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Abstract: Our research team surveyed members of the mathematics education scholarly community to gain insight into the community's perceptions of the two-column proof and its role in high school geometry. We created an email survey, which we sent to email addresses gathered from public lists of conference participants and speakers from recent mathematics education conferences. We asked study participants to describe the value of the two-column proof and discuss whether they would be in favor of eliminating it from the high school geometry curriculum. There was a wide range of diversity in the responses, and we present several themes that we observed. We conclude with a discussion regarding the implications for research.

Keywords: Two-Column Proof, Geometry, Proof, Proving, Mathematics Education Research

Introduction

Reform documents (e.g., Common Core State Standards Initiative, 2010; National Council of Teachers of Mathematics, 1989, 2000) have long advocated for mathematical proof be a meaningful part of K-16 students' mathematics education. However, despite these efforts, proof still has a marginal role in schools (Stylianides, Stylianides, & Weber, 2017). Historically in the United States, students' experience with proof is often limited to geometry; and the most common way of engaging students with proof is through the two-column format (Herbst, 2002b). This format developed as a way to meet a Committee of Ten mandate, introduced in 1893, that all students write proofs in geometry (Herbst, 2002b). The intention of this mandate, decreed by college professors, was to prepare students for college and create opportunities for students to

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develop mental discipline through the art of proving. See Figure 1 for an example of a two-column proof.

The Isosceles Triangle Theorem states that if two sides of a triangle are congruent, then the angles opposite the sides are congruent.

Given: $\overline{AB} \cong \overline{AC}$

Prove: $\angle B \cong \angle C$

Statements	Reasons
$\overline{AB} \cong \overline{AC}$	Given
Construct a bisector of $\angle A$	Every $\angle$ has 1 $\angle$ bisector
$\angle BAD \cong \angle CAD$	Definition of $\angle$ bisector
$\overline{AD} \cong \overline{AD}$	Reflexive Property
$\triangle ABD \cong \triangle ACD$	SAS Postulate
$\angle B \cong \angle C$	CPCTC

Figure 1. Example of a Two-Column Proof

The two-column proof format first emerged in textbooks in the 1910s (Herbst, 2002b; Kelley, 2013). Since the 1910s, the two-column proof has continued to have a substantial place in United States secondary geometry textbooks and classrooms (Kelley, 2013; Stylianides et al. 2017). But the format has received a substantial amount of criticism from mathematics education scholars. For instance, based on his observations of high school geometry classrooms in New York, Schoenfeld (1988) remarked,

> The use of such a form for proof strikes most students as being arbitrary and capricious. The teacher usually overcomes this initial resistance, but at the cost of a significant amount of time and effort. At the beginning of the term the form is presented as something that must be used, something that the students will simply have to get accustomed to. (p. 158)

And Bass (2011) commented,

> The things that students are asked to prove are intuitively obvious, without surprise or sense of discovery, so that ritualistic execution of the two-column proof has become more an academic etiquette than a purposeful mathematical practice. (p. 98)

In the 1989 standards, NCTM suggested the two-column proof be deemphasized in geometry classrooms. In spite of the criticism and reform suggestions, when students prove in high school geometry classrooms the two-column proof format remains the format used most often in the US (Stylianides, Stylianides, & Weber, 2017). Would it be reasonable, as was done by NCTM (1989), for current mathematics education researchers to suggest that the two-column proof format be deemphasized in secondary mathematics? Or even to suggest that two-column proofs be eliminated entirely from the secondary curriculum? These questions were the topic of conversation amongst several attendees of the 2016 North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA) conference. This conversation led us to review mathematics education literature related to the two-column proof, and ultimately to conduct the study described in this paper.[2]

In this study, we seek to understand how the current members of the mathematics education scholarly community view the two-column proof. We believe that the teaching and learning of mathematical proof is important to many members of the mathematics education scholarly community. Mathematics education researchers typically have classroom experience, conduct research in classrooms, interact with practicing teachers, and publish research that aims to influence policy. Many mathematics education researchers also teach proof either to undergraduate mathematics majors, to high school students, or to future and practicing teachers. Our opinions on

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[2] An early version of this paper was published as part of conference proceedings (Pair, Singh, & Strachota, 2018).

the two-column proof matter; and we need to be informed stakeholders. Thus, it is valuable to explore our community's perceptions of the two-column proof.

In our study, we created an email survey which we sent to email addresses gathered from public lists of attendees and speakers at recent mathematics education conferences. We asked participants to describe the value of the two-column proof and discuss whether they would be in favor of eliminating it from the high school geometry curriculum.

