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Variations on Convergence Criteria

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ABSTRACT: The didactical idea of variation is exemplified with examples from convergence of series. This yields a new convergence test similar to the root test.

Keywords: series, convergence criteria, didactical variation

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Introduction

Teaching math at the university has two goals that coexist with some tension. On one hand, a huge body of knowledge must be delivered ti the students, on the other hand students should develop the ability to discover new knowledge on their own. One way to remedy this situation is to let students explore variations of known definitions or theorems. Consider e.g. the laws of the derivative. Some of them cannot be easily found by students themselves. Hence, the lecturer must present them. The idea of variation can be realized e.g. by letting students explore the definition $f^*(x) = \lim_{q\to 1} \frac{f(qx)-f(x)}{qx-x}$ after the standard one. Students can re-prove the laws of differentiation, prove the restricted equivalence and perhaps find more variations [O].

This paper extends the idea to convergence of series. Most calculus courses teach two standard recipes to check convergence of a series, the root criterion (also called root test) and the quotient criterion. In this paper we explore the question if there are other criteria that can be derived in a similar fashion.

1 Basics

A real series $\sum_{k=0}^{\infty} a_k$, for $a_k \in \mathbb{R}, k \in \mathbb{N}_0$ is understood as the limit of its partial sums, i.e. $\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} S_n$ where $S_n := \sum_{k=0}^{n} a_k$. For the geometric series $\sum_{k=0}^{\infty} q^k$ one can easily derive the condition that it converges if and only if $|q| < 1$. By comparison (majorization), this leads to the following two criteria:

Theorem 1. Let $\sum_{k=0}^{\infty} a_k$ be a real series. Then the following tests can establish convergence:

Root test of convergence: The series converges if $\exists q : 0 \le q < 1 \wedge \exists n_0 : \forall n > n_0 : \sqrt[n]{|a_n|} \le q$.

Quotient test: The series converges if $\exists q : 0 \le q < 1 \wedge \exists n_0 : \forall n > n_0 : \frac{|a_{n+1}|}{|a_n|} \le q$.

There are plenty of proofs published (e.g. Marsden and Weinstein, II, Chapter 12), and we won't replicate them. However, it is worth noticing the common ideas: First, a finite initial part of the a_k is irrelevant for the question of convergence. Second, both criteria are fulfilled for the geometric series, and third the test itself is thus a consequence of the majorization test. For the geometric series $a_k := q^k$ and hence $\sqrt[n]{|a_n|} = |q|$ and $\frac{|a_{n+1}|}{|a_n|} = |q|$. The variation idea of this paper came out from the little observation that besides taking roots and quotients one may apply other transformations as well.

2 Logarithmic test

If one takes logarithms of the elements of a geometric sequence, one gets: $a_k = q^k \iff \log(a_k) = k\cdot\log(q)$. Hence, we get:

Theorem 2. Logarithmic test of convergence: The real series $\sum_{k=0}^{\infty} a_k$ converges if

$$\exists L \in \mathbb{R} : L < 0 \wedge \exists n_0 \in \mathbb{N} : \forall n > n_0 : \log|a_n| \le n \cdot L$$

Proof. $L < 0 \wedge \log|a_n| \le n \cdot L \Rightarrow |a_n| \le \exp(n\cdot L) = \exp(L)^n = q^n$ with $q := \exp(L), q < 1$. □

Example: Does $\sum_{k=0}^{\infty} \frac{\sqrt{k}}{2^k+1}$ converge? Yes, because of $\log\left(\frac{\sqrt{n}}{2^n+1}\right) = \frac{1}{2}\cdot\log(n) - \log(1+n^2) < \frac{1}{2}\cdot\log(n) - \log(2^n) = \frac{1}{2}\cdot\log(n) - n\log(2) \le -\frac{1}{2}\cdot n$, hence $L = \frac{1}{2}$.

The usefulness of this test in practice is limited but it works!

3 Difference test

For the geometric sequence $a_k = q^k, 0 \le q < 1$ is a non-negative sequence of numbers that monotonically approaches 0 (which is a necessary condition for series convergence anyway) and one has $a_k - a_{k+1} = q^k - q^{k+1} = q^k \cdot (1-q) = a_k \cdot p, p := 1-q$. Hence, we get:

Theorem 3. Difference test of convergence: If $(a_k)_{k\in\mathbb{N}}$ is a monotonically sequence of non-negative number then the series $\sum_{k=0}^{\infty} a_k$ converges if

$$\exists p \in \mathbb{R} : 0 < p < 1 \wedge \exists n_0 : \forall n > n_0 : a_k - a_{k+1} > a_k \cdot p$$

Proof. Set $q := 1 - p$, then $0 < q < 1$ and $a_{k+1} \leq a_k \cdot (1 - p) = a_k \cdot q$. Then the quotient test can be applied. $\square$

Example: $\sum_{k=0}^{\infty} \frac{1}{2^k}$. A simple calculation shows that $p = 1/2$ works.

This may boot be a very useful criterion but the didactical idea is to let students explore such issues. Mathematics has a pragmatic dimensions and my impression is that most students underestimate this!

4 Conclusion and Remark

This paper presents a very simple example of a variation. The experience of using such variations in courses is promising and I think it is worth to collect more examples. I acknowledge any suggestions!

References

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