacher, are presented, including excerpts from the transcript of the video-recorded sessions. Our analysis of the transcript from the observed lessons, drawing on the theoretical notions outlined above, is presented in section 6.

Students' work

This subsection presents how the participating students worked on the task of defining and drawing a parabola, along with its tangent, using a DGS (Geogebra). As none of the students' construction of the parabola used Construction 2 and Construction 3, we also present the follow-up tasks provided by the teacher about arguments for why these constructions work. Finally, a construction suggested by one student drawing on the technique of parametric curves is presented.

Defining a Parabola

All students wrote that a “parabola is the graph of a function”, suggesting $y = x^2$ or a slight variation of this formula. Several students gave the general equation of a second-degree polynomial $y = ax^2 + bx + c$, $a \neq 0$. When the teacher prompted for alternative definitions, a few students gave a definition based on the equidistant property, as in the following excerpt.

S (a student): There was something else, something geometrical, I forgot.

T (the teacher): What was the geometrical definition? Who remembers?

S: There was a line and a point…. then we take all points equidistant from the two.

Ss (several students): Oh, yes, I recall it now!

T: and the line is called…

S: directrix.

T: and the point is…

Ss: the focus.

That the students needed a prompt to remember that they had encountered a geometrical definition suggests that it had not been much used in for example exercises and then “forgotten”. Indeed, while it was the functional description that was provided by the students as response to the task, the focus-directrix was not the only alternative approach:

S: I know one more! It’s one of the curves …. when you slice a cone.

Then also some other definitions were suggested, including the following:

S: A parabola is a curve with the property that every ray parallel to the *y*-axis reflects form it to the focus [Figure 6, right].

S: A parabola is a curve which has a vertex and two symmetric branches.

The reflective property and also the parabolic trajectory of a “stone thrown at an angle to the horizon” were known to some of them from physics. In Figure 6 two student definitions are shown. In summary a few students had heard about alternative definitions of the parabola but still the majority had not.

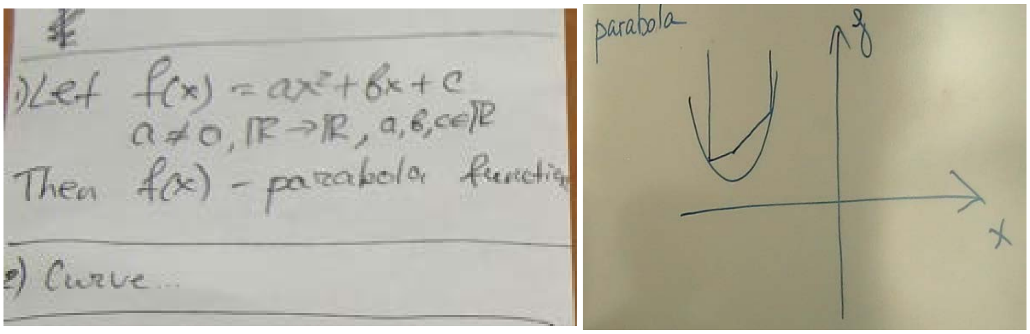

Figure 6. Examples of student definitions of a parabola

Students' solutions for drawing a parabola and its tangent

All students start by plotting the graph of the polynomial $y = ax^2 + bx + c$ (for some a, b and c) and then try to figure out about the tangent line. Some of them use their smartphones, some laptops or computers. The following excerpt illustrates the most common type of solution for the construction of a parabola and its tangent in a DGS proposed by the students, drawing on a functional approach (*Method 1*):

T: Does anybody have a solution?

S: Simply type $y = x^2$ [in the software]. This is a parabola. The *x*-axis is the tangent line.

T: At which point?

S: At zero. I mean $x = y = 0$.

T: Good! What if I choose another point, say $x = 1$?

S: There is a formula for doing that. We need to take the derivative, which is $f' = 2x$.

T: OK. And then?

S: Then we write $y_1 - y_0 = f'(x_1 - x_0)$

T: What are these notations?

S: $x_0 = 1$ and $y_0 = f(x_0) = 1$. [Pause]

T: What is f'.

S: This must be the slope. [Pause] The slope of the tangent at this point.

T: But you said $f' = 2x$?

S: No it's a number. [Pause] Ok, I need to take $f'(x_0) = 2$. Here you go! So we get $y_1 = 2(x_1 - 1) + 1 = 2x_1 - 1$

T: So what should we do to see a parabola and its tangent at $x = 1$ on the screen?

S: Type $y = 2x - 1$. Should work.

The procedure is quite straightforward, though supported by a few adjustments prompted by the teacher. In Figure 7 a student's plot of a parabola and its tangent, using this method, is shown.

