The Mathematics Enthusiast

Volume 18
Number 1 *Numbers 1 & 2*

Article 19

1-2021

The Suuji Approach to Multi-Digit Addition: Using Length to Deepen Students' Understanding of the Base 10 Number System

Ryota Matsuura

Olaf Hall-Holt

Nancy Dennis

Michelle Martin

Sarah Sword

**The Suuji Approach to Multi-Digit Addition:
Using Length to Deepen Students' Understanding of the Base 10 Number System**

Ryota Matsuura[1]
St. Olaf College

Olaf Hall-Holt
St. Olaf College

Nancy Dennis
Prairie Creek Community School

Michelle Martin
Prairie Creek Community School

Sarah Sword
Education Development Center, Inc.

Abstract**:** We describe the Suuji representation of numbers which aims to deepen elementary students' understanding of the base 10 system. ("Suuji" means "number" in Japanese.) This representation takes a two pronged approach of (1) making the place value more explicit and (2) using *length* to represent numbers, thus allowing students to reason spatially. We taught multi-digit addition using the Suuji representation to 20 second and third grade students. The article uses lesson descriptions and student work to illustrate the Suuji approach, as well as its impact on student learning.

*Keywords***:** Base 10 number system, Multi-digit addition, Place value, Measurement-based approach, Spatial reasoning, Representation, Physical manipulatives.

Introduction

Students typically learn multi-digit addition in second or third grade (CCSSI, 2010). Some struggle with this work when they do not have deep understanding of place value in the base 10 number system. For instance, a student might consider 37 as thirty-seven ones, rather than as three tens and seven ones. And when faced with 37 + 56, a student might resort to counting by ones (**figure 1**).

[1] matsuura@stolaf.edu

Figure 1: Student adds by counting by ones.

To alleviate this and other related issues, we developed the Suuji representation that makes each place value explicit. ("Suuji" means "number" in Japanese.) For example, H3 R1 S4, which is a short-hand for 3 houses, 1 room, and 4 shelves, corresponds to the base 10 number 3.14.

We taught multi-digit addition using the Suuji representation to 20 second and third grade students at a rural public school. In 2016–2017, the school had 180 students, with 14% minority, 16% free/reduced lunch, and 17% special education students. These students had a wide range of prior experiences. Some added by counting, without considering place values (**figure 1**). Others proficiently added multi-digit whole numbers.

The article uses lesson descriptions and student work to illustrate the Suuji approach. We share pre- and post-assessment results to describe effects on student learning. While we used decimal numbers with our students, the article also describes how our approach may be adjusted for whole number addition.

Note: We developed physical manipulatives, partner games, and exercises through which students engaged with the Suuji representation. For sample lesson plans, as well as templates for these manipulatives, contact the first author via email.

Background

The Suuji representation aims to deepen students' understanding of numbers by making the place value more explicit. In this regard, our approach resembles often-used manipulatives such as the Base Ten blocks and the abacus (e.g., Cotter, 2000). However, the Suuji approach distinguishes itself by representing numbers as *lengths*, allowing students to reason spatially. More specifically, the Suuji blocks (described in **Lesson 1**) emphasize height as the underlying physical quantity of interest.

The Suuji representation was inspired, in part, by the work of Russian psychologist Vasily Davydov. During the 1960s, Davydov and his colleagues developed an early elementary mathematics curriculum based on the *measurement* notion of numbers, rather than the traditional approach that uses counting (Davydov, 1990). In Davydov's curriculum, children study scalar quantities such as length, area, and volume, which can be experienced visually and tactilely (Schmittau, 2005). For example, early first grade students might compare two lengths and make them equal by adding to the smaller or subtracting from the larger. Shown below is a schematic diagram of their thinking (Schmittau, 2005, p. 19):

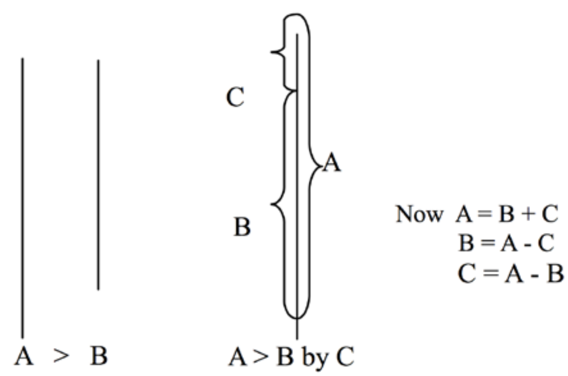


Figure 2: Measurement notion of numbers.