Our study provides insight into perceptions of the two-column proof held by members of the mathematics education scholarly community. Some of the views expressed by our participants are reminiscent of the views that other researchers have expressed within the mathematics education literature. However, within this literature, there is not a complete synthesis of the research results related to the two-column proof.

Literature Review

In the following literature review, we synthesize research on the two-column proof. We found the literature contains contradictory arguments and findings about the role of the two-column proof in learning and instruction. We begin by summarizing some of the explicit critiques of the two-column proof. We then move into discussing a body of research that considers how teachers use and perceive the two-column proof. We then discuss students' interactions with the two column-proof and the role of the two-column proof in the curriculum.

Critiques of the Two-Column Proof

Critiques of the two-column proof have been discussed in the mathematics education literature for decades (Balacheff, 2008; Bass, 2011; CadwalladerOlsker, 2011; Ness, 1962; Schoenfeld, 1988; Usiskin, 1980). A 1962 *Mathematics Teacher* piece by Ness (1962) argued that the format is overly reliant on memorized steps; and it does not require the student to demonstrate any true understanding. Similarly, several authors criticized the format for being too procedural and thus lacking meaning and substance (e.g., Balacheff, 2008; Bass, 2011, Schoenfeld, 1988). In regards to the two-column format, Usiskin (1980) wrote that “we are misteaching our students into believing that this is what proof is” (p. 410). CadwalladerOlsker (2011) expressed concerns that by only writing two-column proofs, students will view proving as a purely formal activity and lose sight of whether a proof is valid or convincing (because they are overly focused on the formalities of the structure).

We are interested in the validity of these claims in the context of teaching and learning. The following sections focus on research that investigated the role of the two-column with teachers and students and within the curriculum.

The Two-Column Proof and Teachers

In a paper outlining the historical evolution of the two-column proof, Herbst (2002b) described how the format first began to appear in U.S. textbooks in the 1910s. By bringing to the fore the logical aspects of proof (as deduction from givens to the conclusion), the two-column format “brought stability to the course of studies in geometry by making it possible for teachers to claim that they were teaching students how to prove and for students to demonstrate that their work involved proving” (p. 283). Herbst raised the concern that this stability came with a price. Namely, “dissociating the doing of proofs from the construction of knowledge” (p. 307). That is, the two-column format reduces proof to a formal process of deducing conclusions from givens, rather than as a natural component of mathematical activity and means of constructing mathematical concepts (Lakatos, 1976).

Within the context of a ninth-grade integrated mathematics course, Herbst (2002a) continued his study of the two-column proof, describing the “right angle episode” in which students had difficulty completing a two-column proof they were assigned for homework. The teacher, Andie, decided to have the whole class work together to complete the proof. At first, Andie solicited suggestions from students regarding the steps for the proof, but when students could not supply them, she provided the main ideas. Using the concepts of *didactical contract* (Brousseau, 1984) and *double bind* (Bateson, Jackson, Haley, & Weakland, 1956), Herbst described this scenario as problematic because the instructor took away the students' opportunity to prove. The didactical contract refers to the tacit norms governing the relationship between teacher and students. Andie and her students understood proofs to be sequences of deductions from givens to conclusions, where each statement is justified by a reason (the two-column format). Furthermore, Andie was responsible for providing fair opportunities for students to write proofs. Herbst claimed the students had difficulty completing the proof because, in order to complete the proof, it was necessary to obtain information from the diagram, violating the norm that a proof is simply deduction from the premises. This constituted a breach of the didactical contract. According to Herbst, the events also put Andie in a double bind as she was faced with two contradictory demands: 1) the task was intended as an opportunity for students to demonstrate their competency in proving; and 2) the need to ensure students found value in the task after not being able to complete the proof on their own. Herbst recommended that teachers look to new meaningful ways of engaging students in proof so that proving is connected to the process of coming to know, as recommended by NCTM (2000). He insisted that teachers refrain from taking over the students' work in proving, and that what counts as proof should be explicitly negotiated by the teacher and students in new contexts.

In subsequent research, Herbst and Brach (2006) again employed the didactical contract as a theoretical lens. Through an interview study with 16 ninth-grade geometry students, the authors sought to understand the tacit norms that governed proving in a classroom. The researchers' method was to provide students a list of tasks and ask the students for which of the tasks they would be expected to write a proof. The authors identified several aspects of the didactical contract related to proving in the classroom; we only briefly discuss a few of the results. The authors found that, in the eyes of the students, the teacher has much responsibility: to provide the conditions and givens in proving tasks, to only ask students to create a proof for recently covered concepts, and to only cover one major concept in a question. Regarding their own roles, students believed that they were responsible for writing proofs, and that a full proof included all statements and reasons. They understood it as their responsibility to recall definitions and properties, to recognize how the diagrams relate to the givens and mark their diagrams appropriately in completing proofs (although they were hesitant to add auxiliary lines). Students also believed they were prohibited from making assumptions and should not assume that diagrams were drawn to scale.

