The Mathematics Enthusiast

Volume 18
Number 3 *Number 3*

Article 3

8-2021

Metasynthesis of Research in Mathematics Education: Foci and Theoretical-Methodological Foundations

Saddo Ag Almouloud

Metasynthesis of Research in Mathematics Education: Foci and Theoretical-Methodological Foundations

Saddo Ag Almouloud[1]
Federal University of Pará

Abstract: In this article, I carry out a qualitative metasynthesis of six doctoral theses that were directed by me, identifying the theoretical and methodological frameworks that support these researches, and how the articulations were made between different theories chosen. I also revisit the results achieved by the studies to highlight their relevance to the Mathematics Education area. This study, although not exhaustive, provides a vision of possible dialogs of Mathematics Education (or education) theories related to teaching and learning of mathematics and the Didactics of Mathematics. The diversity of theories and the specificities of each of them confirm the idea that a single theoretical tendency, or a single model, hardly ever explains and makes explicit all the phenomena involved in the teaching and learning processes of mathematical concepts. All the research works analyzed were used to study the epistemological, ecological, and economic dimensions, to identify the different forms of conceptions of a given mathematical object to help them in the didactic analysis of the findings. This study allowed us to identify, among other aspects, the reasons for being of the mathematical objects and the problems of their teaching. For teacher education, all mapped research, except one, has been supported in teacher education trajectories. The objective of these investigations is to familiarise teachers in initial or continuing education with these training trajectories as a didactic device that has the potential for their professional development, preparing them for an effective transition from the monumentalist paradigm to the world's questioning paradigm. For the teachers' training, the researchers presented didactic devices not based solely on the monumentalist paradigm, and somehow resorted to devices with PEP-FP-type structure.

Keywords: Teacher training, Proof and demonstration, Concept of function, Fractional numbers, Study and Research Trajectory, Anthropological Theory of the Didactic, Theory of Didactical Situations.

Introduction

The investigations proposed by my research group have as thematic axis the study of processes of formation and appropriation of concepts according to the paradigms of the Didactic of Mathematics. We dealt with questions about what is happening in the classroom from the point of view of the students, the teacher and the environment in which the process at stake takes place. Within this context, several dissertations and doctoral theses I supervised were developed. The guiding questions for the works were: *What factors influence the teaching and learning processes of mathematical concepts with the use of alternative resources? How are learning processes characterized in technology environments? What are*

[1] saddoag@gmail.com

the methodological alternatives for investigating the processes of learning and training in these environments?

Searching for the answers, we based our research on the Theory of Didactical Situations (TDS) and the Anthropological Theory of the Didactic (ATD), among others.

The starting point of the TDS (Brousseau, 1997) is the "situation," or better, the "set of situations" that the teacher must organize to enable learning. This notion of "set of situations" allows for the application of the Piagetian idea of development by "balancing" and learning by adapting the subject to the milieu. Here, however, the term is taken in the psychosocial sense: it is the institutional and relational medium of the class where the relationship with the teacher will be privileged (Perrin-Glorian, 1994). The TDS emphasizes the social dimension and the historical dimension of knowledge acquisition. The processes of knowledge acquisition are considered at the level of the classroom subjects: the acquisition of knowledge should result from a process of adapting the subjects to the situations that the teacher organized, and in which interactions with other students will play an important role.

The ATD (Chevallard, 1999) inserts the didactic field into the anthropology field and focuses on the study of the didactic praxeological organizations designed for the teaching and learning of mathematical organizations (MO). It studies the conditions and restrictions of the functioning of Didactic Systems, understood as subject-institution-knowledge relations. Chevallard (1999) states that the theory studies man in the face of mathematical knowledge, more specifically, mathematical situations. One reason for the use of the term "anthropological" is that the ATD situates the mathematical activity and, consequently, the study of mathematics within the set of human activities and social institutions.

The author states that mathematical knowledge organizes a particular form of knowledge, the product of human action in an institution characterized by the production of "things" that are used and taught. In this perspective, the author introduces the notion of habitat of a mathematical object as being the type of institution where the knowledge related to the study object is found, which in turn will determine the function of this knowledge (its niche).

From the point of view of research methodologies, the research works that have been developed under my direction are based on different engineering within the scope of the Didactics of Mathematics. We understand didactic engineering according to Brousseau (2008), who states that it consists of determining communicable and reproducible teaching devices, and evokes the existence of description, study and justifications as accurate and consistent as possible of the conditions of use of those devices.

Didactic engineering accompanies devices produced from a set of studies and analyses that characterize the product according to the theoretical and experimental scientific knowledge of the moment.

Brousseau (2008) explains that, in the field of scientific research, didactic engineering with a phenomenon-technical purpose aims to reconcile the normal teaching obligations and the reproduction and study of well-determined didactic phenomena. This type of research can be undertaken only in specific, complex, and precise organizations, in particular, it is indispensable to study systematically and experimentally theoretical models of learning and teaching devices.

The research methodologies developed and employed in the Didactics of Mathematics, more specifically in our research, are the classical didactic engineering (widely known), which we call the Didactic Engineering of 1st Generation (Artigue, 1990), and the Didactic Engineering of 2nd Generation, according to Perrin-Glorian's point of view (2011), as well as the Engineering of the Study and Research Trajectory (PEP, in the Portuguese acronym) (Chevallard, 2009).

A Didactic Engineering of 2nd Generation (Perrin-Glorian, 1990) aims to develop resources for regular education or teacher education. In this type of engineering, there is a relaxation in teacher decisions, and institutional requirements must be considered.

The Study and Research Trajectory (PEP) is part of a generating question Q_0 proposed within a didactic system $S(X;Y;Q)$, where X is the group of students, Y is the supervisor (the teacher) of the study and Q is a generating question for the study and other questions. The author affirms that the objective of building this didactic system is to study Q, looking for an answer R that meets some *a priori* restrictions by confronting it *with a suitable adidactic milieu*. The synthesis expected from X's work under Y's supervision can be written, according to the author, as $S\,(X, Y, Q) \Rightarrow R$, with R being the answer to question Q, and R must satisfy the set of *a priori* restrictions.

The synthesis of the studies I present later shows the different uses and conceptions of these methodologies, sometimes considered scientific research methodologies, sometimes methodologies involving various processes and procedures for professional training and/or the elaboration of learning objects.

