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Using Records of Practice to Bridge from Teachers’ Mathematical Problem Solving to Classroom Practice

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Using Records of Practice to Bridge from Teachers' Mathematical Problem Solving to Classroom Practice[1,2]

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Abstract: It is often the case that high quality mathematics education professional development involves enhancing both teachers' content knowledge and their pedagogical skills, specifically for teaching mathematics. However, when teachers are immersed in their own learning of mathematics, they are often unaware of the facilitators' instructional decisions and moves that influence their own learning outcomes, as well as how these might apply to their future teaching. Thus, while the teachers can appreciate their new understanding of content, they may not have added significantly to their understanding of the instructor's pedagogical moves that facilitated their growth. As a result, teachers may leave even high quality professional development without assurance that they will be able to adjust their teaching in ways that support their own students' meaningful learning of mathematics. In this work we describe one way in which professional development can both enhance teachers' subject matter knowledge <u>and</u> help to transform these new understandings into pedagogical content knowledge; the mathematics content sessions provide the platform for reflection on pedagogy. To facilitate this reflection, a "record of practice" is created by facilitators, and thereafter utilized for participants and facilitators to identify and analyze critical moments in the mathematics content session. This paper offers two specific examples of records of practice and how they were used, as well as teachers' reactions and insights. It also discusses various formats of records of practice, the logistics of developing them, and ends with the potential benefits of using records of practice in professional development for teachers.

Keywords: Mathematical problem solving; Mathematical knowledge for teaching; teacher practice; pedagogical content knowledge

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Introduction

A large group of teachers had just finished a 2-hour problem-solving session facilitated by the mathematician as part of a professional development program. The mathematics educator began facilitating a reflection on the session by asking the participants: What was your instructor doing while you were working on the problem? The teachers' initial response: "Nothing…" After a bit of laughter and some reflective silence, some teachers tentatively began offering up other thoughts: "She asked what (an aspect of my work) meant"; "she encouraged me to continue developing my calculation"; "she suggested that I consider additional approaches"; etc.

It makes sense that greater content knowledge should correlate positively with better instruction; after all, one cannot teach what one does not know. On the other hand the relationship between the two is not entirely obvious, and quite a bit of work has been done to identify, define, and study "mathematical knowledge for teaching " (MKT) (Hill, Rowan, Ball, 2005) and its relationship to instruction. If increased MKT automatically led to improved instruction, one might depict the relationship as follows:

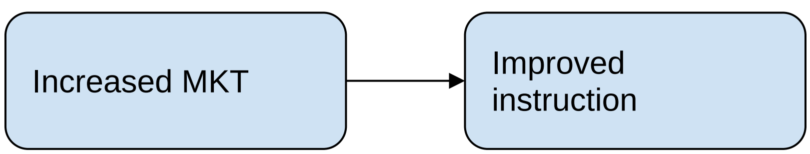


Figure 1. Direct relationship between MKT and instruction

However, as the dialogue in the first paragraph illustrates, when teachers are immersed in their own learning of mathematics, they are often unaware of the facilitators' instructional decisions and moves that influence their own learning outcomes, as well as how these might apply to their future teaching. These issues are part of what Lewis (2007) calls the "invisible work of teaching". Many aspects of the work of teaching are actually visible to the eye but escape notice nonetheless. Even when veteran teachers experience a professional learning session, the facilitators' specific teaching actions are often difficult for them to recognize or understand their effects. Thus, while the teachers can appreciate their new understanding of content, they may not have added significantly to their understanding of the instructor's pedagogical moves that facilitated their growth. As a result, teachers may leave even high quality professional development without assurance that they will be able to adjust their teaching in ways that support their own students' meaningful learning of mathematics. This attention to teachers' content understanding and how this knowledge might or might not impact their pedagogy has not typically been a focus of research (Schifter, 1998; Garet, Porter, Desimone, Birman, & Yoon (2001). Schifter (1998) suggested that there is a need to investigate the kinds of understanding that teachers need in order to enact new teaching practices and how those understandings connect to their pedagogy. As he states, "Ultimately, students benefit when teachers connect their mathematical understandings with appropriate pedagogical moves (Schifter, 1998). Thus, the diagram in Fig. 1 tells only part of the story; we modify it to include pedagogy:

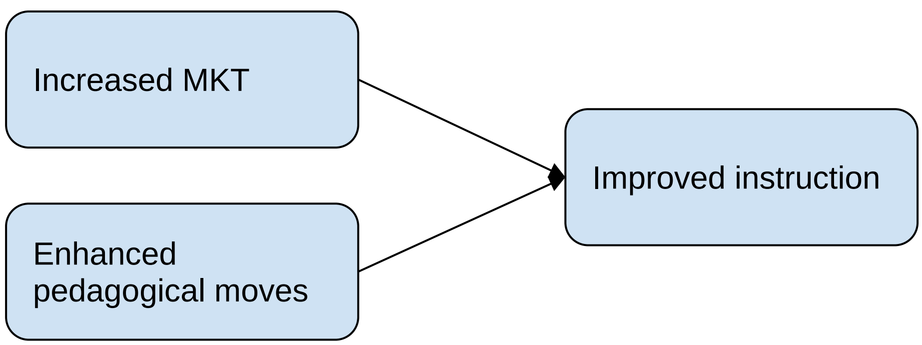


Figure 2. MKT and pedagogy improving instruction independently

However, Figure 2 seems to imply that "Increased MKT" and "Enhanced pedagogical moves" are independent of each other and are each learned separately from the other. We contend that enhanced pedagogy is not independent of content, but both supports content and builds on it; growth in pedagogy is a mediating factor between increased MKT and improved instruction. This philosophy is depicted in Figure 3:

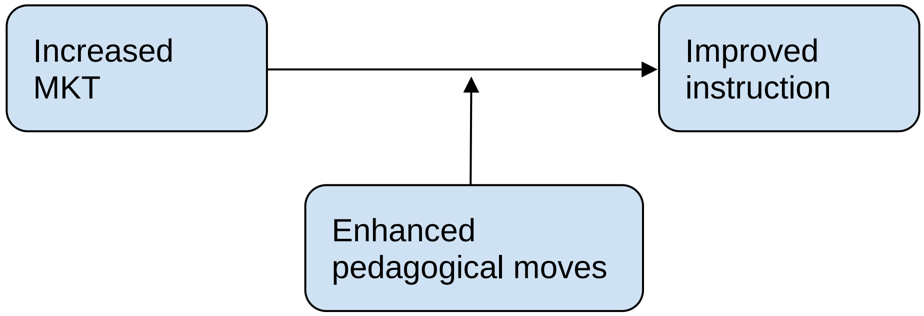


Figure 2. Enhanced pedagogy supporting increased content in improving instruction