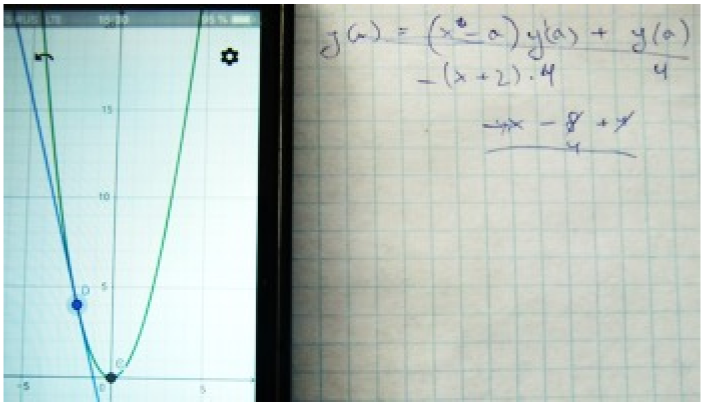

Figure 7. A student's plot of the parabola $y = x^2$ and its tangent at $x = -2$.

Several students used the sliders (available in the DGS) and made an applet showing the parabola with general parameters *a*, *b*, and *c* and a dynamic tangent line moving with the *x*-value of the tangency point. It became apparent that all students in the class were familiar with this way of producing a parabola and its tangents (*Method 2*).

One student suggested, from memory, the construction shown in Figure 8. It also uses a functional approach drawing on the slope of the tangent as the value of the derivative at the point of tangency.

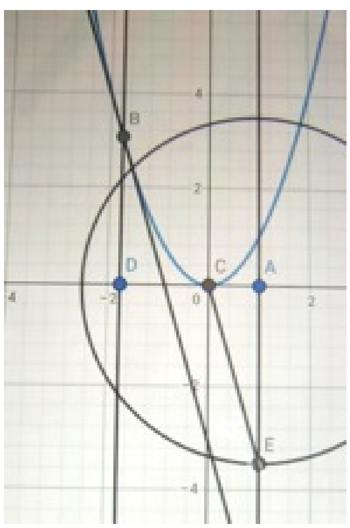

Figure 8: One student's approach to draw a tangent to a parabola.

S: Let $y = ax^2$, *CA* = 1 and *AE* = 2*a* · *CD*. The slope of *CE* is the slope of the tangent at B.

Also some other methods were proposed for drawing a tangent. One student employed an algebraic approach by using the assumption that the tangent line and the parabola intersect only at one point and solved a system of equations (*Method 3*). The following calculation summarizes the student's solution:

Let $y = ax^2 + bx + c$, and the tangent be $y = px + q$, where p and q are to be found. Subtract the equations to get $ax^2 + (b - p)x + (c - q) = 0$. This equation must have only one solution $x = x_0$, and thus $(b - p)^2 = 4a(c - q)$ and $x_0 = (p - b)/(2a)$. Therefore, $p = 2ax_0 + b$ and $q = -ax_0^2 + c$.

Drawing a parabola in DGS using Construction 2

As none of the students suggested Construction 2 as a solution it was demonstrated by the teacher. The students were then asked to explain why this method works. After working individually for about 20 minutes, they presented the following three types of explanations.

Type 1: As C lies on the perpendicular bisector to *BF*, we have $FC = CB$ (Figure 2). Since *CB* is the distance between point C and line *l*, C is equidistant from F and *l*. Thus C lies on the parabola by the second definition discussed in class.

Type 2: Suppose that *l* is the *x*-axis (Figure 2). F is fixed so we say that it has coordinates (0,1). And B is moving along *l* so we say that it has coordinates $(x, 0)$. The slope of *FB* can then be found to be –1/*x*. Since *l'* is perpendicular to *FB*, its slope is *x*. It is apparent from the applet that *l'* is tangent to the curve $y = F(x)$ at point C, where C has coordinates (x, y). If we solve $F'(x) = x$, we obtain $F(x) = x^2/2 + const$. To find *const* we observe that if $x = 0$, the curve passes through point V with coordinates $(0,1/2)$. Thus *const* = 1/2. We got $y = x^2/2 + 1/2$, which is a parabola.

Type 3: Draw the line *l'''* perpendicular to *l'* (Figure 9). Note that since *l'* is the angular bisector of angle FCB and angles FCB and FCD are supplementary, *l'''* is an angular bisector of FCD. Let *l''* be a ray of light. It reflects from the line *l'* at point C in the direction of F because *l'''* is an angular bisector of DCF. Line *l''* is perpendicular to *l* and thus, parallel to *FV*. So, a ray parallel to *FV* will focus (after reflection from the curve) at point F, which is a property of a parabolic mirror with focus at F. Thus, the curve is a parabola and *l'* is a tangent line.

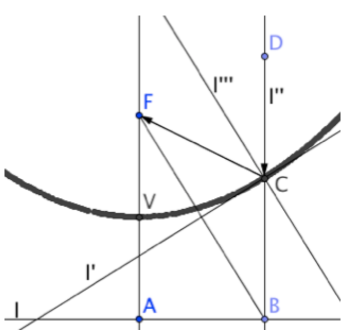

Figure 9. The reflexion property of a parabolic mirror.