In this article, I do a qualitative metasynthesis of some doctoral theses that I supervised, highlighting the theoretical and methodological frameworks that support those works. I analyze the articulations made between different theories of the Didactics of Mathematics and other theoretical perspectives in the analysis of the findings, and revisit the results achieved by these investigations to highlight their relevance to the mathematics education area. For this study, I chose the following theses: *Investigating knowledge of teachers of elementary education with a focus on fractional numbers for the 5th grade*[2] (Silva, 2005), *Teachers' knowledge on function: an investigation on the praxeologies*[3] (Rossini, 2006), *Proof and demonstration in plane geometry: conceptions of students from a mathematics teaching degree in Mozambique*[4] (Order, 2015), *The teaching practice and its influence on the construction of geometric concepts: A study on teaching and learning of orthogonal symmetry*[5] (Silva, 2015), *A didactic organization in quadrilaterals that brings the student closer to the geometric demonstrations*[6] *(* Ferreira, 2016), *Research and professional training device Sudy and Research Trajectory for Teacher Education (PEP-FP,* in the Portuguese acronym*): Constitution of the teaching knowledge for the teaching of plane analytical geometry of the point and the line*[7] (Freitas, 2019).

---

[2] Investigando saberes de professores do ensino fundamental com enfoque em números fracionários para a quinta série

[3] Saberes docentes sobre o tema Função: uma investigação das praxeologias

[4] Prova e demonstração em geometria plana: concepções de estudantes da licenciatura em ensino de matemática em Moçambique

[5] A prática docente e sua influência na construção de conceitos geométricos: um estudo sobre o ensino e a aprendizagem da Simetria Ortogonal

[6] Uma organização didática em quadriláteros que aproxime o aluno de licenciatura das demonstrações geométricas

[7] Dispositivo de pesquisa e formação profissional Percurso de Estudo e Pesquisa para a Formação de Professores (PEP-FP): constituição do conhecimento docente para o ensino de geometria analítica plana do ponto e da reta

These six theses were chosen because they represent the theoretical-methodological perspectives that guide the research I carried out and that I am still conducting.

Also, all these theses focus on the initial or continuing education of teachers. Figure 1 shows the focus of the investigations and some of the theoretical and methodological foundations that guided those studies.

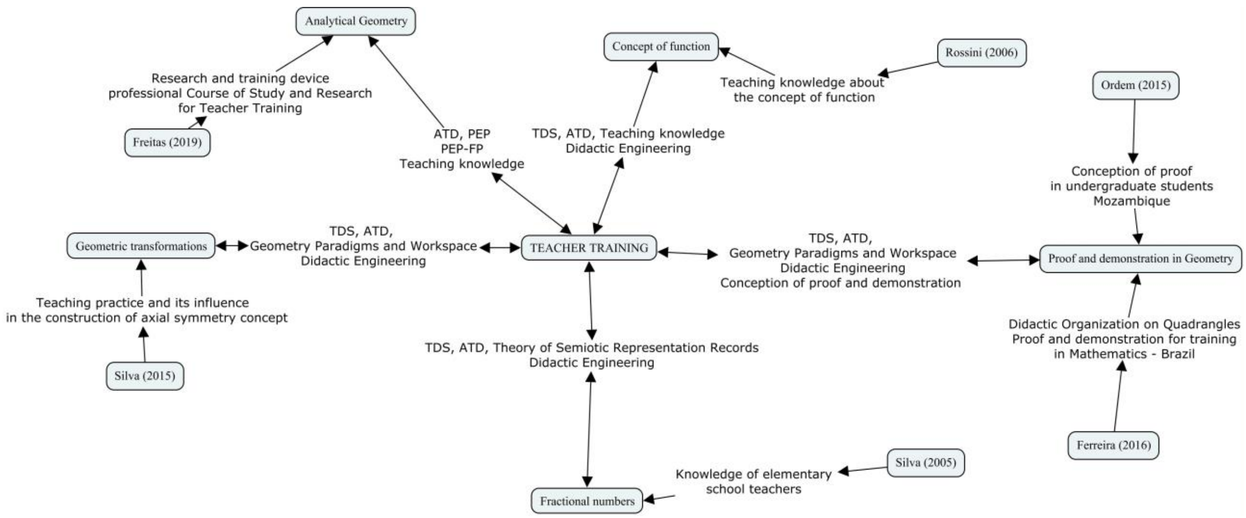

Figure 1: Map of the research works selected (through Cmap Tools 6.04)

One of my objectives is to re-read and analyze some of the predominant aspects of this research, such as the formation of teachers based on the ATD (Chevallard, 1999) and/or TDS (Brousseau, 1997), among other theories, and the types of engineering adopted.

Map of the research works selected

In this part, I present the doctoral theses chosen, pointing out the research focuses, the contributions of the theoretical referential used and the methodologies employed, and the main results of each research. At the end of the article, I discuss the different approaches and research perspectives from the theoretical and methodological point of view.

Investigating knowledge of elementary school teachers with a focus on fractional numbers for the fifth grade (Maria José Ferreira da Silva, 2005)

This research focused on the study of teacher knowledge and deals with conceptions of a group of mathematics teachers of fractional numbers and the learning of fifth-grade students, autonomy and difficulties in possible changes of those conceptions in continuing education.

The research was guided by the following questions: *What Didactic Organization do teachers build to teach fractional numbers to the fifth grade of elementary education during their training? Is it possible to guide mathematics teachers to make reflections that allow changes in their students' conceptions, giving them a new place in the school institution? Is it possible, in continuing education, to promote actions that allow teachers to change their practice of teaching fractional numbers to a fifth-grade class?*

The theoretical framework was based on the ATD (Chevallard, 1999) to model, as Mathematical Organization (MO) and Didactic Organization (DO), the conceptions of fractional numbers: part-whole, measure, quotient, ratio, and operator.

ATD contribution

The assumptions of the ATD (Chevallard, 1999) include elements that are needed to model the contents of fractional numbers for the fifth grade. This theory guided the researcher in the choice of suitable situations for the teaching of fractional numbers, and of techniques likely to be mobilized for their resolutions. Those techniques were justified by the conceptions of fractional numbers: Part-whole, measure, quotient, ratio, and operator, associated with each of the situations. The choice of an MO consisted of deciding what to teach from fractional numbers, what the fifth-grade student should learn. This choice involved, directly, the elaboration of a DO, consisting of how to put into practice the teaching of the specific MO in the classroom.

Through those theoretical choices, Silva (2005) carried out some preliminary studies for the desired teacher training. The first study dealt with the question of terminology used to refer to the mathematical object under discussion: fractions, rational numbers or fractional numbers, which led her to adopt the term "fractional numbers" to refer to any expression written in the form of fraction, including those involving complex or polynomial numbers.

The author conducted an epistemological study supported by the ATD, aiming to identify the types of tasks that justify the raison d'être of the fractional numbers, based on the concepts of fractional number, that could be associated with these tasks. This study showed that three needs caused the fractional numbers to appear: measuring, distributing, and comparing, which led to the development of tasks for measurement, distribution, and comparison.

Another preliminary study was the choice of the MO, which served as reference for the teacher training and the elaboration and analysis of the DOs that the preservice teachers presented for fractional teaching in the fifth grade. This option considered both the results of the epistemological study and some research results on fractional numbers.