Davydov's approach provides opportunities for early algebraic thinking (Bass, 2015). Various studies, including those conducted in American schools, have shown that his approach is effective in fostering young children's mathematical understanding (Schmittau, 2003; Venenciano, Slovin, & Zenigami, 2015). Suuji mathematics, with its emphasis on length, espouses this measurement-based approach, bringing coherence to the mathematics that students learn and providing them access to algebraic thinking at an early age.

The Suuji representation takes the "best of both worlds" route by combining Davydov's measurement-based approach with an explicit emphasis on place value. As described in **Lesson 2**, the Suuji blocks allow exchanges between ones and tens, or tens and hundreds; but in distinction from the Base Ten blocks, these exchanges occur in the context of towers of equal height. Thus, students can use not only their grasp of discrete exchanges, but also their spatial reasoning with *lengths*, to undergird their learning about the base 10 number system.

The Suuji representation was developed in collaboration with classroom teachers, who were looking for more than what their students were getting out of the Base Ten blocks. The teachers chose the representation using neighborhoods, houses, rooms, and shelves, because they thought it would be fun and relevant to their students. The key here is not “houses” or “rooms,” however. Instead, it is the way in which the Suuji representation allows students to visualize and work with numbers and number relationships in a meaningful way.

In teaching these lessons, our goal was not simply to have students perform multi-digit arithmetic. Rather, we designed activities that allow them to develop richer understanding of the base 10 system *through* multi-digit addition, even as they gain fluency and understanding of multi-digit operations. Using the Suuji blocks, students learn to name numbers (such as H3 R1 S4 that we saw in the introduction), compare numbers, rewrite numbers (analogous to grouping in base 10), and add numbers. We view and treat these concepts not as separate skills to acquire, but all part of an interconnected whole. In this regard, our approach and this paper serve as an illustration of Skip Fennell’s rendering of how counting, place value, comparing and ordering, and operations are interconnected (Fennell, 2015).

Lesson 1: Towers and Tower Names

We describe five lessons on Suuji addition, each lasting 40 minutes. Lesson descriptions have been edited for clarity.

In this first lesson, students learned about *towers* and *tower names*. Towers are made from *building blocks* that correspond to powers of 10. The block types and corresponding decimals are shown:

- Neighborhoods tens
- Houses ones
- Rooms tenths
- Shelves hundredths

For example, a tower containing 2 houses (2 ones) and 3 rooms (3 tenths) corresponds to 2.3. **We did not reveal this tower-decimal correspondence until later.**

Note: This correspondence relies on general place value structure, rather than on particular place values. For example, if we had been working with whole numbers instead of decimals, we would have had neighborhoods, houses, rooms, and shelves correspond to thousands, hundreds, tens, and ones, respectively.

We began the lesson by showing the two towers in **figure 3**. Blocks are stacked in right-alignment. Tower name describes how its blocks are arranged from the bottom. For example, the tower R1 H1 R2 H1 (**figure 3**, right) has one room at the bottom, followed by one house, two rooms, and one house. In this lesson, students worked with towers containing houses and rooms only.

Students played a partner game. Each student received a set of houses and rooms made of foam boards and cards with tower names. During each round, students built the tower whose name was written on the card. Then they studied their partner's tower and wrote *its* name. This gave students practice in converting between towers and tower names. After each round, partners compared their towers to determine which is taller. Though we did not tell the students, the cards were designed so that partners built towers of equal height in each round (e.g., H2 R3 and R1 H1 R2 H1).

Figure 3: Towers H2 R3 and R1 H1 R2 H1.