Solutions of Type 1 were the most common. The explanations of why $FC = CB$ varied, but all students drew on basic ideas from Euclidean geometry. Once articulated, all students in the class agreed that this approach is correct. They also noted that *l'* is the tangent line to the parabola.

Two students used this observation to produce a Type 2 solution and one student to get a Type 3 solution. The teacher asked the students to explain in more detail why this line is a tangent and write it on a piece of paper to prompt the students to think about properties of a tangent line to a curve. Five students found a justification as illustrated by the following dialogue with one of them.

S: It looks like the line *l'* is tangent to the curve at point C [Figure 2].

T: How do we know that this is indeed so? Can you justify?

S: Sure. I temporarily drop the prime notation and will call my line *l*, ok? I assume that there is another point D on *l* that belongs to the parabola [refers to Figure 10].

T: Why do you do so?

S: I assume that the parabola and the line intersect at two points C and D. I want to show that this is impossible. Then I will conclude that there is only one point of intersection, and so *l* is a tangent not a secant line.

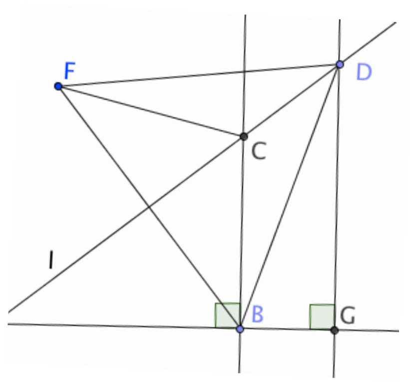

Figure 10. Proof by contradiction that the line *l* is tangent to the parabola at C.

T: What do we call this type of proof?

S: This is proof by contradiction. I want to find a contradiction by assuming something that should not happen.

T: Ok, go on.

S: So, suppose D is on the parabola. Then *FD* = *DG* by the definition of the parabola.

T: Yes, I agree.

S: But D is also on l, which is the perpendicular bisector of FB. So, *FD* = *DB*.

T: This is correct.

S: But then I get *DG* = *DB*, which is a contradiction.

T: Why?

S: Because DGB is a right triangle, so the hypotenuse *DB* must be longer than the leg *DG*. In the plane geometry. [Laughing]

T: OK, and so what do you conclude?

S: I conclude that unless D and C coincide, there is a contradiction. Thus the line l and the parabola intersect in only one point C.

T: Very nice!

Having the tangency of the midpoint perpendicular thus established, the solution of Type 2 would then link Construction 2 to Construction 1. Attempting to highlight the connection between Construction 2 and Construction 1, the teacher gave the following hint.

T: What if we use the definition we started with, that is, a parabola is a graph of a function $y = ax^2$? Let us look at Figure 2 again and introduce Cartesian coordinates so that l has equation $y = -p$, and the coordinates of the points are $F(0, p)$, $V(0,0)$ and $C(x, y)$. Without loss of generality we assume that $p > 0$. Can we derive a relation between *x* and *y*?

This idea led to the following explanation (Type 4) produced by a student (Figure 11).

Type 4: Drop a perpendicular from F onto *BC* at D. Consider the right triangle FDC. By the Pythagorean theorem, $FC^2 = x^2 + (y - p)^2$. Observe that $FC = CB = y + p$, and after simplifications one gets the equation $y = x^2/4p$. This equation describes a parabola. At the same time the slope of FB is $-2p/x$, so the slope of *l'* is $x/2p$. This result agrees with the calculus approach, by which the derivative is $\frac{d}{dx}(x^2/4p) = x/2p$. Finally, the equation of the tangent line at point $(x_0, x_0^2/4p)$ is $y = \frac{x_0}{2p}x - \frac{x_0^2}{4p}$.

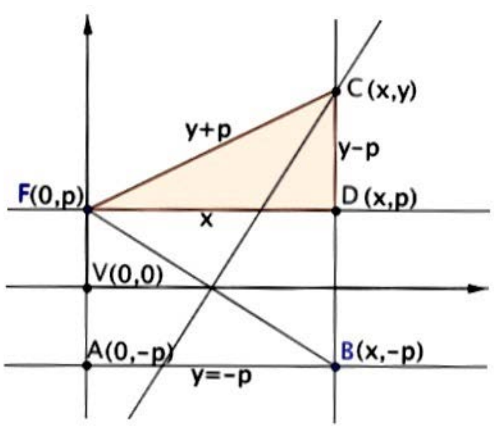

Figure 11. Proof by introduction of Cartesian coordinates that the line *l'* is tangent to the parabola at C (cf. above the method regarding Connections between Construction 1 and Construction 2).

Drawing a Parabola in DGS using Construction 3

In the final part of the meeting the teacher demonstrated Construction 3, as nobody had suggested it, and asked the students to provide an explanation for why it produces a parabola and its tangent. One type of explanation suggested by the students was the following (with reference to Figure 3).

T: Can you reformulate what exactly you need to show?