As an instrument for the analysis of the DOs, Silva (2005) used the Didactic Moments (Chevallard, 1999), which allows for the description of the construction of a DO - which is characterized by assays, reformulations, impairments, and advances. Thus, according to the definition of each of the six moments, the author analyzed the MOs that were likely to be mobilized in the DOs presented by the teachers.

She also used the notion of Degree of Completeness of an MO (Bosch, Fonseca & Gascón 2004) to assist in the analysis of those DOs. This theoretical construct enabled the evaluation of the degree of rigidity of an MO, observing in particular the types of tasks that allow to mobilize different techniques, besides reversible or open tasks and the incidence or not of the technological element.

The use of the ATD showed that, in the elaboration of a DO, it is possible to avoid the choice of tasks of the same type or tasks that request the mobilization of the same technique that would represent the same know-how-to-do that would characterize a repetitive action for the students.

The MO chosen for the training seems to have been a contribution to the school institution to explain a variety of types of tasks and techniques that allow for the conceptualization of fractional numbers for the fifth grade. The MO also allows the teacher to analyze their own choices to verify what types of tasks or techniques they need to modify, add, or remove.

Methodology Contribution

The didactic engineering used involves two aspects: the research and development of teacher education objects. It adds some assumptions of action research that postulates the explicit interaction between researchers and subjects of the research situation to establish the priority of the problems to be treated, anticipating the follow-up of decisions, actions, and all intentional activity in the process (Barbier, 2004).

In this sense, Silva (2005) states that teachers participated actively in the work and some decisions, such as the choice of the fifth-grade group with which the work would be developed, the teacher who would lead those works, the choice of the didactic variables for the DO that would be applied, among others that were part of the discussions of each particular meeting.

The author assumed that she was in a process of continuing education, planned and developed in a university institution that, from a pragmatic point of view, promoted reflections on how the teacher guides their actions. Hence, she sought to trigger processes of changes in attitudes, conceptions, and practices in an environment of collaboration among the participants that created conditions for individual and collective reflection. In this perspective, she prioritized the following global formative actions:

- Individual production of a sequence for fractional teaching for the fifth grade. This action is part of the first stage of the training, allowing the researcher to characterize the study problem for the teachers, and being responsible for the teachers' first contacts with their non-knowledge of fractional numbers.
- Sequence group production based on the individuals with the same objective. This action permeated three stages of the training: one, which consisted of the elaboration in groups of the sequence; in the other, the specific training on fractional numbers was developed and, finally, the last stage, in which the works for collective elaboration of the sequence of teaching were resumed. Each one highlighted the actions arising from assessments and decisions taken throughout the process.

During the development of this training, several specific actions had to be decided to enable the continuity of the activities. One of them occurred when the teachers perceived their no-knowledge (Silva, 2005), which brought out emotions, through complaints from the students and parents regarding their education, etc., requesting quick formative actions to reverse this picture.

- The application of the DO built during the training in a fifth-grade group, developed in the sixth stage of the formation, was, in part, according to the author, responsible for the new gaze the teacher addressed the students, from observing the teacher trainer's action in the classroom, applying some of the DO sheets prepared. Thus, the freedom given to students to express themselves and produce made teachers discover the importance of the objective observation of students in action.

The researcher analyzed this months-long formation based on the record of the observations made throughout the meetings by at least three observers, focusing on the teachers' construction of two conceptual maps with the keyword "fractions" and the Didactic Organizations they elaborated during the course of the training.

Main results

Silva (2005) observed through the findings that the teachers emphasized techniques and nomenclatures related to the fractional numbers, and that their anxiety when they face their lack of knowledge of a mathematical content they thought they mastered prevented them from explaining the concepts they had before and the ones they acquired after training.

Generally, the author's analysis indicated that there were some changes in the conceptions of fractional numbers, not so much because the teachers can promote effective formative actions with autonomy for their students' learning, but by raising the awareness of the group about the limitation in mastery that they had regarding the content, besides the non-effectiveness of rules-based education, without understanding.

She found the possibility of changing the teachers' gaze to their students, when she allowed the teachers to observe the practice of flexible and interactive teaching methods that gave the students the effective place of apprentices.

She also realized that the lack of education for decision-making that would emerge with the development of some autonomy is one of the probable causes of the teachers' difficulties. This is probably because they are accustomed to playing their role based on the notions and principles available in the school institution that were not created by them and are often unknown because they were instituted in different historical moments.

Teachers' knowledge of Function: an investigation on the praxeologies. (Renata Rossini, 2006)

Rossini (2006) investigated teachers' conceptions of function and their difficulties with the subject, and how they overcame them throughout a process of continuing education. More specifically, she sought answers to the following questions: *Which MOs are mobilized during the construction of a sequence for teaching functions to an 8th grade of Elementary School? How do teachers (re)build their teaching knowledge of the concept of function?* The methodology adopted is the action-research in the sense of a collaborative research (Barbier, 2004) since it provides the interaction between researcher and teachers and their practice in training and in action. The theoretical foundation was based on the ATD (Chevallard, 1999) to model the concept of function in terms of the MO and DO, associated with the concepts of function: interdependence of quantities, input and output machine, analytical expression, regularity pattern of geometric sequences, correspondence between sets.

Contributions of the theoretical and methodological foundations

Rossini (2006) was based on the ATD, which allowed her to model the concept of function in terms of mathematical organizations and analyze each of the mathematical organizations around the conceptions of function in textbooks of the eighth grade.

The author asserts that the contribution of the ATD was crucial to follow up and analyze, step by step, and in a structured way, the work of the teachers involved in a project of continuing teacher training. The follow-up and analysis were based on the MOs the teachers' mobilized around the conceptions of function.

The teachers themselves chose the contents that were to be developed in the MO addressed to teach function to 8th-grade students. This choice led to the elaboration of the DO, which was the answer to the question: *How do you put the teaching of those MOs into practice in the classroom?* To do so, they needed to look for answers to their own questions, such as how to start a sequence, how to write a statement or a task, what tasks to ask, how to ask for a justification. Or address questions such as: *Should the student be asked (or not) to explain the resolution of a task? What is the best order of tasks for a given activity? What is the best order of activities to organize the sequence?*

The author noted that, initially, the teachers' decisions of "what to teach" and "how to teach" were based on their own view on mathematics teaching. To better understand these aspects, the author used Fiorentini's (1995) description and identification of the different pedagogical trends of mathematics teaching in Brazil. Some teachers were reluctant to abandon their form of work, which is in line with Ponte's explanation (1992), for whom the conceptions are a backdrop that organizes the concepts, constituting "mini theories."

In this research, the content knowledge and the pedagogical content knowledge of functions (Shulman, 1986) were revisited in terms of the MO and the DO.