Afterwards, we discussed whether or not *towers with different names could have equal height*. Students offered these ways to prove that two towers have equal height:

- The two towers contain the same number of blocks for each type. Example: H2 R3 and R1 H1 R2 H1 both contain two houses and three rooms, so their heights are equal.
- We can rearrange the blocks in one tower to form the other. Example: Students said R1 H1 R2 H1 is "wobbly." Moving its houses to the bottom forms the more stable H2 R3.

During the game, partners built towers H1 and R10. Students observed that 1 house and 10 rooms have equal height, and we wrote H1 = R10. In the Suuji unit, the equal sign indicates equality of heights. The towers themselves need not be the same. Although we did not delve further into this issue with our students, this way of using the equals sign sets up a reasonable way of relating two objects as equivalent but not necessarily "the same."

Lesson 2: Standard Towers

This lesson focused on *standard* towers. A tower is standard if (1) its larger blocks are below the smaller blocks and (2) it uses as few blocks as possible. To standardize H2 R8 H1 R5, for example, rearrange its blocks with the largest type (houses) at the bottom. This yields H3 R13. Then reduce the number of blocks in this tower, because 13 rooms have height equal to one house and three rooms (R13 = H1 R3). Combining these yields H4 R3 (**figure 4**). Standardizing does *not* change a tower's height, since the only operations involved are rearranging blocks and exchanging 10 blocks with one block of equal height (i.e., 10 rooms with 1 house).

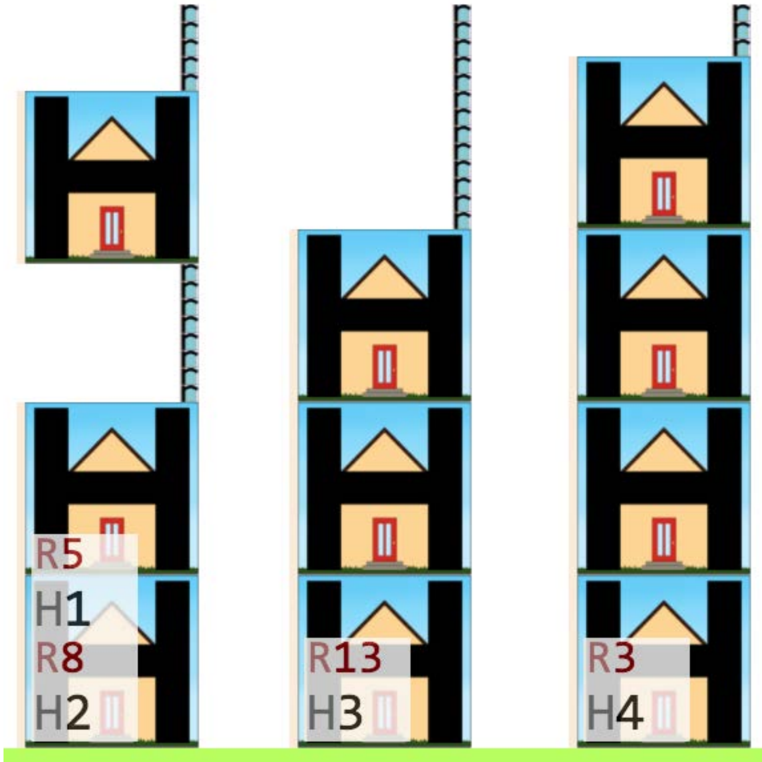


Figure 4: Towers H2 R8 H1 R5, H3 R13, and H4 R3.

We began the lesson by recalling the pair of towers with equal height, H2 R3 and R1 H1 R2 H1 (**figure 3**). This lesson's theme was an "invariant," or a property that does not change. Students would manipulate a tower to look like another without changing its height—thus, the height is an invariant.

Students played another partner game. In each round, the "builder" built two towers of equal height whose names were written on a card. Then the "manipulator" manipulated the first tower (tower A) to look like the second tower (tower B). They switched roles after each round. By design, tower B was always standard—so, students standardized tower A in each exercise. Standardizing involved rearranging houses and rooms, and exchanging 10 rooms with 1 house. The rounds were sequenced so that students first grappled with individual skills of rearranging and exchanging, then gradually combined them.