Clearly the goal of increased content understanding for mathematics teachers is a valuable aim of mathematics teachers' professional development. What we are proposing in this work is one way in which professional development can both enhance teachers' subject matter knowledge and help to transform these new understandings into pedagogical content knowledge, "the ways of representing and formulating the subject that make it comprehensible to others" (Schulman, 1986; Ball, Thames, & Phelps, 2008). It is our premise that it is possible--and in fact preferable--for mathematics teachers' professional development to support increased teacher content understanding as well as enhanced ability in pedagogical skills for conceptual learning and development of mathematical practices, with the mathematics sessions providing the platform for reflection and pedagogical growth. Engaging teachers in problem-based lessons can serve as the basis for both types of learning if a purposeful focus on both content and pedagogy is integrated. As Schifter (1998) stated, "...as participants explore mathematics content, the mathematics lessons themselves provide grist for reflection." Problem-based lessons provide in addition to deep content learning, particularly rich material for reflection, as they actively support also development of the "habits of mind of a mathematical thinker and problem-solver" (Conference Board of the Mathematical Sciences (2012).

In providing professional development experiences, we aim to design and implement a flow from teachers engaging in problem-solving and content development to pedagogical development that begins with a reflection on "what did I need as a learner?", continuing to "What did my facilitator(s) do to support my learning?", and finally to a discussion of "What does this mean for me as a teacher?"

In order to engage in meaningful reflection, teachers need an opportunity to identify and analyze critical points in the design and implementation of the session from the perspective of an instructor. In this paper we describe one way to create a "record of (PD) practice" (Ball, Ben-Peretz, & Cohen, 2014), and suggest ways to use this record for teacher reflection with the goal of growth in both mathematical knowledge for teaching and in pedagogy for teaching mathematics.

The project's use of scripting and photos of participant work completed in summer 2013 developed in response to participants' feedback and our own need to reflect on the sessions; subsequently we noted the Ball, Ben-Peretz, & Cohen (2014) paper which has an in-depth analysis of records of practice as a means to "support public discourse about education" and to "make possible special opportunities for engagement with practice, and, in particular, for the construction of collective professional knowledge." The research described in that article focuses primarily on records of practice of K-12 instruction, but we note that the records used in our PD share many of the same characteristics: the combination of a script of the session and photos of participants' work provides a record of practice that is structured, highly detailed, concrete and specific, not guided by theory, and of necessity, incomplete. This record of practice provided a helpful resource for participants' reflection on pedagogy and how it might apply in their classrooms.

Project Context

The sessions described below took place in the context of a 5-year NSF-funded professional development project involving a total of 116 teachers of grades 4-8. The teachers participated in a 2-week summer institute, academic year monthly seminars, monthly self-facilitated learning communities, and 10 days of lesson study annually. The summer institutes were originally designed to focus primarily on developing mathematical knowledge for teaching (MKT) through inquiry and problem-solving, with the academic year activities meant to focus on a more even balance of content and pedagogy.

Participant feedback and our own observations prompted us to modify the plan in order to incorporate more discussion of pedagogy in the summer institutes. We wanted to do so in a way that was tightly linked to the mathematical content of the session, and in a way that supported participants in analyzing the effects of facilitators' pedagogical decisions on participants' own learning.

The process we utilized for this purpose was as follows:

1. Participants engage in problem-solving around carefully selected content (as originally planned); the facilitator is a mathematician or a mathematics educator;
2. A mathematics educator scripts the session, and a third facilitator takes pictures of participants' mathematical work as it happens;
3. After the problem-solving session, the mathematics educator reviews and analyzes the session in order to select critical decision points in the design and implementation of the session (this step varied based on the goals of the analysis);
4. On the following day, the mathematics educator conducts a debrief and analysis activity with the participants, designed to help them analyze the session's work from the perspective of the facilitator. To do so, she utilizes a script of the session with images of participants' work.

The results were fascinating: participants and facilitators alike found the debrief/analysis activity to be very useful: for teachers, it helped them understand more about how to implement a problem-solving lesson in their own classrooms; for facilitators it helped them understand participants' perceptions of the session and how they might improve on their facilitation.

As a lovely side-benefit, the activity served to broaden and deepen the collaboration between mathematicians and mathematics educators, to help them understand one another's views, and to serve as a springboard for further collaboration.

Below is a description and analysis of two such pairs of problem-solving session and debrief/analysis session. We used the following notation: T for teachers (the participants in the professional development), M for Mathematician (facilitator of the problem-solving session), E for Educator (facilitator of the reflection session). The quotes of participants' and facilitators' discussions are taken from the video of the sessions, and their transcripts.

Content and Pedagogy: Painted cubes problem

Description of the problem and motivation

A well-known problem is the "painted cube" problem: Suppose you have a cube constructed of centimeter-cubes, with dimensions $4 \times 4 \times 4$, and you dunk this cube in a pail of paint. When you remove the cube from the paint, how many of the centimeter-cubes will have paint on at least one side?

This kind of problem is frequently used as a springboard for a problem-solving session with important mathematical content, including topics such as algebraic expressions with and without variables and equivalence of such expressions. We wanted to utilize the problem and its generalization to an $n \times n \times n$ cube in this way, but also for two additional goals:

- To develop a transition to abstract thinking, and
- Subsequently to the problem-solving session, to facilitate teachers' analysis of pedagogy of the lesson, to consider when such pedagogy might be appropriate, and to support teachers' ability to utilize this sort of pedagogy.

Approach to the instruction

In previous sessions, we had encouraged teachers to use a variety of manipulatives to represent the problems, to develop an understanding of the problems, and to experiment with possible solutions. Scaffolding included:

- Providing many types of manipulatives to help solve problems by representing the problem and solving it through working with the concrete objects such as cubes, chips, dice, tape, and construction paper.
- Asking teachers to first solve problems for small numbers that can be represented by the manipulatives available, and then for larger numbers that are not easily represented by the manipulatives available, as a step toward abstraction

- Asking teachers to make drawings of their concrete representations, as an aid to visualization and a step toward thinking abstractly about the problem.

In this session, we wanted to help teachers to deepen their abstract thinking skills by developing the ability to imagine the concrete representations in their minds, and to utilize these representations to think about solutions. In order to do so, we did not provide any concrete objects, though teachers were still free to use colored pencils and paper. To extend their thinking more abstractly, after considering the problem for a $4 \times 4 \times 4$ cube, teachers were asked to solve the same problem for an $n \times n \times n$ cube. We note that the teachers had used physical cubes for other problems in previous years, and the removal of this tool was only temporary to support further development. Ultimately, we expect teachers and students to use any tools at their disposal to solve problems.

Implementation: The problem-solving session

The session consisted of three parts: a warm-up activity, the first problem which involved a $4 \times 4 \times 4$ cube, and its generalization to a $n \times n \times n$ cube.