S: When N moves then H moves and C moves. … C is on a parabola …. So $NC = (VN)^2$ … this is what I need to prove …

S': Not exactly! There could be a coefficient k, so $NC = k(VN)^2$.

S*:* OK. From similar triangles $VN/VL = NC/LK$. So we can find $NC = VN \cdot LK/VL$. But $VL = const$ and $LK = VN$! So we have shown that NC is proportional to $(VN)^2$. Done!

Here, one student expressed interest in possible origins of mathematical ideas:

S: Yes, I agree. This is clear. But. But how one could invent this method? Where does it come from?

This led the teacher to prompt, as an historical remark, the use of the idea of the application of areas expressed in Euclid I.43 by asking them to find rectangles with equal areas in Figure 3. One student then marked two rectangles (see Figure 12), which led to the following discussion:

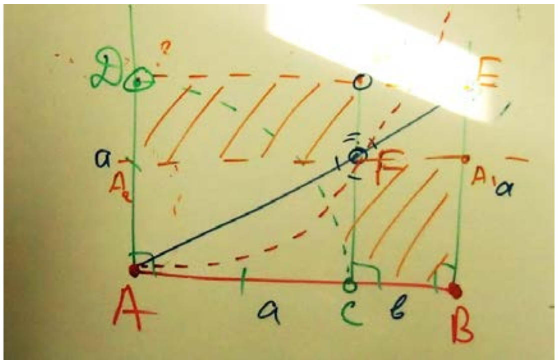

Figure 12. A student's proposal for which areas might be equal.

T: Why do you think so?

S: It looks like that…. If any two are equal it should be those… Ok, I can prove it! A diagonal divides any rectangle in halves, right? So here we go.

S': I do not understand. What do you mean? Ok, ABE and ADE are equal. And what?

S'': Also ACF is equal to AF… what is this point? Call it A_2? And FHE is equal FEA_1. Same idea. Then just subtract.

S': What to subtract?

S'': Same small areas from same big areas. So what is left are the two rectangles and they are equal areas.

S'. Okey… now what? We need to express these areas algebraically… and set them equal… let this segment be x and this is y, and these are a and b. [S7 marks then on the board and tries to write an equation. $S_1 = S_2$. So… $ax = by$; see Figure 13] … Oh, too many variables… Wait, $x + y = a$ so $y = a - x$, we can eliminate y... where does it go now?

S'': Here. I have another idea. Take $x = a - y$ and sub in $ax = by$. So $a^2 - ay = by$. So $a^2 = y(a + b)$. And $a + b = const$! So y is proportional to a^2. This is what we need!

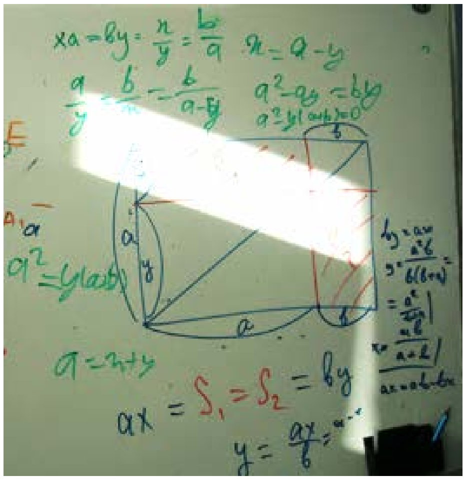

Figure 13. Students' work on the equation of a parabola based on the equality of areas.

Another type of explanation, which like the previous one did not deal with the tangent line, involved the introduction of coordinates C(x,y), L(q,0), N(x,0), H(0,x), and E(x,x). From the equality of areas QCEH and NLJC (as in Figure 3), one then finds $x(x-y)=(q-x)y$, which simplifies to $x^2 = qy$, identified as the equation of a parabola. Then one student proposed the following reasoning, involving the tangent:

> S: Note that *MP* is tangent to the parabola at point $C(x,y)$ and its slope is $\frac{x}{\frac{q}{2}} = \frac{2x}{q}$. Thus, if $y = f(x)$ then $f'(x) = \frac{2x}{q}$, so by integration $f(x) = \frac{x^2}{q} + const$, and the curve is a parabola.

This reasoning is, however, based on the visual appearance of *MP* as a tangent line. In order to prove the tangency one could differentiate the result found in the explanation above, that is $y = x^2/q$, and observe that the derivative and the slope of *MP* have the same value $2x/q$.

An emerged solution

One more construction was proposed by one of the students after the lesson was completed. This student observed that the first method of drawing the parabola demonstrated in class differed from the other two because it "produced the parabola on the screen at once, while the other two methods required dragging a point in order to obtain the parabola as a trace of another moving point". Based on this reasoning the student proposed the following construction (*Method 4*; see Figure 14):

> Introduce the slider t (available in the software) and then draw the lines $x = t$ and $y = t^2$. Let A be the intersection point of these lines. Then as the slider t is moving, the point A traces the parabola. This approach is linked to the idea of a parametric description of curves, an idea not taught to the students but rather spontaneously emerged from the demonstrations of the second

and third constructions of the parabola involving DGS. In addition, the vector $(x', y') = (1, 2t)$ shows the direction of the tangent line at the point A(t, t^2).