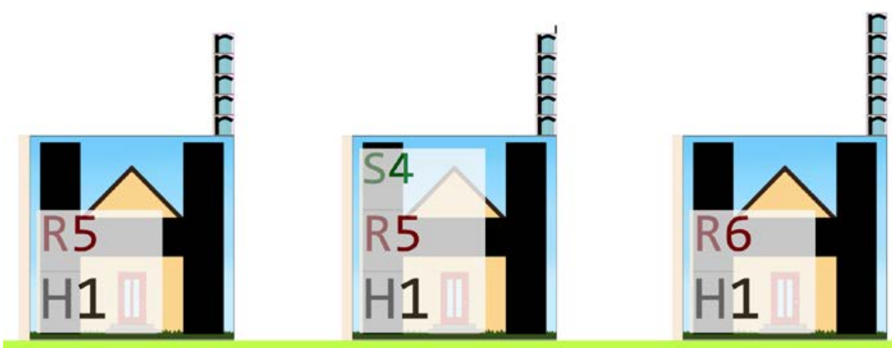
During the whole class discussion afterwards, we created H2 R8 H1 R5 and issued a challenge: "Make a tower of equal height that uses as few blocks as possible." Students suggested rearranging the houses to the bottom, then exchanging 10 rooms with 1 house, obtaining H4 R3. Then, we formally introduced the notion of standard towers and pointed out that students had made standard towers throughout the partner game.

Although we did not yet connect towers and decimals, some students noticed that houses and rooms behaved like base 10 numbers.

Lesson 3: Standardizing Tower Names

Students began the transition from visual to symbolic, to standardize towers using only tower names, without the visual support of blocks. We also introduced a new block type called *shelves*.

We began by showing students the visual image in **figure 5a**. We said, "This middle tower is taller than H1 R5 but shorter than H1 R6. The four tiny blocks are *shelves*." We discussed how 1 room and 10 shelves have equal height (R1 = S10), just like 1 house and 10 rooms (H1 = R10).



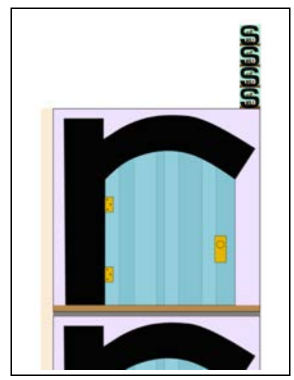


Figure 5a (left): Towers H1 R5, H1 R5 S4, and H1 R6.

Figure 5b (right): Zoomed-in look at the top of the tower H1 R5 S4.

We demonstrated how to standardize H1 R5 S4 H1 R8 S7 using tower names only (**figure 6**). Students suggested combining houses, rooms, and shelves, with the largest blocks at bottom—this yielded H2 R13 S11. We called such a tower a *stable* tower and explained that it is a helpful step before standardizing. Then we exchanged the 11 shelves with 1 room and 1 shelf, and the 14 rooms (13 original plus 1 extra) with 1 house and 4 rooms, obtaining H3 R4 S1.

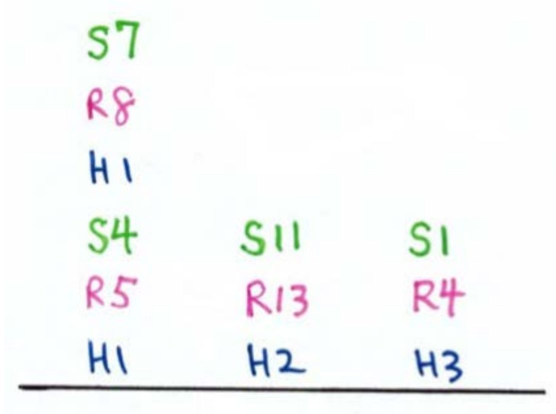


Figure 6: Standardizing H1 R5 S4 H1 R8 S7 using tower names.