In their discussion, Herbst and Brach critiqued some of these norms. For the authors, the students wrote proofs but were not engaged in *proving* as an authentic mathematical practice. Moreover, they argued the two-column proof is only the visible part of a much larger set of problematic classroom norms involved in teaching proof. Furthermore, they claimed that we should not remove proof from geometry courses or even deemphasize the two-column format, but instead incorporate more "geometric substance into students' opportunities for proving" (p. 117-118) as well as more opportunities to prove in other mathematics content areas.

In similar studies, (e.g., Herbst & Chazan, 2003; Nachlieli, Herbst, Gonzalez, 2009; Weiss, Herbst and Chen, 2009) Herbst and colleagues investigated the tension around the two-column proof and the related didactical contract by documenting 26 secondary teachers' perspectives on authentic mathematics and the two-column proof. Teachers were shown a video of a classroom episode in which a teacher[3] allowed a student who was using the two-column proof format to make an assumption in a proof without immediately justifying it. The student said he could complete the proof if only he could state a particular claim, but the student did not have a reason for his claim. The teacher suggested the student make the statement as an assumption and fill in the reason later. The authors shared this episode because they viewed it as a breach in the didactical contract, violating the norm that students always provide a reason for a statement before moving on in a proof.

Weiss et al. (2009) and Nachlieli et al. (2009) documented teachers' responses to the aforementioned classroom episode. Specifically, Weiss et al. (2009) considered the teachers' responses to the episode according to their own conceptions of authentic mathematics, outlining four different conceptions of authentic mathematics: the view that authentic mathematics is grounded in the real world; the view that authentic mathematics mirrors the discipline of mathematics as a logical body of knowledge; the view that authentic mathematics occurs when students are engaged in the practices of mathematicians (e.g., making conjectures, proving in order to convince, etc.); and the perspective that authentic mathematics consists in honoring students' mathematical ideas and methods. Interestingly, the teachers who described authentic mathematics as grounded in the real world criticized the teacher's move, arguing that in the real world it is not wise to make ungrounded assumptions. Conversely, other teachers who described authentic mathematics as the practices of mathematicians claimed that the work of a mathematician can be messy; and thus, it would be appropriate for the student to make an ungrounded assumption. Similarly, Nachlieli et al. (2009) found diverging views. That is, some teachers thought the

[3] The teacher was Herbst (Herbst & Chazan, 2003).

teacher's move was problematic while others thought it helped the student to focus on the crux of the argument.

In both studies, teachers' interpretations of the breach were clearly influenced by the different conceptions of authentic mathematics. However, the two studies differed in how the authors interpreted the teacher's move. Nachlieli et al. (2009) reiterated that the teacher's move was a breach of the didactical contract, particularly that "it is normative for each statement to be justified before proceeding to make the subsequent statement" (p. 30), whereas Weiss et al. (2009) interpreted the breach positively, claiming that educators might consider how the two-column proof can play a positive role in students mathematics learning. Weiss et al. (2009) further explained that the two-column proof has complementary implications. In some cases, "it acts as a watchdog, preventing students from making unsupported claims. At other times, it enables the making of such claims, in much the same way that a safety net enables a circus acrobat to make certain daring maneuvers that would otherwise be too risky" (Weiss et al., 2009, p. 289).

In closing, Weiss et al. (2009) stated that the two-column proof is likely not going anywhere and suggested that rather than debating whether or not the format should be used, educators should consider the productive ways in which teachers can use the two-column format. With that in mind, Cirillo and Herbst (2012) considered ways in which a teacher could modify two-column proving tasks so as to engage students in the authentic processes of proving, including mathematical practices such as conjecturing, or creating representations in the form of diagrams. The authors recommended that the two-column format be one of a variety of formats used by students and argued that by using a variety of formats, students' attention could be focused on creating sound mathematical arguments rather than on the form of the arguments. The authors offered the pros and cons of each of four different formats of proof: two-column, paragraph, tree, and flow. For instance, the authors claimed that some teachers find students' use of the paragraph proof problematic because of students' tendencies to omit too many reasons from their proofs (Cirillo, 2008).