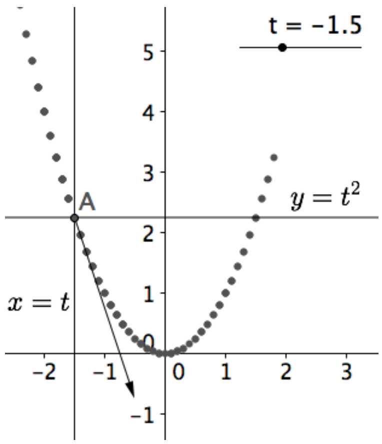

Figure 14. A parabola as a parametric curve with the tangent vector.

6. Discussion

Familiarity and accessibility of various techniques to draw the parabola and its tangent

The task given to the students of drawing a curve and its tangent is of a type commonly studied in current Calculus courses in upper secondary school. As part of a 'standard' Calculus reference praxeology the technology includes the derivative $f'(x_0)$ as representing the slope of the line tangent to the curve $y = f(x)$ at a given point $x = x_0$ and the equation $y - b = k(x - a)$ of a straight line passing through a given point (a, b) with a given slope k. Construction 1, employing a functional approach, therefore presents a standard technique for the participants of this study to solve this type of problem and was indeed recognized by them as familiar. Methods 1 and 2 above proposed by students are variations of this technique. Method 3 does not use the relation between the slope of the tangent and the derivative but is still algebraic in nature. Method 4 also employs a functional approach.

Even if none of the students suggested Construction 2 as a possible technique for solving this task, some participants did recall a geometric definition of a parabola in terms of focus and directrix. Once the students were shown Construction 2, drawing on this geometric definition of the parabola, they were able to provide a justification of its validity within a praxeology drawing on Euclidean geometry, as well as connecting it with Construction 1 by employing a functional approach after introducing a coordinate system. One student produced a proof by contradiction to show that the perpendicular bisector used for constructing a point on a parabola also is tangent to the parabola at that point, remaining within the same praxeology as the construction.

The technique involving the application of areas (Construction 3) was unfamiliar to all students. Similarly to Construction 2, however, for Construction 3 they were able to find a justification at the logos level of why the curve is a parabola, while referring to visual observation afforded by the software regarding the tangent line. In both geometric constructions, however, the justification for the curve being a parabola moved beyond the geometric praxeology by a use of the algebraic representation already familiar to them (e.g., as the graph of a quadratic function in a Cartesian coordinate system). Here we observe a presence of what could be called *semiotic traces*: students use a construct including a semiotic register familiar to them in order to justify a new one which is based on a different praxeology and technically does not require any reference to Cartesian coordinates or functional representation. When referring to the reflection property of the parabola, students also seem to implicitly think of the *y*-axis as being (parallel to) the axis of the parabola. To remain within the (Euclidean) geometric praxeology while working with Constructions 2 and 3, one would need to use one of the constructions as a *definition* of a parabola and justify the other (see the above section *Bridging different constructions of the parabola*). The students, however, produced connections between different praxeologies, drawing mainly on the theory of functions employing an algebraic register. It seems that the teacher did encourage the use of the functional approach when giving a hint leading to a type 4 explanation of Construction 2. This outcome reflects students' previous subjective "curricular" experiences and that for them the notion of parabola had been "endowed with cultural values and theoretical content that are refracted in the semiotic means to attend to it" (Radford, 2006), in this case the functional-algebraic approach.

The Role of DGS in this Study

In the first approach (Construction 1), in addition to observing the parabola, its tangent line and the point of tangency (Figure 1), by suggesting two equations students could verify their formulas by immediate graphing of the functions in the Cartesian plane. In the second approach (Construction 2), by using DGS for dragging the point B along the line *l* (Figure 2), it was possible for the students to follow the point C as it traced a curve to which the line *l*' is tangent. They could also observe how the equidistance condition shaped the path of the parabola and the ray reflection property (Figures 2 and 9). Students could observe how the shape and position of the parabola depend on the relative positions of the focus and the directrix. In the third approach (Construction 3), by moving the point C along the segment *VL* (Figure 3) students observed that C traces what looked like a parabolic curve with the segment *MC* remaining tangent to it. The shape of the parabola could be controlled by varying the length of the segment *VL* (the parameter of the parabola; cf. Bergsten, 2015). While not done in the teaching intervention, the DGS could also have been used to construct the focus and directrix, given the parabola had been constructed by the application of areas, as well as to find the parameter for the application of areas given the parabola had been constructed by the equidistance property (cf. Figure 5). Conversely, in the construction described in Figure 5 it is possible to observe the invariance of the segments *VF* and *NB* when dragging the arbitrary point N along the segment *VL* (and its extension).