Warm-up activity.

The warm-up to this session asked teachers to view images of a $3 \times 3 \times 3$ and a $4 \times 4 \times 4$ cube and to consider what was the same and what was different between the two. No manipulatives were provided. The intent here was to develop some abstract thinking, to make connections between a 2-dimensional image and the 3-dimensional object it represented, and to encourage a variety of approaches to thinking about the cubes, thus establishing a starting point for participants to think about the main problem of the session from a variety of perspectives.

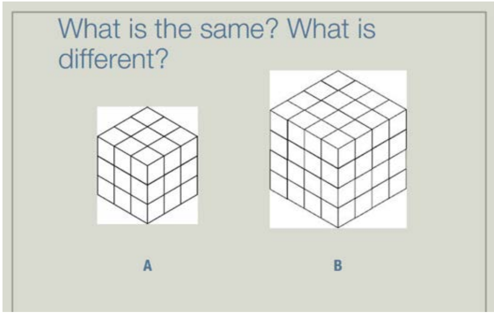


Figure 3. Image projected for warm-up activity

First problem.

M: "For the following problem, put your pencils/pens down. Respond without writing, without talking, and without counting one by one, consider the following problem.

You have a $4 \times 4 \times 4$ *cube made up of unit cubes, and you dunk it in a pail of paint. When you take it out, how many unit cubes have paint on them?*

INDIVIDUALLY: Write down your answer and your reasoning.”

Subsequent to working individually on the problem, M asked the participants to share, compare, and contrast their strategies in their small groups, and then facilitated a whole-group discussion sharing and comparing strategies.

While the participants were working on their solutions and sharing solutions, one of the facilitators took pictures of their work, and another, the Mathematics Educator, scripted the conversations. Below are some excerpts of the whole-group discussions that took place after the table discussions, along with images of teacher work.

One teacher realized her improved ability to imagine concrete representations in her mind, working from the outer layer of cubes to the inner sections:

T: (See Figure 5.) I usually can’t work without the object in front of me. This time, I looked at the picture of the cube and imagined it in my mind. That was a big leap for me. I think I was able to do it this time. The top and the bottom have 16 squares in it. Then I went to the sides. On the sides, what hasn’t been touched by paint were 8 that hadn’t been touched by paint. On the other sides, there were 4 that weren’t touched by paint. This is what they did, but I did it in my head…What was left, were 8 in the middle…The other sides, 4 and 4.

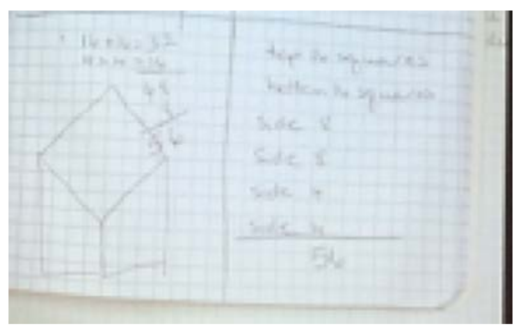


Figure 4. Participant work: abstraction.

Another teacher decomposed the image differently, visualizing cross-sections:

T-(See Figure XX.)Red represents the paint. We can think of the 4 stacks of cubes.

M-Tell me what you’re counting.

T-There's a total of 64. Looking at the sides. This particular cube is painted. (Colors in all visible parts of cube.) This cube is painted…Here are the cubes that are not painted. (See Figure XX.)

M- Look at that. They both got to the same point…The last student peeled the layers off like an onion. This one took cross sections…

T-We know the whole top and bottom layers are already counted because they're painted…Looking at the second row, this row is counted…Just the 4 cubes in the center are not.

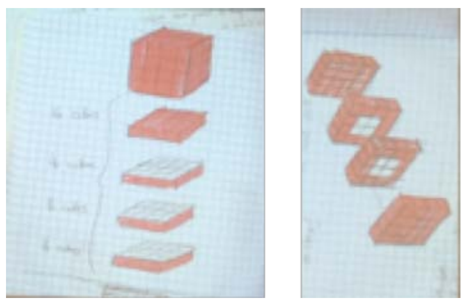


Figure 5. Participant work: layers.

A third decomposition was demonstrated by another teacher, in which a misunderstanding was depicted and explored through discussion:

T-(See Figure 7.) We were way off. We misread the question. We saw it as how many full cubes rather than faces…

M-Bring what you did. Let's see… You only painted faces of the cubes, not entire cubes…

T-We figured out on this first piece here (top layer), each corner has 3 faces that are painted (blue). There were 32 on this top…96 faces divided by 4 which gave us 24.

M-You have all the ingredients to answer the problem. How would you do it now?

T-Why did they divide by 4?

T-They have…Each cube has 4 faces.

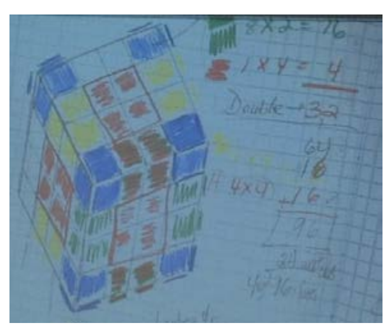


Figure 6. Participant work: corners and sides

M-Oh, there is the explanation…Should be 6 faces.

(M writes whole group's calculation of the actual problem from this group's picture.)

Generalized problem and wrap-up discussion.

Following the decomposition and discussions of the $4 \times 4 \times 4$ cube, the facilitator posed the same problem for an $n \times n \times n$ cube. We utilized the general problem both to highlight aspects of the problem-solving process and to discuss mathematical concepts and practices that appeared during the work.

After some small-group discussion about the $n \times n \times n$ cube, the group as a whole raised and discussed questions arising from the attempt to generalize. Some examples of these questions:

- What does it mean to "generalize" a formula or other mathematical result? What aspects of the problem stay constant, and what aspects change under the generalization attempted here?
- How can we represent a general $n \times n \times n$ cube visually?
- What (if any) is the relationship between size and shape?
- How might changes in size and shape influence the results?

Eventually, the group developed two expressions for the number of cubes with paint on at least one side:

$$n^3 - (n-2)^3$$

$$2n^2 + 2(n-2) \times n + 2(n-2)(n-2)$$

These in turn contributed more food for thought to our ongoing reflection on the meaning of equivalence, particularly in the context of algebraic or arithmetic expressions.