To define the different knowledge present in the teaching practice, the author used the typology proposed by Tardif (2002), who states that teacher knowledge is a plural knowledge, formed by the amalgam of knowledge derived from professional training, disciplinary, curricular, and experiential knowledge, in which *the temporal dimension* cannot be disregarded. Thus, Rossini (2006) identified several components of the professional life of the teachers participating in the project: where and when their initial education took place; their involvement in other projects of continuing education; time of service in teaching; previous experience in teaching functions and, furthermore, the reach of their curricular knowledge.

The author sought to base her study on Lastória and Mizukami's (2002) training practices, who consider the construction of instructional materials as a path for teacher's learning, and on Nóvoa (1992), who stresses that continuing education must be engaged in collective practices.

Main results

Rossini (2006) assumes that the collective elaboration and analysis of a didactic sequence on functions and its subsequent application in the classroom could trigger a process of building a set of teacher knowledges. They regard the mathematical object, the pedagogical content knowledge, the reflection on the genesis of the mathematical object, the importance of that knowledge within the curriculum, and the knowledge of the students' potentialities.

The complex architecture of teacher continuing education assembled by the author seems to have provided conditions for them to build a teaching knowledge of the concept of function, which considers the content knowledge the pedagogical content knowledge, and the knowledge of the historical evolution of the concept. Also, acting as observers and/or trainers enabled them a new gaze toward the students' potentialities, and toward themselves.

Rossini inferred, in her analysis of the findings, that this form of work allowed teachers to reconstruct and expand their pedagogical knowledge of *function* as an mathematical object, simultaneously to the elaboration of the didactic sequence for the teaching and learning of functions. She also observed that the pedagogical content knowledge was strengthened in the teaching organizations formulated by the teachers.

According to the author, the intricate work of experiences, collective works, and reflections led teachers to elaborate statements and tasks for the activities that compose the teaching sequence, and to perceive and present an articulation of types of tasks around the concept of function, a discourse that goes beyond what is found in the textbooks analyzed. Thus, the teacher became a producer of knowledge and not only a technician who applied a task proposed by a textbook.

She still affirms that the teachers found it quite challenging to build and apply a didactic sequence to teach functions. In the end, the teachers, protagonists of the research, assessed the special meaning of this experience: self-assurance to deal with functions in the classroom, to talk about the topic with the colleagues; satisfaction for having managed to create activities, with the dynamics of the training itself, which, according to the teachers, enabled them to unite fragmented knowledge.

Proof and demonstration in plane geometry: conceptions of mathematics teaching degree students in Mozambique (Jacinto Ordem, 2015)

Ordem (2015) aimed to analyze the conceptions of proof and demonstration in plane geometry of mathematics teaching degree undergraduates at the Pedagogical University of Mozambique, through a sequence of tasks that require the production of proofs and demonstrations and the assessment of methods of proof by the research subjects. The same subjects are interviewed about their productions. To carry out this part of the research, each subject talk to the researcher about what he/she did. The author seeks to highlight the meaning of those productions. Nineteen undergraduates of the 4th year of the Mathematics Teaching Degree for the final grades of the elementary education – Secondary Education – from the Nampula and Beira delegations participated in the research. Ordem (2015) also earned from the didactic analysis (*a priori* and *a posteriori*) of tasks designed for data collection. As a theoretical reference, this author used the ideas of Geometrical Paradigms and Geometrical Working Space proposed by Houdement and Kuzniak (2003, 2006); the Types of Proofs proposed by Balacheff (1987) and the Proof Schemes advanced by Harel and Sowder (1998, 2007).

Contributions of the theoretical and the methodological foundations

Although they are not the subject of explicit teaching as are several mathematical concepts, the notions of proof and demonstration are inseparable partners of those concepts for their validation. However, several studies carried out by Ordem (2015) show that these notions are among the most difficult in all the mathematics teaching. The difficulties are manifested, above all, by the conflicts between the meaning the mathematicians attributed to those notions, their value within professional mathematics, including researchers in mathematics education, with the meaning often built by the students and students in the process of their appropriation, and sometimes, as a result of the teaching practices of their teachers.

The geometrical paradigms and the geometrical working space (Houdement & Kuzniak, 2003, 2006) allowed the analysis of the research subjects' answers related to the geometric category, as well as the geometric object and its properties mobilized, or those which constitute or not the repertoire of those who acted in a particular way to solve a given research task.

The types of proof (Balacheff, 1987) or the proof schemes (Harel & Sowder, 1998, 2007) were fundamental to classify the proofs explicitly presented by the subjects according to the validation procedures or the justifications presented during the task resolution.

To dispel misunderstandings in the interpretation of the meaning of some terms, recurrent throughout the thesis, Ordem assumed a theoretical posture that differentiates proof from the demonstration according to Balacheff (1987).

Regarding the methodology, Ordem opted a qualitative research because he noted that little or nothing was written about Mozambican students' conceptions of proof and demonstration, particularly about the conceptions of proof and demonstration of mathematics undergraduate students. He chose methodological procedures that were based on sequence of tasks and semi-structured interviews on the productions presented, to allow for a method triangulation cross-checking. The author claims that this procedure was useful, because the set of tasks of proof and demonstration that the subjects should produce or assess served to have their actions recorded on paper during the resolution of the tasks of the research, and the interview served to explain the thought behind the resolutions, because it is not always easy to have them registered on paper for several reasons: lack of writing habit, space limitation, time and action savings, etc.

This procedure was crucial, because in some cases there was a clear-up of misconceptions that arose during the resolution, which helped to give a better interpretation of what was put forward and consequently to change or improve the message of both the transmitting and the receiving (in this case, the subject of the research and/or the researcher).

Main results

Taking into account the method triangulation cross-checking that consisted of comparing the answers to the questionnaire and the answers obtained in the interview to the resolutions of the respective sequence of tasks, the research indicates that the research question: *What conceptions do mathematics undergraduate students present in situations involving proof and demonstrations in plane geometry?*, desired by Ordem (2015,) was answered, based on the following arguments:

1. The facts Ordem identified present some evidence of conceptions of the investigated subjects. Those conceptions lead, in most of the tasks presented, to errors regarding the validation of the properties and/or claims involved in them. He identified, for example, that the mathematics teaching degree students who participated in the research see proofs and demonstrations as a simple ritual dissociated from their function, which is to validate true properties and conjectures, or to refute false conjectures.

2. The study reveals that among the research subjects there are empirical methods that validate geometric properties, even if they are not demonstrations, and empirical methods that do not validate geometric properties, depending on the type of instrument used. From the perspective of conceptions according to Artigue (1990), this can simply mean that until the fourth year of college, those students stick to a meaning of demonstration that escapes the notion that usually prevails in mathematics.

3. The results of the study show that, despite having some participants who rejected this or that method based on examples, they do not generally know that it is problematic to use empirical evidence as a means of generalization.