Students worked on a series of exercises in which they saw a tower name like H1 R5 S4 H1 R8 S7 and found the name of the standardized tower, i.e., H3 R4 S1. The exercises were scaffolded by gradually increasing the complexity of towers. In the early exercises, students were also shown images of tower blocks with tower names. When students were ready, we removed the visual support, and students worked purely symbolically. Many students devised their own approaches for standardizing (**figures 7a** and **7b**).

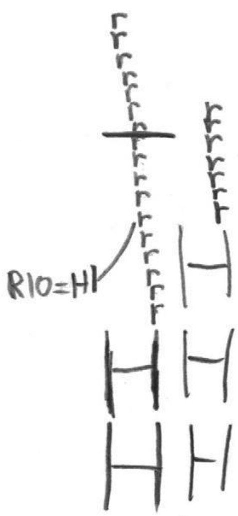


Figure 7a: Standardizing H2 R17.

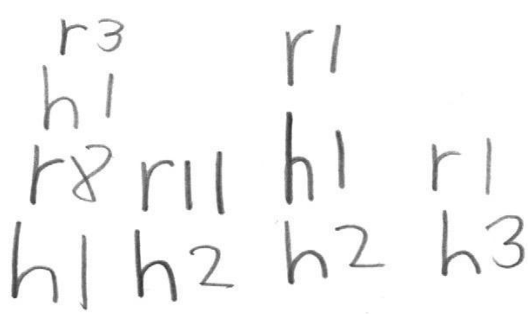


Figure 7b: Standardizing H1 R8 H1 R3.

Lesson 4: Tower Addition

Students learned to add towers. Example: To find H2 R8 + H1 R5, stack the two towers on top of each other, obtaining a new tower H2 R8 H1 R5; standardizing that yields the sum H4 R3 (**figure 8**).

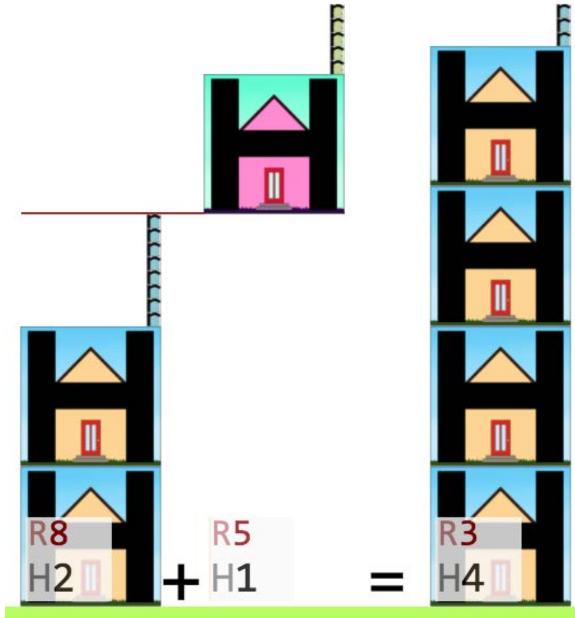


Figure 8: Tower sum H2 R8 + H1 R5 = H4 R3.

Since students had previously standardized towers such as H1 R5 S4 H1 R8 S7 (which corresponds to the sum H1 R5 S4 + H1 R8 S7), tower addition felt familiar. This lesson also provided more experience with standardizing.

Students worked on a series of exercises in which they saw two tower names and found the sum. Again, the exercises were scaffolded to introduce sums of gradually increasing complexity. Tower blocks were shown in early exercises, and this visual support was later removed.

During the whole class debriefing, we used the sum H8 R2 + H6 R1, where the two towers together contained more than 10 houses, to introduce a new block type called *neighborhoods*. 1 neighborhood and 10 houses have equal height (N1 = H10), thus H8 R2 + H6 R1 = H14 R3 = N1 H4 R3.

Note: Tower addition is essentially regrouping. For example, to find H2 R8 + H1 R5, first create a stable tower H3 R13; this is analogous to combining the digits in each place. Then exchange the 13 rooms with 1 house and 3 rooms to obtain H4 R3; this exchange corresponds to the regrouping process.