The session was then wrapped up in whole group discussion, highlighting mathematical concepts and practices:

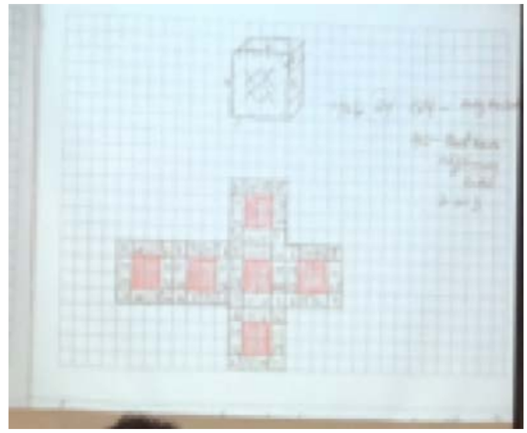


Figure 7. Participant work: diagram/net

Visualization and imagination: teachers were asked to solve the problem with no concrete objects, but were expected to make the transition from a verbal description to a visual image in their own ways. As a result, we saw a wide variety of approaches to the problem, as described above. Some of the teachers commented on this explicitly, for example:

T-: I had a hard time visualizing this, so I drew a diagram.

Equivalence (of arithmetic and algebraic expressions): Throughout the discussions of the $4 \times 4 \times 4$ cube, M had been collecting teachers' calculations on the board (see Figure 9). She referred to the list and subsequently to the algebraic expressions the group had developed, and asked the teachers to reflect on connections. Ultimately the group concluded that all the arithmetic expressions were equivalent to one another, and the two algebraic expressions were equivalent to each other.

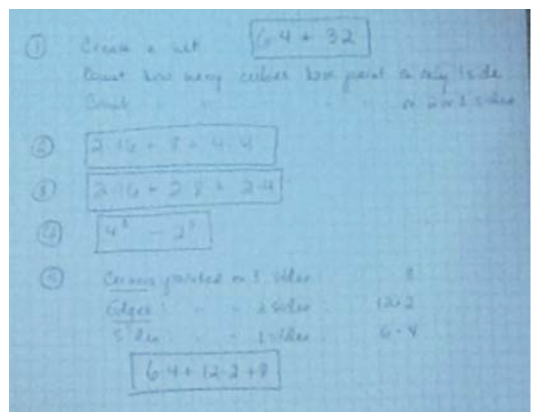


Figure 8. Data for equivalence discussion.

Scaffolding: Scaffolding for problem-solving is a tricky balance between providing sufficient support so that students will persevere, but not so much support that a problem becomes an mere exercise. The group discussed how our structuring of the work (for example building up from a simpler problem, collecting, sharing and analyzing the structure of and relationship between multiple solutions) supported their thinking about the general problem and allowed them the opportunity to solve a problem that otherwise most would likely not have been able (or willing) to attempt.

Pedagogy: The Analysis and Reflection Session

Preparation

Our focus on pedagogy took place the following day. In preparing to utilize the painted cube lesson as the foundation for reflection, we reviewed the teachers' written work and also a finalized version of the script of the session, which included images of work shared by various teachers and the facilitator during the session. Our review of this record of practice helped us, as instructors, to identify critical moments from the lesson that could prompt teachers' reflection on the lesson's pedagogy and its relation to their own learning. In their analysis of the critical moments, teachers were encouraged to use their own notes from their own problem-solving work and from the whole-group discussion during the painted cube session, thus utilizing a second type of record of practice for reflection.

Two critical instructional moments from the session were selected by the mathematics educator (E) as prompts for teachers' consideration:

1. The instructional planning decision to have teachers work on the problem without the use of concrete objects in this lesson.
2. Teachers' own recognition that they were misunderstanding the problem.

Each of these topics was presented with specific discussion prompts that included quotes from the script. Each topic, in turn, was presented for consideration and discussion facilitated by the educator, who also invited comments from the mathematician.

Session Analyzing Pedagogical Connections to the Content (Problem-Solving) Session

Critical Moment 1.

The first critical moment was projected with quotes from the script and question prompts to allow for teachers to consider why manipulatives were not allowed in this lesson:

> M (in presenting the problem): "We purposely did not provide you with cubes. Keep in mind that this is on purpose."

- How did this approach help or hinder your learning?
- Why do you think "M" made this instructional decision?

As described above, the teachers were accustomed to having access to concrete materials in our professional development sessions, and many valued them as a learning tool. We wanted to use the omission of materials as a means to promote participants' ability to think abstractly and then to generalize in the last phase of the lesson, and to help them consider prior experiences as they imagined the cubes in this problem. In this part of the follow-up reflection session, we wanted teachers to reflect on this instructional decision, its purpose, and to consider a variety of options for planning and implementing any particular mathematics lesson. Through reflection on their own learning in this session, we hoped that they would recognize how our learning purposes had guided our instructional decisions, and consequently how they might use learning goals as a driver of both pedagogy and content in their own teaching

After teachers had discussed their reactions to these questions within their small groups, the educator asked them to share their thinking.

> E: "...Here, she didn't let you use manipulatives. Why would she make that decision? It would've been a lot easier just to count it…"
>
> T: "It makes it harder to generalize after that."
>
> T: "...Yeah, because you just come up with the solution...She forced us to think about, why is this? If you count..."
>
> T: "I would've just come up with a solution...I wouldn't have even had to think about the "why,"...it might be that M really wanted us to think about the "why" to force us to visualize, to have an understanding and then take it to another level, which is all this generalization wonderfulness, but for us to feel what the students might feel when they need to express and see the patterns and see some connection to explain the "why." It's not just about getting an answer. I think it's more than that. It's about understanding how to get the answer."

> T: "I actually think there's even more to it than that...Algebra is more abstract, so if you do it for yourself, each person has to come up with their own representation... but I think it's great, too, that each person can go ahead, and [consider] how could they represent it. We saw different kinds of representation in drawings...a little bit of their own individual thinking process, but I do think that you really have to figure it out. You may have to do it more than one way if it's an abstract pinnacle piece... seek to the algebraic thinking."
>
> T: "...one of our themes this week is relationships, so putting it all together abstractly, you were trying to get us to make relationships. When you were saying, "Take the bookends off," I needed to hold that in my mind, right? Instead of individually counting, so I needed, and am forced to pay attention."
>
> M: "Right, so the relationship between the size of the bookends, the size of the longer ends, the size of the shorter ends, we have a lot of relationships going on here, which is also what makes it hard. You have a lot of things to keep in your mind to put all of this together, and sometimes I go right for the pencil... Even when we think about writing an essay or a report or a letter, some people, it all goes on in their minds, and then they put it down and it's perfect. I can't do that. I write it, then I rewrite it, and then I rewrite it, so I rewrite it. I do my math that way, too, but I think that's a drawback because I sometimes have to force myself just to think about things, put them in places that—think about them differently. It helps you get into a different way of thinking that can help stretch your mind."