In each case there were certain observations, which called for justification at the logos level of a praxeology. We see these combinations of observed and then 'explained' geometric properties as constituting the most viable moments of the students' exploration of various constructions of the parabola. While dragging specific points was essential for most of the demonstrations, making the emergence of the curve dynamic, this instrumented process has the potential to lead to a *dynamic* conceptualization of the parabola in contrast to the more rigid image afforded by plotting coordinates (found by calculating the squares of numbers) in a rectangular grid. The inventive approach proposed by one participant (Figure 14) is another indication that students started to view the parabola as a family of points resulting from the trace of a moving intersection point (i.e., the notion of *locus*). The use of a DGS did indeed support this type of dynamics.

Meaning development support by various constructions of the parabola and its tangent

In the classroom session about the parabola students became involved in more than one approach (technique) to solve the task. However, for a viable meaning construction to occur it was also necessary for students to see relations between the different approaches. As the students were able to establish connections between the formulas and steps given in different techniques for drawing the mathematical object *parabola*, linked to different historical framings and representations, within a short period of time, in line with a semiotic-culture approach their meaning construction was supported by a *semiotic objectification of knowledge* (Radford, 2006). A Cartesian graph and two geometric constructions aim at “the *same* mathematical object, different intensities of a flash with which we light it” and are “meaning conferring intentional experiences […] out of which a particular […] meaning is attained” (Radford, 2006, p. 50). For instance, a learner exploring the geometric definition of a parabola managed to obtain the same result as in Calculus (see type 4 explanation of Construction 2). Students also found that the equality of areas involved for a construction of a parabola can be linked to a quadratic function (Construction 3). A learner equipped with such experiences may develop a more *versatile* meaning (cf. Tall & Thomas, 1991; Thomas, 2008) of the mathematical object parabola and might be able to use alternative representations and properties of a parabola and its tangent line, as well as of other mathematical objects, depending on the tasks they need to solve. As discussed above, by developing representational flexibility of mathematical objects students can become better prepared for solving non-routine problems.

In this connection, our analysis and data imply two curricular recommendations. First, it seems to be more appropriate to start with studying Constructions 2 and 3 and links between them within Euclidean geometry. This would enable students to use respective praxeology consistently without reference to additional mathematical tools and concepts. Second, since Construction 1 emerges within a new praxeology, at this point it would be useful to revisit Constructions 2 and 3 and establish necessary links between geometrical and functional approaches. This scenario will be consistent with the instructional recommendation by Acevedo Nistal, et al. (2009) to encourage active comparison and evaluation of representations. In addition it will take into account the idea that in problem solving besides the characteristics of the task, also student characteristics affect the choice of concept representation. In particular, if students are not comfortable with a geometric description of the parabola, they may not use this representation even if it could be more efficient in a particular problem situation. Establishing firm grounds of each representation using respective semiotic means seems to be an important condition for its success in problem solving (cf., Acevedo Nistal et al., 2009).

Curriculum issues

Students' unfamiliarity with geometric constructions of the parabola prior to the lesson discussed here certainly relates to curriculum issues. Indeed, Bergsten (2015) observed that in Sweden the study of the parabola in secondary school was embedded in a local mathematical praxeology of analytic and Euclidean geometry during the 1960s, while since the 1980s it has been embedded in a local praxeology of functions. Similarly, in Canada the geometrical definition of conics no longer has a place in secondary school mathematics, neither does it get much attention at the university level. This situation was reflected in the results from a study involving students graduating from a Canadian university with a mathematical degree (Kondratieva, 2015) with results very similar to those described in this paper. The lesson described above took place in Russia. As mentioned earlier, according to the ordinary teacher in the two classes, students attending Russian schools non-specialized in mathematics study parabola only within the functional approach. However, the students in the two classes taking part in this study (specialized in mathematics) met geometric definition of the parabola in an earlier grade, but did not work extensively with this definition. At the same time the study of functions was current and extensive. Our results need to be interpreted within this context and thus partly support the proposition about necessity and usefulness for learners of working with various representations of concepts establishing links between them.

7. Final remarks

The students' experiences of the parabola lesson described here were not formally evaluated by a questionnaire or classroom discussion. However, from the spontaneous feedback given by a few students, the following communication (via email to the first author) may well summarize these experiences:

> S: Despite the fact, that topic about parabola looks simple I learned a bunch of stuff about it. […] this math lesson showed me, that even simple entities in mathematics can be more complex than we think before digging into it. I really appreciate your idea about making this lesson.

The "digging" is here a metaphor that may be used in a kind of archeological sense, referring to the different tools needed to analyze an object which is not yet well understood. It also reflects an element of engagement and curiosity.