4. The answers of the research participants revealed they did not know the importance of justifying each step of a demonstration. A relevant detail is that the meaning of demonstration built by the participants seems more attached to textbooks that misrepresent the demonstrations than to mathematical ideas.

5. From the point of view of the theory of the geometric paradigms and working space, the results achieved show that, in general, those students acted guided by the concrete geometry (dominated by the observation and manipulation of concrete objects), and the concepts and properties of the plane geometry do not seem to have been properly assimilated. In terms of type of proofs or proof schemes, the pragmatic proofs, or perceptual proofs, were the most prevalent.

Teaching practice and its influence on the construction of geometric concepts: a study on the teaching and learning of orthogonal symmetry (Cleusiane Vieira Silva, 2015)

Silva (2015) investigated how an environment of action and reflection that involves pre-analysis, reflections on pre-analysis, experimentation with middle-school students, post-analysis, and reflections on post-analysis related to a didactic sequence on orthogonal symmetry interferes with the mathematics teachers' knowledge at that level of teaching. She wanted to answer the following question: *How can an environment of action and reflection scheduled in the period intended for Complementary Activities (C.A.) influence the teaches knowledge of middle-school mathematics teachers on orthogonal symmetry?*

The methodology used was based on the assumptions of the Didactic Engineering (Artigue, 1995) and on Schön's contributions (1995; 2000). The theoretical framework was based on the TDS (Brousseau, 1997), to study the influence of the didactic variables chosen in the procedures and answers of middle-school mathematics teachers and their students, and on Margolinas (2002); 2004), for the analysis of the teacher's activity, to understand how this professional develops his/her teaching practice and how it influences the students' learning. She also used the Geometrical Paradigms (Parzysz, 2001; 2006) framework to analyze the nature of the geometric work developed by teachers when solving and analyzing problem situations and by students when interacting with those same problem situations.

Contributions of the theoretical and methodological foundations

The choice for the Didactic Engineering (Artigue, 1990) helped in the general formatting of the research. The preliminary studies made it possible to characterize the mathematical object orthogonal symmetry. In the light of the ecology of the didactic, the author could have a notion of the reality that surrounds the teaching of this object. Also, the literature review allowed access to the research already carried out on the teaching and learning processes of orthogonal symmetry.

The construction and *a priori analysis* of the didactic sequence on orthogonal symmetry allowed, through the theoretical framework chosen, the prediction of possible resolution procedures and answers to be presented by teachers and students involved in the research, and the difficulties that could arise during those procedures.

Silva (2015) also based her sequence implementation on the concept of reflection on the practice, proposed by Schön (2000).

She collected the data through questionnaires, followed by fortnightly meetings in the form of collective debates. Part of the data was collected through audio recording. The audio recorded in those meetings was transcribed, analyzed, and used in the form of questionnaires in the next data collection instruments in the following meetings, in a process of knowledge-in-action, reflection-on-action and action-on-reflection (Schön, 2000).

The study of the orthogonal symmetry in the light of the Ecology of the Didactic and the analysis of official curricular documents (National Curriculum Parameters, Mathematics Curriculum Guidelines for Elementary School of the state of Bahia) and of some collections of Mathematics textbooks for middle school promoted reflections on the suggestions of progression from one cycle to the other proposed for geometric transformations.

Regarding orthogonal symmetry, the author observed that several relationships can be proposed between this and other geometric objects. Depending on how they relate, orthogonal symmetry will be seen as a mathematical object that must be studied from its definition and its properties. On the other hand, those properties have intrinsic relationships with other geometric objects, in which case orthogonal symmetry serves to feed their study.

The TDS has leveraged relationships between the roles of the teacher and the student in each of the situations in the learning phases, and, on the other hand, the Geometrical Paradigms provided a study of the subjects' records from a geometric view.

Main results

Through this research, Silva (2015) confirmed some of the results obtained by Grenier (1988) related to the conceptions of middle-school students about orthogonal symmetry, and identified others that seem to be specific to the group of students investigated. For this group of students, she observed the following conceptions:

- the image of a horizontal, vertical, or oblique segment is a segment of the same direction on the sheet, regardless of the direction of the symmetry axis;
- the orthogonal symmetry is a transformation of a semiplane into another semiplane, delimited by the symmetry axis, caused by the association of two didactic variables, the intersection of the object-figure with the symmetry axis and by the complexity of the object-figure.
- the distance from the object-figure to the edges of the sheet (lower, upper or lateral) will be retained for the line of the symmetrical figure, regardless of the direction of the symmetry axis on the sheet, motivated by the position of the symmetry axis on the sheet (when it does not divide the plane into two semiplanes of the same dimensions).

The author also noted that the fact that the students ignored the orthogonality and sometimes the conservation of distance as the properties of orthogonal symmetry led them to strengthen the conceptions related, since the domain of validity of those subjects was restricted to perception through visualization.

As for the teachers participating in the research, she states that they have engaged their previous knowledge to solve the proposed problem situations, but this did not prevent them from using their perception in the procedures and responses to the set of activities proposed in the didactic sequence. This resulted in the construction of wrong symmetry axes and symmetrical figures, possibly for ignoring the orthogonality.

Regarding the teaching of orthogonal symmetry, Silva (2015) identified three conceptions in the teachers, participants in the research:

- the use of terms such as mirroring, or reflection, can facilitate the understanding of learners when resolving activities about orthogonal symmetry.
- if the student knows how to draw the symmetric point, he/she can trace the symmetric of any figure.
- the squared paper is a facilitating agent for the construction of a symmetric figure.

Regarding the relationships established in the study of the conceptions of students and teachers, she observed that in the analysis of the didactic sequence, the teachers identified some of the didactic variables that could influence their students' procedures and answers. When Silva (2015) analyzed the records of the students' pairs, some of her hypotheses were confirmed.

By comparing the pre-analysis of the didactic sequence and the post-analysis in the records of some pairs provided teachers with reflections on the importance of taking into account the didactic variables and their values in the construction of teaching situations, and evaluate their influence on their students' procedures and responses.

When they analyzed the records of the mathematics teachers, she realized that some conceptions of orthogonal symmetry seem to be related to how this concept is presented in the textbooks. During the investigation, teachers assessed their practice and thought about the teaching methods they adopted, to observe whether such methods are being effective for their students' learning. The author states that an environment of action and reflection in the school would have influenced the teacher knowledge of the mathematics teachers, although its influence is limited.