A trend we notice across the literature and that is represented in this review thus far is that the initial harsh criticism of the two-column proof in the earlier work was gradually replaced by a more accepting view of the two-column proof. Initially, the two-column proof format was discussed as playing a role in divorcing proof from knowledge construction (Herbst, 2002b) or placing contradictory demands on the teacher (Herbst, 2002a). But later, the two-column proof is considered to play a supportive role in enabling students' reasoning (Weiss et al, 2009), and is suggested as one of a variety of methods of proof to which students should be introduced (Cirillo and Herbst, 2012). Interestingly, these contradictory arguments are mirrored in our results, which we present and discuss in later sections of this paper.

Several other researchers also investigated proof in a variety of forms with teachers and in turn answered questions about the two-column proof specifically. For instance, the two-column proof has been discussed by researchers who have investigated teachers' knowledge and conceptions of proof (e.g., Boyle 2012; Cirillo, 2008; Knuth, 2002). This research reveals that teachers typically view two-column proofs to be rigorous and formal (Knuth, 2002). While teachers value informal justification, some teachers question whether formal proofs are appropriate for all students (Boyle, 2012; Knuth, 2002). In a case study of one high school geometry teacher, Cirillo (2008) found that the teacher favored teaching using two-column and flow chart proofs, as students tended to leave out the reasons for claims when completing paragraph proofs.

Students Perceptions of the Two-Column Proof

Another area of related research provides insight into students' interactions with and attitudes toward the two-column proof (e.g., Alsup, 2004; Chazan, 1993; McCrone & Martin, 2009; Senk, 1985; Toney, Slaten, & Peters, 2013). This research reveals that students struggle with two-column proofs (Senk, 1985), view two-column proof as more sophisticated than other forms (Leddy, 2001; McCrone and Martin, 2009), find proofs more convincing or would receive more credit when they are in the two-column form (Leddy, 2001; McCrone & Marton, 2009), and do not believe two-column proofs generalize beyond the given diagram (Chazan, 1993). Research suggests that some of students' perceptions of the two-column proof may be a result of the ways in which this format is introduced. For instance, some texts distinguish what they call "formal" formats, which include the two-column form, from "informal" formats, which include paragraph proofs (Otten, Males, & Gilbertson, 2014)—a framing that aligns with the findings of Knuth (2002).

Two-Column Proof and the Curriculum

According to textbook studies, geometry textbooks typically introduce a variety of proof formats, including the two-column format, by the end of the first chapter (Otten, Males, & Gilbertson, 2014). Although texts typically introduce the two-column format early on, Battista and Clements (1995) recommended not introducing formal proof until students develop an appreciation for other justification techniques. In recent years, other forms of proof, such as the paragraph and flow proof, have obtained a greater share of the pages in modern textbooks, but the two-column proof still has a dominant place (Kelley, 2013; Stylianides et al, 2017). In closing, we highlight that research on the two-column proof is extensive but recommendations on the role of this format in the teaching and learning of proof vary and are sometimes even contradictory.

Methodology

Data Collection and Sample

What are the current perceptions of the two-column proof held by members of the mathematics education scholarly community? To answer this question, we designed a five-question survey which we sent to approximately 900 e-mail addresses gathered from public lists of attendees of three recent mathematics education conferences. The three conferences were the Association of Mathematics Teacher Educators (AMTE) conference, the Research in Undergraduate Mathematics Education (RUME) conference, and the conference on Inquiry Based Learning (IBL). We believe that the participants from these conferences represent many of the individuals who have a vested interest in the teaching and learning of proof: mathematics teacher educators, proof researchers, and mathematicians. We designed an exploratory study to uncover what the academic community believed about the two-column proof format. We asked participants the following five questions:

Q1: What is your position in the field of mathematics education? Please select the option that best describes your position. (university professor, in-service k-12 teacher, graduate student, or other)

Q2: Is there value in having students write two-column proofs? (yes/no)

Q3: Why or why not? (open response)

Q4: Do you support eliminating the two-column proof from the high school geometry curriculum? (definitely yes, probably yes, might or might not, probably no, definitely no.)

Q5: Why or why not? (open response)

Thematic Analysis

Thematic analysis involves identifying a unit of analysis within data, looking across those units for similarities and differences, and sorting accordingly (Ryan & Bernad, 2003). We engaged in a multi-stage thematic analysis process that involved various levels of sorting and considering themes. In the first stage, each of the three researchers took a broad view of the qualitative data, reading the entire data set and taking the unit of analysis as a participant's entire open response to both questions three and five. We then each loosely sorted these according to similarities and differences. This initial sorting process gave each researcher a general idea of the ideas expressed in the data.