Lesson 5: Big Reveal

We introduced a shorthand for tower names called *decimal numbers*. Example: Decimal number 2.5 corresponds to tower name H2 R5. Each digit in the decimal number corresponds to a block type, and the dot separates houses and rooms. Other examples include 1.54 = H1 R5 S4 and 1.87 = H1 R8 S7. Thus, a decimal sum 1.54 + 1.87 corresponds to H1 R5 S4 + H1 R8 S7. This equals H3 R4 S1 (see **figure 6**), whose decimal form is 3.41. Hence, 1.54 + 1.87 = 3.41.

To summarize, computing a decimal sum like 1.54 + 1.87 involves:

1. Converting the decimals into tower names. Here, 1.54 + 1.87 becomes H1 R5 S4 + H1 R8 S7.
2. Computing the tower sum, i.e., H1 R5 S4 + H1 R8 S7 = H3 R4 S1. The sum must be standard.
3. Converting the standardized tower sum back to decimal form, i.e., H3 R4 S1 becomes 3.41.

Students worked on a series of exercises in which they saw a pair of decimals and computed the sum. Initially, the exercises guided students through the three steps above. Eventually, this support was removed—students were simply given, say, 2.78 + 5.63, and computed the sum (**figure 9**).

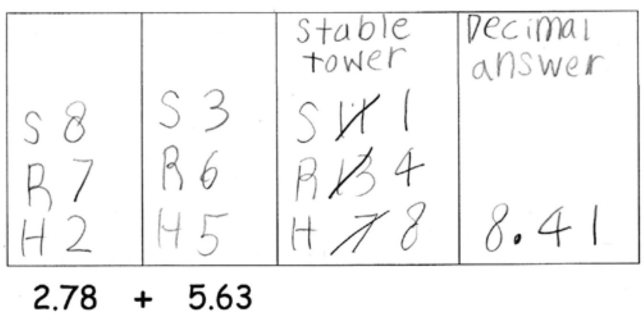


Figure 9: Computing 2.78 + 5.63.

Effects on Student Learning

To measure effects on student learning, we administered pre- and post-assessments. They were administered three weeks apart, at the beginning and end of a larger unit that contained the five lessons. The assessments contained the same ten addition problems such as 37 + 56 and 5.76 + 3.37. On the post-assessment, students demonstrated their understanding of the role of place value in addition. **Figure 10** shows the work of the student who, on the pre-assessment, computed 37 + 56 by counting (**figure 1**).

Here, he thinks about the place of each digit in 37 and 56 by grouping them as neighborhoods and houses. This conceptualization is very different from where he started.

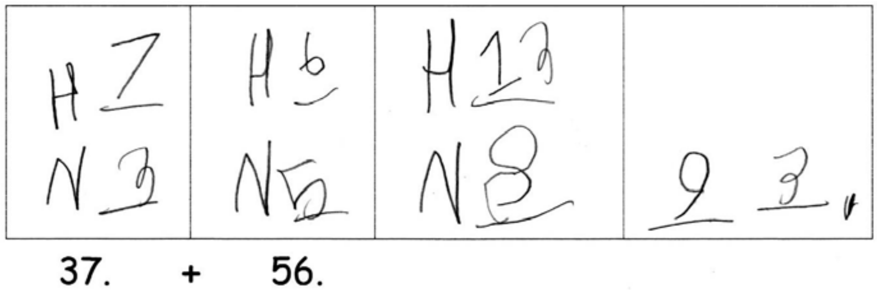


Figure 10: Computing 37 + 56 on the post-assessment.

On the post-test, 84% of students (16 out of 19) comfortably added three- and four-digit numbers, as opposed to 10% (2 out of 20) on the pre-test. We saw *fluidity* in their thinking—students readily transferred their understanding of two-digit addition to the four-digit case. If students are merely building procedural knowledge, four-digit numbers pose more challenge than two-digits, because procedures become more complex. In Suuji addition, more digits did not increase difficulty for students, suggesting that they built *conceptual* understanding of multi-digit addition.