A didactic organization in quadrilaterals that brings the student closer to the geometric demonstrations (Maridete Brito Cunha Ferreira, 2016)

Ferreira (2016) carried out a didactic proposal whose tasks articulate proofs and demonstrations as a methodological alternative to minimize the difficulties related to the topic 'quadrilaterals' in a mathematics teaching-degree course. The tasks involve geometric constructions in a paper-and-pencil environment in which students are asked to construct geometric figures and mathematically justify the techniques used. While dealing with the resolution of the tasks, the students convert representations of different records and mobilize the different apprehensions of a geometric figure (sequential, perceptual, operative, and discursive). To fulfil this objective, Ferreira looked for support of the methodology of Didactic Engineering and was based on the Registers of Semiotic Representation Theory (RSRT), the TDS, and the ATD. Besides these references, the author used the Types of Proof proposed by Balacheff (1988) and the different functions of the demonstration (validation, explanation, systematization, and communication) according to De Villier (2001).

Contributions of the theoretical and methodological foundations

The TDS was fundamental, on the one hand, to elaborate and experiment the didactic sequence, as well as in the didactic analysis of the teaching situation, and, on the other hand, to construct tasks that would allow students to participate in the process of building geometric concepts through demonstrations. The TDS played a key role in the development of a Dominant Model (in the sense of the ATD) that provided students with the simulation of a scientific environment in which they could participate actively in the construction of demonstrations. The tasks of geometric construction required the students to justify the techniques they used mathematically. A learning context was created for the students to test conjectures, formulate, and validate their hypotheses, socialize and communicate their results, mobilizing previous knowledge, such as the congruence of triangles, the theorem of the parallels, and the perpendicular bisector of a segment to rebuild knowledge about quadrilaterals.

For the cognitive analysis of the students' productions, the author used the RSRT (Duval, 2011, 2012). The articulation between the records (treatment and conversion) helped in the analysis of how the students' knowledge about quadrilaterals is mobilized in the different (perceptual, sequential, operative, and discursive) apprehensions of figures from the different tasks proposed.

The ATD has provided elements to analyze textbooks and the different tasks of the didactic sequence, and to model the teaching situation.

The distinction between the terms 'proof' and 'demonstration' (Balacheff, 2000) aims to minimize problems in the interpretation of the term 'proof' by socializing the results of the participants' interactions with the different tasks. According to Balacheff, the levels of proof were fundamental to classify the types of proofs performed by the students.

Regarding the different functions of the demonstration (De Villier, 2001), Ferreira observed that the students demonstrated aiming exclusively at verifying/convincing. The actions taken to minimize this problem provided the students with the conditions to consider other functions of the demonstration, especially the explanation function, which is one of the sources of the mathematics demonstrations.

In the experimental phase of engineering, after identifying the problems that influence the maintenance of the teaching and learning context of demonstrations in the epistemological, cognitive, and didactic dimensions, Ferreira (2016) highlighted the global variables and their values that underpin the construction, the a *priori analysis*, experimentation, and the *a posteriori analysis* of the different tasks that compose the didactic sequence. Based on those variables, she organized and applied her teaching sequence in such a way as that:

- the approach to the tasks should follow the inverse order of the proposed in the textbooks analyzed. The tasks applied are intended to allow students to experience moments of action, formulation, and validation, taking an active role in solving situations, culminating in institutionalization (Brousseau, 2008).

- it provides students with tasks in which they could coordinate between the representation registers and between the apprehensions of the figure (Duval, 2011, 2012), so as not to confuse a geometric object with its representation and to recognize the status of geometric figures, axioms, of theorems, and definitions.

- it requests mathematical justifications of the constructions, demanding that the students practice the demonstration considering, besides the function validation, other functions of the demonstration, such as explanation, systematization, communication, and discovery, as described by De Villier (2001).

Main results

Overall, besides the conceptual difficulties related to the characterization of the notable quadrilaterals and the relations with each other, Ferreira (2016) points out, among other aspects, that the choices of the variables and the values students attributed them mobilized different knowledges, since each choice led to different construction strategies and, consequently, required a corresponding mathematical justification. The choices made allowed the institutionalization of the main properties of the notable quadrilaterals. Ferreira also observed:

- an advancement in terms of geometric knowledge, since the students could already mobilize knowledge of a segment perpendicular bisector, parallel theorem, triangles congruence, quadrilateral properties, bisector, and isosceles triangle in the validation of the technique used;
- a reduction in the use of empirical arguments to validate techniques;
- that the students seemed to have become aware of the limitations of the perceptual apprehension and started to carry out a discursive interpretation of the figure, which evidences the awareness of the status of the geometric figures, axioms, theorems and definitions;
- an evolution from pragmatic proofs to conceptual proofs (Balacheff, 2000), and in the writing of the demonstration and the structure of well-structured demonstrations;
- other functions of the demonstration, besides the validation function, such as those of explanation, systematization, and communication (De Villier, 2001).

The PEP-FP /ATD research and professional training device: the constitution of the teaching knowledge for the teaching of plane analytical geometry of the point and the line (Rita Lobo Freitas, 2019)

The research developed by Freitas (2019) is part of the mathematics teachers' initial education, specifically in Brazilian Mathematics teaching degree, and aimed to answer the following question: *What new (or not) knowledge in plane analytical geometry (PAG) can be acquired by preservice teachers, with the help of a Study and Research Trajectory for Teacher Education (PEP-FP), and what benefits can they obtain to design (conserve) this knowledge in high school?* To answer this question, Freitas set up a PEP-FP/ATD-methodological device, through which she developed thirteen study and research sessions with students (preservice teachers) of a course of Mathematics teaching-degree course in Bahia, attending the supervised-practice courses. The theoretical reference adopted was the ATD, with the support of studies on teacher knowledge.

From the point of view of the methodological paths, the author called for the Study and Research Trajectory for Teacher Education (PEP-FP) developed by Chevallard (2009) and collaborators. This methodology is based, in the first place, on the study of the three dimensions of the didactic problem (Gascón, 2011) of PAG. They are the epistemological, ecological, and economic dimensions. The analysis of the epistemological and economic-institutional dimensions of a didactic problem is based on historical development, allowing for the identification of the different forms of conceptions of a given mathematical object that may favour the didactic analysis. This study allows for, among other aspects, the identification of the reasons for being of that mathematical object and the problems of its teaching. The analysis of the economic dimension is necessary to identify how praxeologies (the tasks, the ways of solving them, and their mathematical justifications) behave in a given institution (in mathematics, in the pedagogical projects of the university courses, in the most used textbooks, etc.). It is based, among other things, on issues relating to the conditions governing the organization and functioning of such practices in the reference institution (the university, for example), that is, on issues relating to the system of rules, principles, and laws (rules) governing its institutional life. The institutional-economic dimension permeates the ecological dimension, since "birth," "life," and the possibility of "withering" and/or "resurging" dispense with economic conditions. The ecological dimension allows, for example, to locate, from the didactic point of view, ***the habitats*** and ***niches*** of the mathematical object investigated in the ecosystem of the teaching at stake. ***The habitats*** will be the conceptual environments where a particular object of mathematical knowledge meets and experience its practices. *The habitats* will be the sectors of an ecosystem where the curricular components shelter praxeologies with mathematical objects. *The niches*, in turn, will regard its functionalities and praxeologies, which are evidenced by the practices that, in relation to a teaching object, are indicated in a given *habitat* of a specific ecosystem, interacting with the niches.