After discussing these broad initial findings amongst our research team, we then returned to the data and analyzed it on a finer grain size. Instead of taking an entire open response as our unit of analysis, the lead author broke up each qualitative response to questions 3 and 5 into individual statements. For example, in response to question 5 one participant wrote,

> I think that the two-column proof is a good entry point for some middle school and early high school students as they begin studying logic and proof. I think the problem is when the only kinds of proofs they see and create are in that format. So even though I think that teaching with two-column proofs has value, I do think they should be less exclusively focused on.

This response was broken into three statements that we believed expressed three different opinions. In this case, the three statements coincided with the three separate sentences highlighting the three ideas: 1) Proof is a good entry point for students beginning to study logic and proof; 2) Problems arise when students only see and create two-column proofs; and 3) Even though they have value, they should be less exclusively focused on.

After breaking up each response into component statements, the researcher then sorted the individual statements according to similarities and differences. This lengthy process led to the construction of several categories of ideas that were represented across the participants' responses.

In the third and final stage, we assigned a code to each of the various ideas expressed by participants. Then at least two researchers together coded the data according to these categories, again taking as our unit of analysis the individual responses to both questions three and five. As our participants often expressed many opinions in the same response, most participants' responses were assigned several codes. This involved much discussion amongst the coding researchers. After completing the coding, we tallied the ideas that were expressed the most by our participants as represented by our codes and identified representative quotes for the various themes that emerged. In the results section, we present our results and discuss some major themes.

Results

We obtained 177 responses from the approximately 900 emails that we sent out. Of the 177 respondents, there were 138 University Professors, 23 Graduate Students, and 5 In-service K-12 Teachers. Given these responses and the nature of the conferences to which our participants attended, we can be confident that our responses represent that academic community of mathematics educators: including mathematics teacher educators, mathematics education researchers, and mathematicians.

A majority of participants, 79%, stated that there is some value in the two-column proof. Conversely, 21% of participants stated that there is no value in the two-column proof. In response to question four, "Do you support eliminating the two-column proof from the high school geometry curriculum?," 16% responded definitely not, 27.5% responded probably not, 21% responded might or might not, 19.5% responded probably yes, and 16% responded definitely yes (1% did not respond). Note that about 35.5% were probably or definitely in favor of eliminating the two-column proof, 21% might or might not, and 43.5% probably or definitely were not in favor of eliminating the two-column proof. See Figure 2 for a summary of these results.

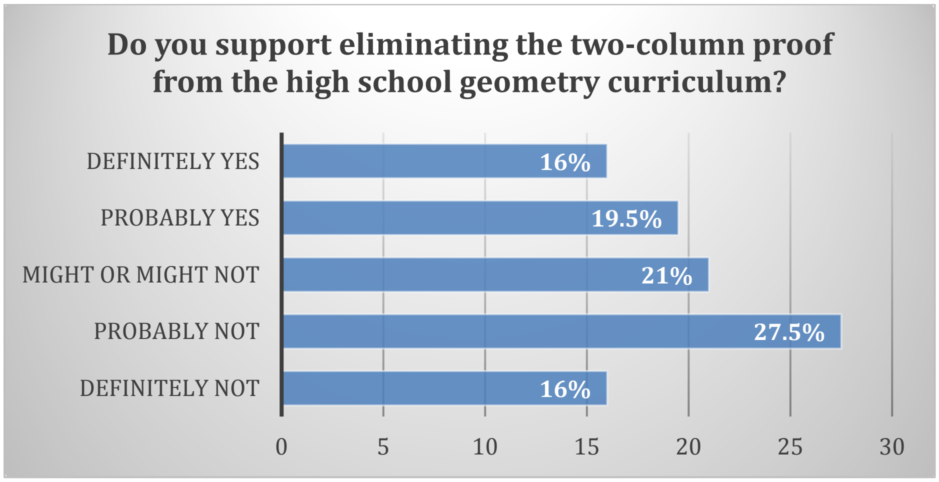

Figure 2. Participant Responses to Question 4

We now present the themes expressed in participants' open responses together with a description, an example of data quotations to illustrate each theme, and the percentage of our participants whose responses expressed that theme. We consider the ideas that were expressed by at least 5% of our participants to be themes. Table 1 shows themes related to general praise of the two-column proof. Table 2 lists the ways in which our participants critiqued the two-column proof. Table 3 contains neutral opinions related to the two-column proof. Recall that we often coded an individual participants' responses with several codes, so the tallies are not expected to sum to 100%. Rather the tables show the percentage of participants who expressed a particular idea (some participants expressed several ideas).