We gave students opportunities to develop their own understanding, which is empowering. Several students who had previously lacked confidence in mathematics said they felt they are now good at math. A student who usually struggled mathematically said, “I like this Suuji math. It’s just at my level.” He was doing far more complex addition than he had been doing prior to the Suuji unit.

Suuji Subtraction

After the Suuji unit, we worked with one of the students on subtraction. We asked her to compute 823 – 286, although she had not subtracted three-digit numbers before. She tried converting 823 to a tower name, but realized that another block type was needed to represent the 8 in 823. She chose *boroughs*, where 1 borough and 10 neighborhoods have equal height (B1 = N10). Thus, 823 – 286 became B8 N2 H3 – B2 N8 H6. She could not subtract H6 from H3, however, so she exchanged one of the neighborhoods in B8 N2 H3 with 10 houses; then she exchanged one of its boroughs with 10 neighborhoods. Thus, B8 N2 H3 became B7 N11 H13, after which she subtracted (**figure 11**).

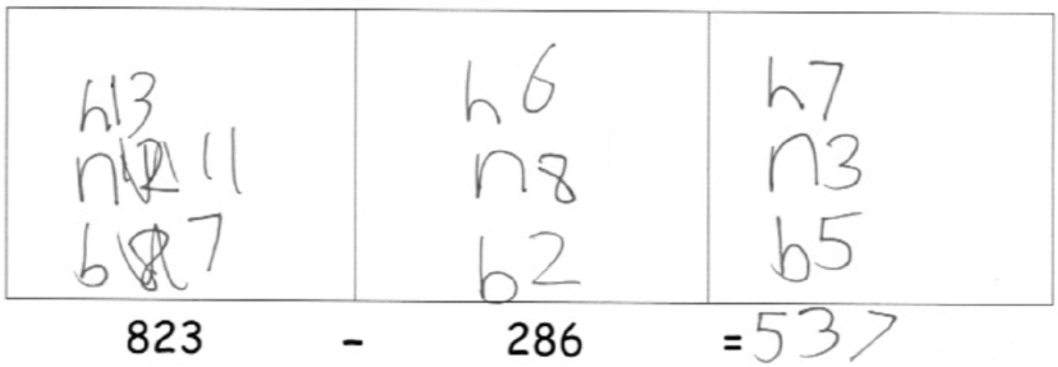


Figure 11: Computing 823 – 286.

As this episode illustrates, Suuji lessons provided students with a sense of ownership and empowerment—this particular student applied her understanding of Suuji concepts to solve an unfamiliar problem. We explicitly discussed with all students that the Suuji approach was designed to help them understand how numbers worked, and this gave them license to think about extending the model. If we had presented it as a *fait accompli*, they may not have felt as empowered to tinker and explore.

Concluding Remarks: Benefits of the Suuji Approach

In this section, we describe the various benefits of the Suuji approach in deepening students' understanding of numbers and the base 10 representation.

The Suuji representation such as H3 R1 S4 makes each place value explicit. Cotter (2000) notes how the English language—with words such as *eleven*, *twelve,* and *thirteen*—blurs the patterns for counting and obscures the groupings that occur in base 10. In contrast, many Asian languages have predictable counting patterns. For example, the number 37 is named 3-ten 7 in Japanese (san-ju shichi, where san, ju, and shichi mean three, ten, and seven, respectively). The Suuji representation follows Cotter's recommendation of learning the base 10 system with an approach that highlights such counting patterns.

We have discussed (in the Background section) how Suuji mathematics takes the "best of both worlds" route by combining Davydov's measurement-based approach with an explicit emphasis on place value. This connection is highlighted by the fact that there is a corresponding visual depiction of numbers, in the form of towers. Thus, students can see concretely that, for example, H3 R13 and H4 R3 have equal height, which offers a visual confirmation for the grouping that occurs. Such a visual cue is a powerful way to *experience* the relationship between these number representations.