To study the economic and ecological dimensions of the didactic problem, Freitas (2019) used a reference epistemological model (REM) built from the study of the epistemological dimension of her didactic problem. The study of the economic and ecological dimensions allowed the construction of the dominant epistemological model (DEM) for the teaching of the PAG in the different institutions (high school and university), in consistency with its REM and DEM, and, based on them, Freitas built an alternative epistemological model (MERA, in the Portuguese acronym) of what means "learning" mathematical knowledge in this field. The MERA constructed served as support for the development of the experimental phase of its PEP-FP.

Contributions of the theoretical and methodological foundations

Freitas (2019) articulated the two benchmarks for the creation of a device for research and teacher education. Having developed an exploratory qualitative research, whose methodology was PEP'S Didactic Engineering and epistemological models, the ATD, as a theoretical reference, is inseparable from this discussion.

The REM constructed enabled the reconstruction of the main aspects that contributed to the constitution of what we know today as Analytical Geometry, especially the Plane Analytical Geometry.

For Freitas's research, the REM fulfilled the role of epistemological nature, within the scope of the AG, in the sense of the ATD.

Regarding the constitution and analysis of the DEM, having a REM to highlight the ecological transformations of the study objects allowed, according to Freitas (2019), first, to enlighten the researcher regarding the didactic problems arising from the way that this DEM is inserted in the educational system. Secondly, the conclusions imposed on the DEM have brought important elements for the construction and experimentation of the PEP-FP.

MERA was a balanced mix of the relevant aspects of the REM that were not covered in the DEM and vice versa. Freitas (2019) states that, in this perspective, the MERA "has fulfilled its role of being an "alternative" to the proposal for research and teacher education on the PAG, seeking to overcome the paradigm of a visit to the works and the vector approach of the AG." She also claims that the PEP-FP, as a methodological reference for research and teacher education articulated to the ATD was configured as a theoretical-methodological device that allowed us to question the world, more specifically, the world of a part of the PAG (Freitas, 2019).

The comprehensive character of the device began to be drawn in the construction and analysis of the epistemological models and the *a priori* description of the study sessions. Freitas (2019) shows that although it is not possible to predict some of the subjects' behaviors regarding the activities that were proposed, the *a priori* description of the experiment was fundamental, since it often allowed us to redesign the situations that were proposed in the experiment and build new ones according to the needs of the research.

From a macro perspective, the experimental device made it possible for subjects to (re)signify some teacher knowledge. This result leads us to resume the essential elements of the investigation of the findings, linking the answers to the research question.

Main results

Freitas (2019) stated that the praxeologies the students participating in the research adopted throughout the experiment indicate that there were changes in their praxeological equipment. Also, their praxeologies revealed several knowledges placed in action for the development of the different types of activities and mathematical tasks, activities inherent to the teacher's profession.

To identify the elements of teacher education, Freitas (2019) was based on the categories of teaching knowledge from the works by Shulman (1986, 1987), Ball, Thames, and Phelps (2008), Mishra and Koehler (2006), and Silva and Lima (2015). This perspective allowed the author to define four essential categories to articulate the mathematical and the didactic in the training proposal with the preservice teachers: the (mathematical) content knowledge (CK), the didactic content knowledge (DCK), the pedagogical content knowledge (PCK) and the didactic-pedagogical-technological content knowledge (DPTCK).

The author affirms that the results obtained show that several of those knowledges were mobilized during the experiment, and some were constructed when the students had no mastery, while others were reconstructed from the didactic view incorporated into the study.

The training proposal based on the MERA for the teaching of the PAG by a theoretical-methodological device (PEP-FP /ATD) seems to have had the potential to make participants in the training appropriate some professional teacher knowledges related to the education and learning of the high school AG. On the other side, they also revealed their weaknesses that were overcome in the face of the knowledges studied. They disclose that the training carried out both in the mathematical content learned and in the aspect of the didactic approach aimed at high school education was something completely new and that it gave practical meaning to the knowledges they already had. The theory-practice dialectics promoted in the different stages of the formation does not seem to have been experienced by them in the teaching degree until then.

Final notes

This text aims to make a metasynthesis of six doctoral theses developed under my supervision, discuss the theories of the Didactics of Mathematics and other theoretical constructs developed in the area of mathematics education. The map I have presented does not claim to comprehend all the triangulations developed in the various works. It brings to light some theoretical foundations of the Didactics of Mathematics in the context of different research works that dialogue with each other and/or complement some theoretical constructs of mathematics education in the analysis and interpretation of factors that interfere in the teaching and learning processes of mathematical concepts.

Although not exhaustive, this study provides a vision of possible dialogues of Mathematics Education (or Education) theories related to teaching and learning of mathematics and the Didactics of Mathematics. The diversity of theories (Figure 2) and the specificities of each of them confirm the idea that a single theoretical tendency, or a single model, hardly ever explains and makes explicit all the phenomena involved in the teaching and learning processes of mathematical concepts (Almouloud, 2019). The authors of the works mapped here sought to know very well the main ideas of the various theories and to identify which of them they could use to reference their research theoretically.

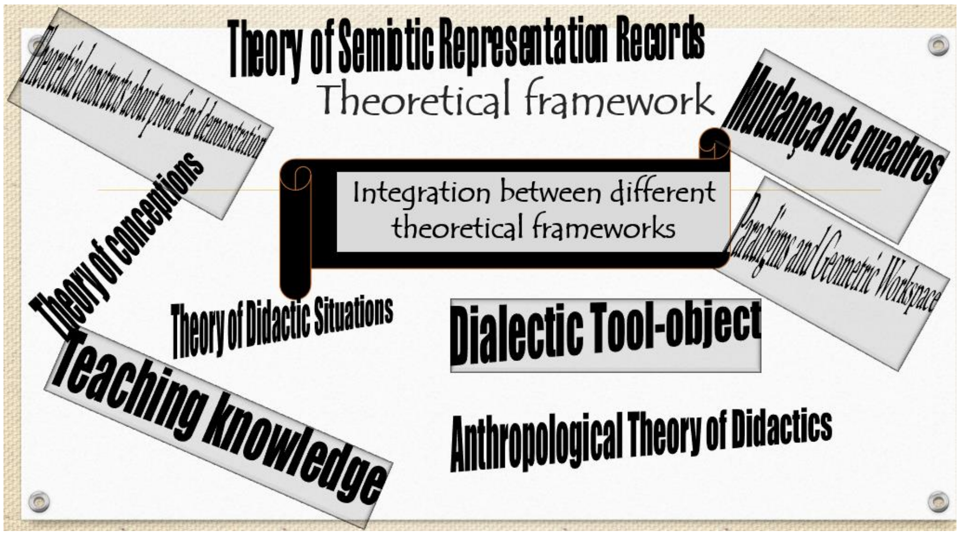

Figure 2: Theories and theoretical constructs used in different research works (adapted from Almouloud 2019, p. 172)

The results and suggestions obtained from the research works we revisited highlight the challenges of choosing situations that could be pertinent to teacher education and the teaching/learning of the mathematical objects chosen. The theories and/or theoretical constructs allowed the authors to choose the elements necessary to model the mathematical content at stake in the different works. The ATD, for example, allowed the authors to analyze mathematical praxis as products of an institution, according to the types of tasks that members of that institution must fulfil, of the techniques they mobilize to solve those tasks, the technology they use to justify such techniques, and the theory that justifies those technologies.