This discussion sequence demonstrates that teachers were able to extrapolate from the record and then to articulate many of our intended purposes for not allowing them to use manipulatives in this lesson. Teachers saw that using blocks to solve the 4x4x4 problem might indeed lead to a solution of that specific problem, but might not necessarily lead to sufficient understanding of the "why" in order to be able to solve for other sizes of cubes, or the general nxnxn problem. They understood that constructing their own visualization required an understanding of that "why" and was a means to more abstract thinking, and to recognizing relationships and their connections to generalizations--a goal they recognized we'd been addressing in prior sessions.

Responses in teachers' written reflections on the session indicate that these insights into their own learning connected to teachers' plans for their own students. Some sample responses to the prompt "How will you use this in your teaching?" were:

- *Attempt to allow students to develop the ability to think abstractly. Have some days where they use manipulatives and some days when they don't.*
- *I learned that sometimes providing concrete models can hinder critical thinking. For example, if we provided cubes for the 4X4X4 problem, we might just count the cubes rather than come up with the many ways of finding the answer. I need to start thinking what I want my kids to get out of the lesson and choose my tools to support it rather than start with manipulatives and wrap my lesson around them.*
- *Allow time for students to work out problems, and discussion is vital.*

Critical Moment 2.

We projected the following critical moment prompt to encourage teachers' recognition that some of them had misunderstood the question in the cube problem, as evidenced by a teacher's statement from the script. We were hoping that this part of the discussion might lead to teachers' consideration of ways that they might try to prevent some misunderstandings (and possibly elicit misconceptions), check for understanding, and respond to misunderstandings in ways that increase understanding. Teachers were encouraged to access their notes from their work during the problem-solving session as a record of practice regarding the wording of the problem, their entry into the problem and their understanding of the problem question.

> T: "We were way off. We misread the question. We saw it as how many full cubes rather than faces…"

- What happened here?
- How did M turn misunderstanding into a learning opportunity for the small group? For the entire group?

> T: (After discussing at tables and referencing notes) "...we think of a cube as one unit with different faces; top, bottom, and sides, so until, I think, people began to draw and to think about the complete situation, too—it's situational. You're linking these, so is the entire cube what I'm thinking of as one? No, so I think it's just trying to get you to break it down and think about it."
>
> E: "Why didn't M realize that you were misunderstanding as she was walking around?"..."She couldn't capture that from what she was seeing you draw or write. She was trying to see that everybody was with her. Then we come up to the point where people are sharing, and people were answering a different question, right?...Don't you want everyone to know the question before they start to work on it?"
>
> T: "Yes, we do."

The teachers and the mathematician discussed various aspects of what may play a role in a teacher not realizing that someone has misunderstood a problem's question. For example, having a large number of students in class makes it difficult for the teacher to assess each student and group. They also discussed a teacher's responsibility in assuring that students understand the question versus students' responsibility in asking for clarification when they don't understand. Teachers related this to their own efforts in lesson study (Lewis, Catherine & Tsuchida, Ineko., 1999) to design questions that had little ambiguity and could be understood by all their students, only to find that some students still did not understand. They noted that (similarly to themselves) sometimes these students were not aware that they were misunderstanding the question, and, consequently, did not seek clarification. They seemed to agree, that although they felt a responsibility to check for understanding and assure that all students were spending their time on the correct question, sometimes, despite their best efforts, some students still move through a lesson without adequate understanding of the question.

The educator then transitioned the discussion to prompt consideration of how to create learning opportunities in response to students' misunderstandings.

E: “Then that second question, how did she turn misunderstanding into a learning opportunity, not only for the group, but for all of you? She didn’t just say, “Oh, well, you all misunderstood the question. You did the wrong thing,” and then move on to other people who were with her. How did she use this as an opportunity?”

T: “She took it, ‘Okay, skip one word to interpret it that way. Let’s see if your solution works.’”

E: “How did that make the people feel who had to deal with this?”

T: “We were happy to do it.”

E: “I was trying to put myself in their place, and I thought, ‘Wow, she’s validating the work we did, even though we weren’t on what she wanted us to do. We spent all this time, and then she took it a step further, where not only did she let them talk about how they were interpreting the problem, but then trying to solve (sic) it in light of a new [correct] understanding of the question.’ Right? She brought them all on track. What’s the value of that?”

T: “She didn’t make you feel stupid. If she would’ve said, ‘Oh, well, that’s not the right way. Here, this is what you’re supposed to do…’”

T: “Those students, the next time, I don’t think they would’ve had the courage to say, and you have to engage your students.”

T: “I think that it would promote a community where students feel comfortable...In perceiving the question that way, she validated back when they did that math chart, as well. I think that’s a way to encourage some people that you rarely hear from...I think I would use this method. I think it would make you a little more comfortable sharing your work.”

T: “To add on to that, I think it’s really important to acknowledge, I mean for everyone, in your thinking, you’re not all wrong or all right. There’s parts of it that are right and parts of it that—it’s not that they’re wrong. You just are off a little bit. In critical thinking, what you have to do is take it apart, and what part do I have that’s okay that can stay as is, and what part do I need to rethink again? I think especially our students are so black and white. I'm all wrong. I'm all right."

T: "That's a huge lesson for everyone, for our kids, especially, to learn. For example, what parts do I have that are okay? Cuz T had many parts that were okay."

M: "That is something that we've discussed before, which is that—remember sometimes in seminars we've shown you a video and I've asked, "What understanding did the students have?" Not what did they get wrong, but what did they understand? You want to look for those understandings, so that you can build on them. You want to look for what's right, instead of [just] pointing out what's wrong."

"Then the other aspect that I was thinking of—I think T was speaking, and T, I think—was our quote from two days ago on ideas, and why I thought that was so important. Ideas are fragile. We want to nurture ideas, and that comes from validating thinking, validating correctness."

T: "When we started moving to these [types of problems], and I knew they were gonna require reading, I knew that that was kind of what we were talking about earlier, about that they're so used to...just right or wrong, so they're not going to talk...I started doing, thinking, "How am I gonna change that?..." ...Instead of saying right or wrong, I just started taking those things purposely, and then, afterwards, say, (sic) "Thank you for the opportunity to learn." I would thank them, and all of a sudden, "She's thanking me for doing something wrong?" All of a sudden, the whole dynamics in the classroom started to change, and they were willing to just take a risk and say it. For a while there were a lot of mistakes they were making, but, eventually, the mistakes dwindled down because the thinking was altered, so I'd thank them for it."

T: "...It's just today that I thought back to when we started [this project] in 2010, and...you gave us that YouTube video we watched. The Japanese students, they were talking, and one didn't know, and they were helping each other. It was okay cuz it was community, and it's such a different way than we taught our kids to learn, and that's what

> was happening. It was okay for those students to be wrong. They would say, "Oh, no-no-no," and they would show each other where they made a mistake or make a suggestion, and it was normal. The kids were not afraid…"

This dialogue suggests that teachers recognized many benefits in M's use of some teachers' version of the problem as a basis for discussion. As learners, they felt encouraged and validated by her recognition of their attempt. They weren't embarrassed by their error but instead seemed to be pleased to have their thinking and their work shared with others. It appears that the interaction helped them reconsider what some believed is the common practice of calling on students to share, only when they have the right answer, or, at the very least, are on the right track. They recognized M's instructional move as one that was new to them, and one that they might pursue with their own students in the future. Foundationally, they saw its value in helping them to create a safe and encouraging learning environment for their own students' problem-solving work.