At a theoretical level, the design of the parabola lesson, including multiple approaches and representations, and the way the students reacted to it, seem to have supported a working mode affording a "complementarity between meaning and rigour" (Bergsten, 2015, p. 42) by an expansion of the praxeologies referred to, involving for example a geometric context being embedded in a functional or vice versa. Similar working modes could be evoked also in the context of more “standard” elementary topics in current curricula such as optimization (e.g., maximizing the area of a rectangle with a constant perimeter, or minimizing the perimeter of a rectangle with constant area; see Bergsten, Häggström & Lindberg, 1997, pp. 118-121). While the experiencing of meaning can be deepened by a search for rigour, a focus on rigour remains void without involving components of meaning. Linking multiplication of decimal fractions to geometric images of rectangular areas could illustrate this complementarity drawing on numerical and geometric praxeologies. Didactically designed to combine multiple approaches and representational flexibility, all these examples of working modes may serve as contexts of “digging” in the mathematics classroom.

References

Acevedo Nistal, A., Van Dooren, W., Clarebout, G., J. Elen, J., & Verschaffel, L. (2009). Conceptualising, investigating and stimulating representational flexibility in mathematical problem solving and learning: a critical review. *ZDM Mathematics Education*, 41(5), 627–636.

Apollonius of Perga (1952). Conics. In *Great books of the Western world, Vol. 11* (pp. 593–804). Chicago: Encyclopædia Britannica, Inc.

Arzarello, F., Bartolini Bussi, M. G., Leung, L. Y. L., Mariotti, M. A., & Stevenson, I. (2012). Experimental approaches to theoretical thinking: Artefacts and proofs. In G. Hanna & M. DeVilliers (Eds.), *Proof and proving in mathematics education. The 19th ICMI study* (pp. 97–146). Dordrecht: Springer.

Barbé, J., Bosch, M., Espinoza, L., & Gascon, J. (2005). Didactic restrictions on the teacher’s practice: the case of limits of functions in Spanish high schools. *Educational Studies in Mathematics*, *59*, 235–268.

Bartolini Bussi, M. (2005). The meaning of conics: Historical and didactical dimensions. In J. Kilpatrick, C. Hoyles, O. Skovsmose, & P. Valero (Eds.), *Meaning in mathematics education* (pp. 39–60). New York: Springer.

Bergsten, C. (2015). Beyond the representation given: The parabola and historical metamorphoses of meanings. In C. Bergsten & B. Sriraman (Eds.), *Refractions of mathematics education. Festschrift for Eva Jablonka* (pp. 15–47). Charlotte, NC: Information Age Publishing.

Bergsten, C., Häggström, J., & Lindberg, L. (1997). *Algebra för alla* [Algebra for all]. Göteborg, Sweden: Nämnaren, Göteborgs universitet.

Bosch. M., & Gascon, J. (2006). Twenty-five years of the didactic transposition. *Bulletin of the International Commission on Mathematical Instruction, 58*, 51–65.

Chan, K., & Leung, S. W. (2014). Dynamic geometry software improves mathematical achievement: Systematic review and meta-analysis. *Journal of Educational Computing Research*, 51(3), 311–325.

DeVilliers, M. (2012). An illustration of the explanatory and discovery functions of proof. *Pythagoras*, *33*(3).

Dowling, P. (1994). Discursive saturation and school mathematics texts: A strand from a language of description. In P. Ernest (Ed.), *Mathematics, education and philosophy: An international perspective* (pp. 121–139). London: Falmer.

Duval, R. (2006). A cognitive analysis of problems of comprehension in learning of mathematics. *Educational Studies in Mathematics*, *61*(1–2), 103–131.

Fried, M. (2007). Didactics and history of mathematics: Knowledge and self-knowledge. *Educational Studies in Mathematics, 66*, 203–223

Hodgson, T. R. (1995). Connections as problem-solving tools. In P. A. House & A. F. Coxford (Eds.), *Connecting mathematics across the curriculum: 1995 Yearbook*. Reston, VA: NCTM.

House, P. A., & Coxford, A. F. (1995). *Connecting mathematics across the curriculum: 1995 Yearbook.* Reston, VA: NCTM.

Kondratieva, M. (2011). The promise of interconnecting problems for enriching students' experiences in mathematics. *Montana Mathematics Enthusiast*, *8*(1&2), 355–382.

Kondratieva, M. (2013). Multiple proofs and in-service teachers' training. In B. Ubuz, C. Haser, M. A. Mariotti (Eds.), *Proceedings of The Eighth Congress of the European Society for Research in Mathematics Education* (pp. 156–165). Ankara, Turkey: Middle East Technical University.

Kondratieva, M. (2015). On advanced mathematical methods and more elementary ideas met (or not) before. In N. Vondrova, K. Krainer (Eds.), *Proceedings of The Ninth Congress of the European Society for Research in Mathematics Education* (pp. 2159–2165). Prague, Czech Republic: Charles University in Prague, Faculty of Education and ERME.

Laborde, C., Kynigos, C., Hollebrands, K., & Strässer, R. (2006). Teaching and learning geometry with technology. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 275–304). Rotterdam: Sense Publishers.