Major Themes

Positive Aspects. Regarding the positive aspects of the two-column proof, the themes that were expressed by participants are that two-column proofs help students understand that each statement needs a justification, the format may help students organize their thinking, the format provides a useful structure for students' learning of proof, and it is a scaffold to other forms of proof. We see several of these themes in the following representative quote:

> It's a nice way to have students organize their thoughts in a linear manner, and stress that every step needs a reason. I think it's a useful tool. I do not think it should be the only way of writing proofs, and I think that every student should have practice translating the two-column proof into paragraph form. However, I do think it's helpful. Sometimes in my introduction to higher mathematics class, my professor would actually have us write out our statements and reasons to make it clear when we didn't explain a step.

Table 1 Themes Related to Positive Aspects of the Two-Column Proof

Codes	Descriptions and Example Quotes
Statements/ Justifications 34%	Responses indicate that the ways in which the two-column proof emphasize statements and justifications is productive.
	"Two-column proofs help students see that for every statement there needs to be justification."
Organization 21%	Responses indicate that the two-column proof serves as a means for organizing students' ideas.
	"The two-column proof is an efficient way for students to organize their thoughts and reasoning in a similar way that an outline or graphic organizer might help students to organize their thoughts when writing an essay in a humanities course."
Structure 20%	Responses highlight structure. Specifically, they indicate that the two-column format's structure helps students understand the logic of proof.
	"For students new to proof in mathematics, it provides them with a structure upon which to hang their hat."
Scaffold 15%	Responses indicate that the two-column proof functions as a scaffold for other forms of proving (e.g., a paragraph proof)
	"The two-column proof provides a scaffold for students who will eventually write paragraph proofs."
Start 6%	Responses suggest that two-column proofs are an introduction to proof.
	"It provides an entry-way to proof writing."

Table 2 Themes Related to Negative Aspects of Two-Column Proof

Themes	Descriptions and Example Quotes
Procedural/ Algorithmic 15%	Responses focused on the procedural and algorithmic qualities of the two-column proof format.
	"The two column proof has nothing to do with real understanding - just provides an algorithmic format to approach something difficult."
Inauthentic 12%	Responses note that the process of writing a two-column proof is inauthentic because it is not a process in which mathematicians participate, nor does it result in an authentic proof.
	"It's a fake form of a real thing. I know of no mathematician that writes 2 column proof. It was invented to help students work into the real thing but it doesn't do that well."
Constraining/ Rigid 11%	Responses describe the two-column proof as limiting because of its constraining and rigid characteristics.
	"...It also limits creativity of expression and sends the message that mathematics is rigid and prescribed rather than flexible and robust."
Form 7%	Responses critique the two-column proof as causing students to focus on form rather than reasoning.
	"It can also lead to a confusion in which the form of the proof matters more than the content of a proof."
Misconceptions 7%	Responses describe how overemphasis on the two-column proof results in misconceptions about what a proof is.
	"Two-column proofs convince students that proofs must have a particular 'structure' and that they are procedurally defined constructs. They were historically created to achieve uniformity in student proofs and diminished the autonomy of teachers. They are detrimental to students' conceptions of what a mathematical argument is."
Not Communicative 6%	Responses describe the two-column proof as lacking in terms of communication.
	"The deficiency is that the two column proof restricts communication of reasoning (and thus I think also reasoning) to one particular simplistic form."

Table 3 Themes Regarding Neutral Views about Two-Column Proof

Codes	Description and Example Quotes
Variety 28%	Responses indicate that introducing students to a variety of types of proof is important.
	"I think it is one of several 'forms' of proofs that should be taught. Students should be allowed to use the form that makes the most sense to them."
The Need for Proof and/or Informal Justification 19%	Responses highlight the need for proof and communication in the curriculum, perhaps emphasizing informal justification.
	"The basic idea is that there should be proof, but it doesn't have to be two-column proofs necessarily."
Implementation and Perceptions 17%	Responses state that the ways in which teachers and students use and/or perceive the two-column proof determines whether it is useful.
	"It can be [valuable], it depends (like so much with proof in class) how the instructor views arguments, justification, and proof as well as their view and use of proof schemes.
Replacement 10%	Responses address questions or concerns about what tools, forms of proving, etc. would replace the two-column proof.
	"I think it would depend upon what took its place."
Geometry 5%	Reponses highlight the critical role of proof in geometry.
	"I think it is important to introduce students to proof in geometry because proving is an important part of doing geometry."