This segment of the pedagogical lesson seems especially strong in that teachers engaged in rich discussion related to students' misunderstandings of a problem's questions, including:

- Prevention of many misunderstandings through careful wording of problems and their questions,
- The importance of checking for student understanding while proceeding through a lesson,
- A teacher's responsibility for checking on students' understanding versus students' responsibility for letting the teacher know when they don't understand,
- The value of using student misunderstandings as an opportunity for learning,
- Ways to use student thinking to help them be part of a mathematical community, acknowledging that this involves also risk-taking and incomplete thinking, and
- Reflection on teachers' current practice for selection of students to share their thinking, and the role that correct and incorrect student answers play in selection and sequencing of students who share.

Teachers' written reflections on the session provide support that these discussions might lead to change in classroom practice:

- *Making mistakes is ok. Let students make mistakes. Develop a safe environment where it is ok to make mistakes.*
- *Persistence, but knowing it's ok to be wrong...I think the power of persistence has been my overall aha this week...This is what I will take back to class.*
- *I saw M work real hard to clear up all our misconceptions. She didn't just move on because of time. Now I'm determined to take more time with my students no matter if there is a Benchmark test that covers more standards than I've been able to cover.*

Clearly, this discussion resulted in a greater appreciation of many pedagogical moves, including questioning, formative assessment, and responses to student misunderstandings. As instructors, we also gained additional insight into the teachers' content understanding. Even within this rich discussion, some teachers did not seem to grasp that the question they pursued was not a misinterpretation of the problem, but a mistaken understanding of the problem's language, hence of the question. Thus, out of this "pedagogical" session, we recognized that content misunderstandings still existed that we would need to address in future sessions. This provides further evidence of the ever-present dance between content and pedagogy.

A Second Example: The Spanish Flu Problem

In the analysis session described above, the facilitator selected the critical moments upon which to focus the discussion. However, the script could also be provided to teachers in its entirety, with the goal of the teachers identifying critical moments. Below is a brief example of such a use of the script.

The purpose of this problem-solving session was to introduce the topics of mathematical modeling and making sense of graphs describing change. Teachers were provided a mathematical modeling problem which was introduced with a bit of role play: The facilitator introduced "our cousin from Italy", who had a problem which she posed to the class:

> *My brother got the flu and gave it to two other people…After 24 hours, two more people got sick! Now it's on its way to Rome, where I live…I don't want to be sick! I don't want a shot! One person gets sick, and then 2 more people get sick…It lasts for 24 hours, then they are cured again…How long must I stay in my house to try to not get sick? What if I have to leave my house—how many people around me are going to be sick? If people around me are getting shots, how many people need to get shots?*

Teachers then were given time to discuss what they would need to know in order to answer the "cousin's" questions, and to ask clarifying questions. They asked questions such as: how long is a person contagious, how many people are infected by any sick person, and so on. Based on these discussions, but without actual data, teachers sketched preliminary models of how they think the flu might spread. Eventually the group decided on the following assumptions in order to create a model:

- On day 0, only one person is infectious. That person infects two others during the next 24 hours.
- Once infected, a person will be infectious for 24 hours.
- During a 24-hour period, an infected person may infect up to two other people.
- Once recovered, a person who has been infected will no longer be susceptible to the flu.

At that point the facilitator provided a brief historical background regarding the world-wide devastation wreaked by the Spanish flu during World War 1. She then modeled how to collect data on the spread of the flu, and led a re-enactment based on the assumptions the group had agreed upon. The enactment led to additional questions that were discussed, for example: once someone had been infected and recovered, were they still part of the population? Answer: Yes, therefore the probability of being infected is reduced.

Once the data collection was completed, it was graphed and analyzed based on the following questions:

1. What is the greatest number of students who might have been in the nurse's room in one day?
2. When was the flu spreading the fastest?
3. When and why did it start to slow down again?
4. Did everyone get sick? About what percent of our population didn't get sick? That's something that we can actually use to make a prediction for a larger population…

The resulting graphs are provided in Figure 9:

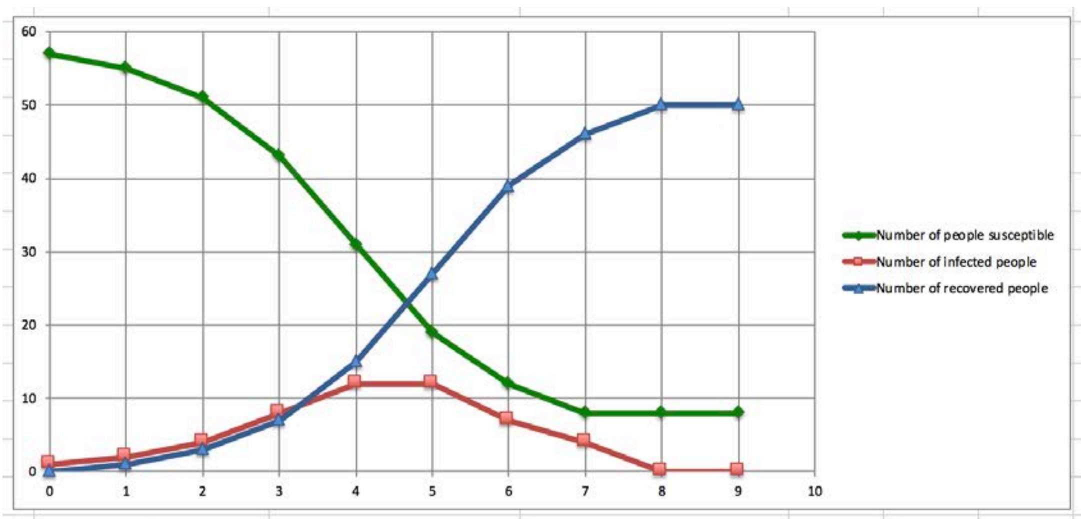


Figure 9. Spanish Flu epidemic graphs

The sigmoid graphs were unfamiliar to the teachers, which led to a discussion of some of its properties. The facilitator noted that although the function rules that describe sigmoid functions are algebraically complicated, the process of collecting, graphing, and interpreting data is accessible to students who are just beginning to explore functions. Sigmoid models are sufficiently complex to allow for multiple levels of problem-solving opportunities.