Leikin, R., & Lev, M. (2007). Multiple solution tasks as a magnifying glass for observation of mathematical creativity. In J. H. Woo, H. C. Lew, K. S. Park & D. Y. Seo (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (V. 3, pp. 161–168). Seoul, South Korea: PME.

Leikin, R., & Levav-Waynberg, A. (2008). Solution spaces of multiple-solution connecting tasks as a mirror of the development of mathematics teachers' knowledge. *Canadian Journal of Science, Mathematics and Technology Education, 8*(3), 233–251.

Mingus, T. T. Y. (2002). A case study of spiraling content and pedagogy through core courses for pre-service secondary mathematics teachers. In M. H. Ahmadi (Ed.), *Readings in innovative ideas in teaching college mathematics* (pp. 21–42). Lanham: University Press of America.

NCTM (2000). *Principles and standards for school mathematics*. https://www.nctm.org/Standards-and-Positions/Principles-and-Standards/ Retrieved 20.04.20.

Radford, L. (2002). The seen, the spoken and the written: a semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics*, *22*(2), 14–23.

Radford, L. (2006). The anthropology of meaning. *Educational Studies in Mathematics*, *61*, 39–65.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher*, *27*(2), 4–13.

Skemp, R. R. (1987). *The psychology of learning mathematics*. Lawrence Erlbaum Associates, Publishers: Hillsdale, NJ.

Sun, X., & Chan, K. (2009). Regenerate the proving experiences: an attempt for improvement original theorem proof construction of student teachers by using spiral variation curriculum. In Fou-Lai Lin, Feng-Jui Hsieh, G. Hanna, M. DeVilliers (Eds), *Proceedings of the ICMI Study 19: Proof and proving in mathematics education* (V. 2, pp. 172–177). Taipei, Taiwan: The Department of Mathematics, National Taiwan Normal University.

Tall, D. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics*. NewYork: Cambridge University Press.

Tall, D., & Thomas, M. (1991). Encouraging versatile thinking in algebra using the computer. *Educational Studies in Mathematics*, *22*, 125–147.

Tall, D., Yevdokimov, O., Koichu, B., Whiteley, W., Kondratieva, M., & Cheng, Y. H. (2012). Cognitive development of proof. In G. Hanna, M. DeVilliers, (Eds.), *Proof and proving in mathematics education: The 19th ICMI Study* (pp. 13–49). New York: Springer.

Thom, R. (1973). Modern mathematics: does it exist? In A. G. Howson (Ed.), *Developments in mathematical education. Proceedings of the Second International Congress on Mathematical Education* (pp. 194–209). London: Cambridge University Press.

Thomas, M. (2008). Developing versatility in mathematical thinking. *Mediterranean Journal for Research in Mathematics Education*, *7*(2), 71–91.

Winsløw, C. (2013). The transition from university to high school and the case of exponential function. In B. Ubuz, C. Haser, M.A. Mariotti (Eds). *Proceedings of The Eighth Congress of the European Society for Research in Mathematics Education*. Antalya, Turkey (pp. 2476–2485). Ankara, Turkey: Middle East Technical University.

Appendix

A point C on a parabola *P* constructed by employing the application of areas (Construction 3) is given (see Figure 15). That the line through M (the midpoint of VN) and C is tangent to *P* at C can be proved as follows:

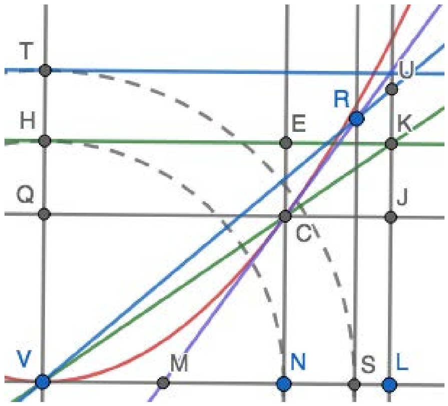

Figure 15. A tangent to a parabola using Construction 3 (notations as in Figure 3, with VL = q being the parameter of *P*).

Select an arbitrary point R on the line through M and C and assume that R is on the parabola *P*. With VL and VH as coordinate axes, let the coordinates of C and R be (x_1, y_1) and (x_2, y_2), respectively. Then, with $VL = q$ the relation $x_1^2 = qy_1$ holds. By similarity of the triangles *MSR* and *MNC* it follows that $\frac{x_2 - x_1/2}{y_2} = \frac{x_1/2}{y_1}$, and thus $\frac{2x_2 - x_1}{y_2} = \frac{q}{x_1}$. If R is on the parabola P, then $x_2^2 = qy_2$, implying that $2x_2 - x_1 = \frac{x_2^2}{x_1}$, which simplifies to $(x_1 - x_2)^2 = 0$, that is $x_2 = x_1$. Thus C is the only point on the line through M and C that is on the parabola *P*. As MC is not parallel to the axis of *P* it must be its tangent.