Negative Aspects. Regarding the negative aspects, the major themes were that the two-column proof is procedural and algorithmic, inauthentic, constrains student thinking, and that the form causes misconceptions. Consider this critical response:

> Two-column proofs emphasize form and layout rather than reasoning and logic. I think the drawbacks (e.g., proceduralization of proof, forcing justifications that are obvious or inconsequential to the main crux of an argument) far outweigh any benefits (e.g., consistent organization to ease grading, reminders to students to have a justification for every

> statement). I think it would be better if two-column proofs were NOT used. I do not think students should learn a format for writing proofs, I think they should just learn to convince skeptics and explain general ideas in mathematics. Two-column proofs lead to an emphasis on form rather than the ideas of proof. They also shift the practice of proving from "establishing and understanding a general claim" toward the less valuable "following the rules and being sufficiently detailed." It also limits creativity of expression and sends the message that mathematics is rigid and prescribed rather than flexible and robust.

Neutral Comments. In terms of major themes that were neutral in nature, we found that participants value proof, and they believe it needs to be part of the curriculum. Of these participants, some believe that two-column proof should be replaced with informal justification, and others stated that at least the two-column proof offered students a limited opportunity for deductive reasoning. Furthermore, some participants thought students should learn a variety of proof formats. Additionally, several participants observed that implementation and perceptions matter. Namely, the two-column proof may not be a problem in itself, but rather how it is implemented that is the problem.

> What's the point? If you remove it, what is the proposed replacement? If you keep it, what are the proposed benefits and ways that it can be made more effective. I think it really comes down to, are we teaching teachers how to use it as a tool for learning? AND much like so many education problems, can we really make any blanket statements about the use of two-column proofs in the US that are universally true?

Discussion

Members of the mathematics education scholarly community have varied perspectives on the two-column proof. Interestingly, we do not know of another mathematics topic, or more specifically tool for learning, that would receive such mixed reviews. It is surprising that one fifth of our participants said the format has no value, and one third of our participants said that they probably or definitely would be in favor of eliminating the two-column proof from the high school geometry curriculum. This suggests that some members of the community believe that the two-column proof's presence in classrooms is problematic. In most responses, however, we found that participants could see the advantages and the disadvantages of the format and were unsure about its place in the curriculum. The tension is captured in this response:

> I am not a big fan of two-column proofs, but I can appreciate that they give a format for making explicit the claim/warrant structure that underlies all proofs. […] I think that two-column proofs can be a useful tool for highlighting the underlying structure of proofs. However, they take on their own life in HS geometry as a particular format of proof that is highly over-emphasized. I wish there were more examples of proof and reasoning outside the context of two-column proofs. I think that important proof formats like proof by contradiction are somewhat clumsy in a two-column proof style. I also think that students need to work on communication skills and sometimes that means choosing when and how to introduce ideas, something that two-column proofs don't really allow room for.

While seeing the potential value in the two-column proof, the participant recognizes its limitations, particularly in terms of implementation. The format itself may not be the problem, but rather how it is used (e.g., as the only proof format, with deduction separate from knowing).

A teacher's perception of proof influences the ways in which the two-column proof is used in the classroom. As expressed by Herbst and Brach (2006): "the two-column format in which proofs are written is only the visible part of a much larger set of norms and dispositions and not by itself the culprit of the anomalies that can be observed in students' work" (p. 117). Just as Weiss et al. (2009) found that teachers interpreted a teaching episode differently according to their different notions of authentic mathematics, we believe that our participants' perceptions of mathematics influence how they view the two-column proof. There is a conflict or tension between different conceptions of mathematics. Some academics will contend that students need to learn that all statements need a justification, believing that this is the core of proof; whereas others will contend that students need more opportunities for conjecturing and informal proving because they understand such to be fundamental to the practice of mathematicians. In short, amongst the members of the mathematics education community, one's views of the two-column proof are influenced by one's perceptions of proof and the nature of mathematics.

The International Use of the Two-Column Proof

After presenting the results of our survey at a conference, some colleagues claimed that the two-column format is not used in other countries, so it should be possible to teach proof without using it. These claims, and the fact that several of our participants mentioned that they were educated in another country or had extensive experience with mathematics education in another country, led us to wonder about the role of the two-column format outside of the U.S.