This type of lesson is even more free-flowing than a “regular” problem-solving session, and was fairly challenging to script. The goal of the script was to capture the major lesson components as well as much of the general discussion, in sufficient detail so that teachers could implement a similar lesson in their classrooms.

Pedagogy: The Analysis and Reflection Session

On the following day, the teachers received the script and were prompted to review the section in which they’d discussed their problem attempts with the facilitator. They were to consider which aspects of the discussion were helpful to their own learning and which aspects might be beneficial for their work with their own students. Pedagogical topics generated by the teachers as a result of their review included the following:

- Selection and preparation of problems for problem-based lessons
    - How can mathematics problems be adapted to be more suited to a teacher’s students?
    - What might be the benefits (and drawbacks) to students of dealing with differently scaled graphs within the same lesson?
    - How does a problem’s wording potentially impact student understanding, especially with regard to English learners?
- Learning goals and outcomes within problem-based lessons
    - How can we select problems that will forward our learning goals?
    - At what stage of a lesson is it appropriate to introduce learning goals and expected outcomes to students? Oftentimes, school expectations require teachers to post and/or state learning goals at the start of their lessons - is this always appropriate?
    - How can we connect measurable outcomes to a lesson’s learning goals, and what is the potential for learning goals to evolve throughout a lesson?
- Addressing student misconceptions

    - Should teachers be planning to elicit predicted student misconceptions?
- Timing of teacher input
    - When might it be beneficial to introduce academic vocabulary after students have defined it in their own words, versus frontloading vocabulary?
    - What are the implications of a teacher questioning students versus telling them?
- Time constraints
    - How important is it to provide ample time for discussion so that students can engage in reflection for deep learning?
    - What compromises do we - or don't we - make in relation to having limited time? "What is the point of doing a problem-based lesson that we don't really finish?" What does it mean to "finish" a problem-based lesson?

This session analysis and debrief took approximately an hour and resulted in strong teacher-made connections between their own content learning, the instructional moves of the facilitators, and what teachers projected as their future practice. The rich discussion connected to our teachers' common beliefs and practices that we were interested in impacting over time. For example, at the start of our project, many of the participating teachers' current practice included avoiding calling on students if they had incomplete or incorrect answers to a problem. This was most likely based upon a belief that students in general would be confused by wrong answers, or even worse, they might learn another student's "wrong" approach. Rather than eliciting common misconceptions or incomplete ways of thinking about a problem, the teachers would attempt to avoid such discussions, a practice that more comfortably aligned with their direct instruction and modeling of procedures. As teachers often stated, "In our first year, I would've said you had to model first. You had to show them how to do it."

The teachers' analyses of the script resulted in their consideration of the facilitator's plan for eliciting a common misconception as evidenced by some teachers' comments:

- *What you said is very interesting...you planned for misconceptions, you planned whether we noticed it or didn't notice it, you planned it both ways.*

- *That’s exactly what happens in our classrooms, too...we’re just chugging along and all of a sudden a student brings up something and, boy, we have to be quick on our feet to think it’s a misconception, whatever you wanna call it and you didn’t plan for it and now I have to shift gears here really fast. That’s the hard part.*
- *That’s where the questioning comes in.*

Teachers noted that if we plan lessons with potential student confusions in mind, then we can prepare ways in which we might respond when they arise, including probing questions that can help students recognize how their approach might or might not address the problem’s questions.

Overall, formative and summative assessment surveys of the teachers provide evidence that the pedagogical analysis sessions utilizing session scripts were fruitful in teachers recognizing how learning goals drive a teacher’s instructional choices; for example

- *As we moved through the week, the big picture came into focus. I enjoyed the sessions...as a debriefing and "teacher talk" concept so that I could understand why certain choices were made with the lessons [with which] we were presented. Again, thank you.*
- *The breakdown of M's lesson was extremely useful in trying to improve the development of my own lessons.*

Some Thoughts Regarding Script Formats

Depending on the primary purpose, and potential ancillary purposes, the script-taking might be conducted in different formats. Excerpts from some sample scripts that we have used are shown below. These are provided verbatim, and the labels are a bit different from the remainder of the paper: “Teacher” or “T” in the script is the “mathematician” or “M” above, and “Student” or “S” is a “participating teacher” or “T” above. The scripts were all created by the educator (“E” above).

<u>(Three) columns, with or without images</u>

This format is geared toward helping teachers understand and plan for some of the complexities of teaching, with multiple aspects of instruction and learning to be considered simultaneously. The selection of the column headers can vary, depending on the specific purposes for the professional development.

This format is also helpful in supporting teachers' thinking about lesson planning. In particular, formative assessment is frequently not targeted as a major component of planning, and we wanted the participating teachers to both see it in action, and consider how they might plan for formative assessment in their own teaching.

The excerpt below is from a session not described in this paper.

Lesson Steps	**Teacher Support of Student Learning**	**Formative Assessment**
8:54 T presents problem and tells students to pursue it.	Show video to introduce problem (prior to showing text of problem) <u>While monitoring students trying to solve the problem:</u> T-*What else did he eat besides the ½? You can work through it and see what happens and then adjust. (Figure)* Student shows totals for each cat. *Did you check it? Show me how you checked it…I'm asking how you checked, not how you solved it. Start from the beginning and kind of follow the story to follow your work. (Figure)* Naming the strategies being tried may	<u>While monitoring students trying to solve the problem:</u> The teacher assesses to identify types of approaches being used by the students. The teacher checks to see if students are confused related to the "remainder" of each cat's snack.

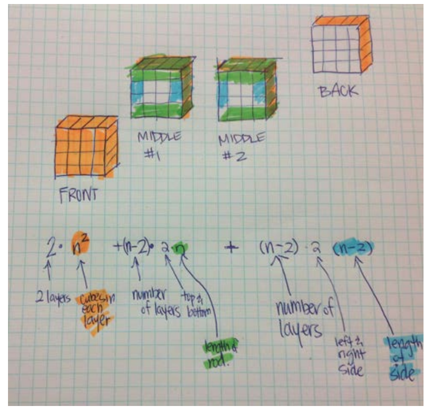
9:02 *What ways are you approaching this problem?* *You are going in an algebraic direction? And some of you are trying diagrams…none of which is very easy with this problem…Ok, yes, some of you are working backwards. Yes, and some of you are trying to make whole numbers…What is the issue here…What is that 1/3 a third of? What is that ½ a half of? What does the one refer to?*	be helpful to some students. It may bring back to mind strategies that they'd used in the past. This may prompt further efforts. The teacher is also stating that some approaches are difficult with this problem. Additional interactions with individual students: …	The teacher checks her observations of types of approaches being used by asking students to call out their approaches.

Another option might be to record the session in a 2-column format, one for teacher actions and the other for student actions. During the reflection stage, participating teachers might be prompted to identify moments of formative assessment (or other important aspects of the teaching), and expand the table to highlight those.