Games and exercises played a prominent role in the Suuji unit. They provided targeted practice with individual skills, while gradually increasing the complexity of the tasks; this also allowed students to work at their own pace. Students worked with many examples, which gave them opportunities to "look for and express regularity in repeated reasoning" (CCSSI, 2010, p. 8). As a result, our students found general methods, such as their own approaches to standardizing.

An essential aspect of our approach is the set of physical manipulatives that allowed students to see and interact with tower blocks. The tactile nature of physical towers made these houses, rooms, and shelves feel concrete and real, even when working with their symbolic counterparts. Moreover, students connected with the playful nature of the Suuji representation. Learning about the base 10 system can be overwhelming and dry for many children. But our students enjoyed building towers with houses and rooms, seeing them get "wobbly," and rearranging and exchanging blocks to make the towers standard. And it is precisely this "playing" with the tower blocks that led students to figure out for themselves and make sense of grouping in base 10.

Our students worked with decimal numbers, although (as described in **Lesson 1**) the Suuji representation can be adjusted to work with whole numbers only. Decimal numbers intimidate many students. They think they have to relearn everything about integers—like how to add them—in decimal setting. By treating decimals as a separate topic, we balkanize students' mathematical understanding. We are telling students there are "special cases" they will learn later. We developed the Suuji approach with a belief that learning about decimals and multi-digit addition together brings coherence to students' understanding. We can teach *fluidity* with the base 10 number system. When learning to add multi-digit numbers, rather than working exclusively with integers, students can learn to view integers (and decimals) as part of a continuum in their understanding about numbers. The Suuji approach gets at the core of how addition is done in the base 10 system, without separating the process into different cases according to different types of numbers.

As students engaged with Suuji mathematics, they bumped into very significant mathematical ideas, used in virtually all areas of mathematics: equivalence, invariance, underlying structures of the base 10 number system, multiple representations and going back and forth between them, and the value of “unpacked” representations and “compact” (or standardized) representations for working with and communicating about mathematical ideas. It is convenient to *build* a tower by simply stacking two towers on top of each other (e.g., H2 R8 H1 R5 in **figure 8**). But when *communicating* about numbers, it is convenient to have a standardized, compact way to describe them (e.g., H4 R3 in **figure 8**).

Lastly, we did not present the Suuji representation as a finished product. Instead, we gave students free rein to extend the model and use it to ask questions beyond the scope of what we covered in class—in essence, using the model as an authentic mathematical tool, thus allowing students to be “tinkerers” and “inventors” (Cuoco, Goldenberg, & Mark, 1996).

References

Bass, H. (2015). Quantities, numbers, number names, and the real number line. In X. Sun, B. Kaur, & J. Novotna (Eds.), *Proceedings of the twenty-third ICMI Study: Primary Mathematics Study on Whole Numbers* (pp. 10–20). Macau, China: University of Macau.

Common Core State Standards Initiative [CCSSI]. (2010). Common Core Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

Cotter, J. A. (2000). Using language and visualization to teach place value. *Teaching Children Mathematics, 7*(2), 108–114.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curriculum. *Journal of Mathematical Behavior*, *15*(4), 375–402.

Davydov, V. V. (1990). *Types of generalisation in instruction: Logical and psychological problems in the structuring of school curricula*. Reston, VA: NCTM. (Original published in 1972)

Fennell, F. (2015, January 5). *Critical Foundations*. Retrieved from https://www.nctm.org/Publications/Teaching-Children-Mathematics/Blog/Critical-Foundations/

Schmittau, J. (2003). Beyond constructivism and back to basics: A cultural historical alternative to the teaching of the base ten positional system. In B. Rainforth & J. Kugelmass (Eds.), *Curriculum and instruction for all learners: Blending systematic and constructivist approaches in inclusive elementary schools* (pp. 113–132). Baltimore, MD: Brookes Publishing.

Schmittau, J. (2005). The development of algebraic thinking: A Vygotskian perspective. *ZDM*, *37*(1), 16–22.

Venenciano, L., Slovin, H., & Zenigami, F. (2015, June). *Learning Place Value through a Measurement Context*. Paper presented at the twenty-third ICMI Study: Primary Mathematics Study on Whole Numbers, Macau, China.