Interestingly this question was not as easy to answer as one would think. First of all, we only identified one study (viz., Sekiguchi, 2002) that considered the role of the two-column proof outside of the U.S. Sekiguchi (2002) claimed that students in Japan do not write two-column proofs, and they would have much difficulty supplying reasons for every statement. Furthermore, we realized that while the Common Core Standards for Mathematics (CCSSM, 2010) does not mention the two-column proof format, it is a common practice in geometry classrooms in the United States (Herbst, 2002b). Thus, we found it difficult to determine if this practice was common in other countries based on curriculum documents. We concluded the most effective way to determine the prevalence of this practice would be to communicate with experienced educators from individual countries.

We contacted our mathematics education friends as well as other noted mathematics education specialists in different countries to gain some insight about the role of the two-column format outside of the U.S. We received about a dozen responses. Based on these communications, we found that in some countries (e.g., Greece and Turkey) the two-column proof is used by some teachers and not by others. In other countries the two-column proof exists only in textbooks that were translated from English, whereas the texts that originated from that country do not include this format of proof (e.g., Argentina). Occasionally, the underlying process of writing a two-column proof was used (distinguishing statements and reasons), but it did not occur in a two-column format, nor was it referred to as a two-column proof (e.g., China, India, Israel, South Korea and Russia). In other countries (e.g., South Africa) the two-column proof was fairly common practice, which seemed similar to the ways in which it is used in the United States. And, in other countries (e.g., Australia, Singapore and Serbia) it was not used. One of our participants, from Germany, wrote: "In my whole education in Germany I had never heard of such a thing as two-column proof."

We recognize that this international survey is informal and includes data from only a few countries and hope that a future study will build on these findings. Perhaps it would be interesting to compare and contrast students' views of proving in countries where two-column proof is used and those where it is not.

Implications for Future Research

We suggest that future research draws upon the specific findings of this study. For instance, many participants valued the way two-column proof's structure emphasizes statements and justifications, believed it help organize students' thinking, and considered it an effective scaffold to other forms of proof. These findings led us to ask, Does the two-column proof indeed help students organize their thinking as many of our participants stated? Does it provide a useful scaffold to paragraph proof? Or not? The current research base generally does not help us answer these questions. Indeed, Weiss et al. (2009) suggested the two-column proof can play a productive role in supporting students' reasoning, but the current research base does not address the specific affordances described by our participants. Similarly, some of our participants claimed that the two-column proof form was algorithmic, constraining, and may cause students to focus on form rather than reasoning. Are these claims, and others like it that we see in the literature, justified? We believe that new research in mathematical proof needs to investigate the possible benefits, and also limitations of the two-column proof.

Based on the fact that many of our participants (28%) advocate for instruction that includes a variety of proof formats we might consider the ways in which different proof formats interact, augment, or hinder students learning. Advocating for a variety of proof formats to be used in the classroom is consistent with calls in the literature (e.g. Cirillo & Herbst, 2012). We believe, as many of our participants do, that the ways in which the two-column proof is implemented influences whether it is a productive practice. Therefore, an avenue of research that would build on these findings might consider questions such as: In what ways do different forms of proof augment and hinder students' learning? And what aspects of students' learning do different proof formats support? For instance, we imagine, as many of our participants suggested, that the two-column format supports students in organizing their ideas, whereas the paragraph format may not have the same positive effect on students' thinking. The results of this study may be the foundation for future investigations that might reveal the pros and cons of using the two-column format in a classroom setting.

Limitations. The purpose of this study has been to understand mathematics education researchers' opinions on and attitudes towards the two-column proof. As all of our participants had recently attended mathematics education conferences, the majority were university professors. We recognize that we have not accounted for several groups of stakeholders in the mathematics education community: classroom teachers, students, and policymakers. Another avenue of research might study these different stakeholders' views of the two-column proof. Ultimately, we believe that academics and teachers must work together to ensure that proving is a meaningful practice for students in schools.

Another limitation is that our study did not discover the differences in opinions between different groups of the academic community. For instance, do mathematicians generally view the two-column proof one way while mathematics teacher educators have a different perspective of two-column proof? Or do experienced members of the academic community have a different view of the two-column proof than those entering the field? These are questions that could be interesting to consider in a future study.

Conclusion

Any suggestion to deemphasize or eliminate a practice of school mathematics, such as two-column proving, would need consensus and the backing of a significant portion of the mathematics education community to become reality. Our research revealed that such a consensus does not exist. As a field we are divided on the issue of the two-column proof. Thus, we need to engage in more discussions about the format. These discussions need to be had both within the academic community and with students and teachers. In the future, we also need more specific research targeting the effectiveness of the two-column proof or alternate proof formats that support students in meaningful mathematics learning.

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