Linear text, with or without images

A linear format, as shown below, is easier to record, since the classroom discussion generally takes place linearly. It is also easier for some teachers to interpret, as it simplifies the reading.

T: Now let's take these visuals, the numbers and see if we can generalize. Let's look at the 16. What would that look like if we're looking at and n x n x n?



S: 2x· n^2…

T: Now what about the red, those bars? ... Where did this 2 come from? ... Is that part of the shape or the size? ... I hear some say size or shape…If we had a 6x6x6, how many would we have in there? ... So in relation to the size of the cube, how many are in there? ... If we had size 3, how many in there?

S: 1.
T: If we had 4, how many?
S: 2.
T-If we had 5, how many?
S: 3. So it's (n-2).
S: Would you explain why it's n-2?
T: Imagine in your head if you had a 3x3x3… Take off the ends…If you had a 4x4x4…

Images: The images of "student" work were found to be very helpful in understanding the flow of the lesson. If possible, we recommend including such images in the script to facilitate the recall of the lesson discussion, and reflection on facilitator and participant actions.

Focus of Script: In this project we took the approach of scripting the full content of the session verbatim. However, depending on the purpose, one might focus the script on actions pertaining to particular aspects of instruction, such as equitable access, classroom climate, questioning strategies, student engagement, and so on.

Varying formats: We believe it is useful to vary the scripting formats, as that would allow for different levels of complexity in the reading, as well as different uses of the script.

Discussion

The work of teaching is complex, and involves a myriad of decisions regarding both content and pedagogy. In order to provide professional development that will significantly improve instruction, we must address each of these areas, both independently of each other, and in relation to each other. What we offer in this paper is a way to support teachers' noticing of and reflection on the purpose and value of the instructional decisions made by the facilitator(s) through scripting (and photographing) the events of a professional development session, and then using the resulting records of instruction to facilitate accurate recall of events and reflection on many aspects of teaching.

The scripts, as we used them, share many of the characteristics described by Ball et.al. (2014). Some important traits of these scripts:

- Each script has a predetermined structure.
- The scripts are highly detailed, concrete, and specific.
- The scripts describe actions as they happened, and are not in themselves interpretive or judgmental.
- Although the Educator who scripted the session included as much of the session actions as possible, the record is necessarily incomplete. For example, not all student work was photographed, and not all images were included in the scripts handed to the participants; not all of the discussion was recorded because not all was heard or events moved too fast.
- The scripts afforded a view of both challenges and successes, without prejudging the value of any particular action.

We now consider a number of questions related to the process of scripting and using the scripts for reflection.

How should the script be created, and by whom? While a transcript of videoed instruction can serve as a record of practice, we found that a script developed during the instructional session itself has several advantages for the purpose of facilitating pedagogical sessions. First, such a script can typically be produced more quickly than a transcript, thus, allowing for script-based reflection to occur in close proximity to the learning experience. Second, in our scripting, unlike a more technical record, we focused on the actions of the instructor and teachers without inclusion of every detail and word spoken. We thought of the script as a factual record of the session (what is seen and what is heard), with prompts, activities, and the instructor's and teachers' verbal exchanges. It enables the facilitators' and/or teachers' analyses of the flow of the session and pertinent interactions while stimulating teachers to reflect on their own learnings in relationship to the instructor's teaching moves. We note that this kind of scripting involves an understanding of the content and pedagogy, and on-the-spot interpretation and analysis, so in order for the script to be valuable, it should be recorded by a mathematics teaching professional, such as (in our case) a mathematics educator with experience in mentoring teachers, who collaborates closely with the facilitator of the content-based session.

How should the portions of script for reflection be chosen, and by whom? The reflective sections described above demonstrate how facilitators used scripts as a record of practice in two ways: (1) to focus teachers' attention on pre-determined critical moments of the learning session, and (2) to support teachers in analyzing a full segment and themselves selecting the moments that were critical to their learning. For the first method, the reflective session came a day or two after the problem-solving session, to give the pedagogy facilitator time to review and select the moments in collaboration with the mathematician. The second method may be employed fairly soon after the problem-solving session, as well as later on. As our approach evolved, within the scripts we included photos of the teachers' work samples adjacent to the related verbal interactions and believe this offers more potential to directly prompt teachers' analysis of their own learning and its relationship to the facilitator's instruction.

What should be included in the script? We intentionally created scripts that were absent of judgement, focusing only on the facts: what was seen and what was heard. Thus, our records of practice provided a means for teachers to reflect on actualities of what occurred rather than on the facilitator/scripter's interpretation of the lesson. Our intention was to allow the teachers to do the work of interpreting what happened, to develop their own understandings, and to make connections to the facilitator's teaching moves and their intended purposes.

What are some side-benefits to teachers of having session scripts?

- During the analysis session: Because the scripts described actions without interpretation, teachers were led to discuss what they felt or how they interpreted certain actions versus what actually took place. They were used to considering objective actions (teacher moves, student work, etc) as evidence for interpretation in the lesson study post-lesson reflection and analyses, but this work of separating fact from interpretation in their own learning was a new experience. As noted in the introduction, the teachers who participated in this project were generally unaware of the facilitator's teaching moves when they were engaged as learners of mathematics. The "invisible work of teaching" (Lewis, 2007) needs to be revealed, and scripts can be artifacts that provide the evidence and prompt the reflection. As Ball, et al.(2014) discussed when citing Shulman (2002), "Close engagement with records of practice affords that possibility of disequilibrium through the surprise one may experience at phenomena not noticed before."

- Outside of the "official" professional development: Teachers wished to retain the scripts so that they could review the process of solution, with its nuances and various methods of solution, as they considered their own teaching of particular problems in their classrooms. The scripts supplemented the notes they had taken as learners with a more comprehensive verbal and visual record of the experience, including by teacher and student actions and words.

How do the facilitators benefit from the scripts? Through this work, we found some far-reaching unplanned side-benefits.

- During the professional development: Where previously the mathematicians focused primarily on content (with some pedagogy), and the mathematics educators focused primarily on pedagogy (with some content), this approach helped us to collaborate much more closely on both, and consequently to do each better.
- In curricular work as university faculty: The collaboration between mathematicians and mathematics educators did not end with the end of the project. The increased understanding of each other's areas of expertise has led to increased sharing of resources and a deeper collaboration on university instruction and curriculum design.

- In personal/professional relationships: It is not often the case that mathematicians and mathematics educators have a deep appreciation of each other's areas of expertise. The shared work to create and utilize the scripts in professional development has resulted in a far greater mutual appreciation, to the benefit of countless numbers of teachers and students.

It is our hope that our work might provide support to others who are planning for projects that require collaborations across diverse colleges such as Natural Sciences and Education.

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