All six studies focused on the teacher (continuing and initial) education. I understand that the didactic analyses and the training actions undertaken helped the teachers who participated in those studies in the construction of problem situations that met the objectives of the teaching and the learning they targeted.

I also understand that knowledge as a personal construction does not occur only in a cognitive way, the subject must identify with what they learns so that they can give their meaning to the relationship they build with knowledge. Learning takes an active sense for the individuals and is linked to the moment and situation in which learning occurs. Thus, the object of analysis - when the learning processes are studied - should be the relationships in which the subjects engage in knowledge. In other words, to question this set of relationships with knowledge is to reflect on the process in which the subject is integrated into their *milieu*.

From the research methodology perspective, the authors of the studies analyzed here use the didactic engineering, or its principles. This methodological perspective seems to have provided the authors with conditions to develop training situations, in which the researcher is guided to describe and analyze the results of its application, taking due care of the degree of generality of the outcomes. This led to the production of resources that the teachers can use in their class, or for the continuing or initial teacher education, to learn mathematics or to teach mathematics.

We used all the research works addressed here to observe the epistemological, ecological, and economic dimensions, aiming to identify the different forms of conceptions of a given mathematical object that can favor the didactic analysis. This study allowed us to determine, among other aspects, the reasons for being of the mathematical objects and the problems of their teaching.

For teacher education, all mapped research works, except for Ordem's (2015), have been supported by a methodological perspective (Didactic Engineering of 1st Generation and the PEP-FP) that aims to familiarize teachers in initial or continuing education with this perspective as a useful didactic device for their professional development, preparing them for an effective transition from the monumentalist paradigm to the world-questioning paradigm (Chevallard, 1999). Teacher education needs didactic devices not based solely on the monumentalist paradigm, and, because of that, we must somehow resort to devices with a PEP-FP -type structure.

The authors conducted different trajectories for teacher education, designed and developed to promote reflections on how the teacher guides his own actions and to trigger processes of changes in attitudes, conceptions, and practices in a collaborative environment among the participants.

In my view, those studies are a significant contribution to the process of continuing and initial education that was intended to develop the teacher's autonomy, both about mathematical content and to the teaching of such content. The works carried out with the teachers emphasized three aspects: content, regarding the research topics, didactic training, and a critical analysis of the teaching practice. From the didactic point of view, I observe that the different works involved types of formation for an epistemological and critical gaze at the discipline that teachers will have to teach. Mathematical knowledge refers to the knowledge they must possess of the mathematical concepts and procedures, problem-solving strategies, and the links that exist between those different components, on the one hand, and between a mathematical notion and another, on the other.

The methodological procedure followed by the training offered by the authors is a contribution to the area, since it allowed us to propose a method of continuing and/or initial education that aimed to develop the teacher's or the preservice teacher's autonomy, both concerning mathematical content and their teaching, that leads them to produce new knowledge rather than just reproducing it. It seems to have allowed the teachers who participated in the research to present their lack of knowledge (Silva, 2005), while the discourse in situations of reproduction can camouflage them.

As a perspective, I suggest a meta-analysis based on the categories that would be built from the synthesis of those investigations. One of the objectives would be to re-read and analyze some of the predominant aspects of those works, the teacher education, for example, from the ATD, and/or TDS, or the didactic engineering perspectives.

References

Almouloud, S. Ag (2019). Diálogos da didática da matemática com outras tendências da educação matemática. Caminhos da Educação Matemática em Revista/Online, v. 9, n. 1, p. 145-178.

Artigue, M. (1990). Épistémologie et didactique. *Recherches en Didactique des Mathématiques*, *10*(2-3), 241-286.

Balacheff, N. (2000). *Procesos de prueba en los alumnos de matemáticas* (P. Gómez, Trad.). Bogotá, Colombia: Universidad de los Andes. Disponível em: <http://hal.archives-ouvertes.fr/docs/00/52/01/33/PDF/Balacheff2000Proceso.pdf>. Data do acesso: 12 de janeiro de 2013.

Balacheff, N. (1987). Processus de preuve et situations de validation. *Educational Studies in Mathematics*, *18*(2), 147-176.

Ball, D. L., Thames, M. H. & Phelps. G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education***,** *59*(5), pp. 389-407.

BARBIER, R. **A Pesquisa-ação**. Tradução: Lucie Didio. Brasília: Liber Livro Editora (Série Pesquisa em Educação, v.3), 2004

Bosch, M., Fonseca, C. & Gascón, J. (2004). Incompletitud de las organizaciones matemáticas locales en las instituciones escolares. *Recherches en Didactique des Mathématiques, 24*(2-3), 205-250.

Brousseau, G. (1997). *La théorie des situations didactiques – Le cours de Montréal*. Disponível em : http://guy-brousseau.com/1694/la-theorie-des-situations-didactiques-le-cours-de-montreal-1997/.

Brousseau, G. (2008). Introdução ao estudo da teoria das situações didáticas: conteúdos e métodos de ensino (C. Bogéa, Trad.). (1ª ed). São Paulo: Ática.

Chevallard, Y. (2009). La notion d´ingénierie didactique, un concept à refonder : Questionnement et éléments de réponses à partir de la TAD. In C. Margolinas et all. (Org.), *En amont et en aval des ingénieries didactiques : XVª École d´Été de Didactique des Mathématiques* (pp. 81-108). Grenoble : La Pensée Sauvage.

Chevallard, Y. (1999). L´analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques, 19*(2), 221-265.

De Villiers, M. (2001). *Papel e funções da demonstração no trabalho com Sketchpad* (E. Veloso, Trad.). *Educação e Matemática*, 63, 31-36.

Duval, R. (2012). Abordagem cognitiva de problemas de geometria em termos de congruência (M.T. Moretti, Trad.). *Revemat: R. Eletr. de Edu. Matem, 7*(1), 118-138. ISSN 1981-1322.

Duval, R. (2011). *Ver e ensinar a Matemática de outra forma: entrar no modo matemático de pensar: os registros de representação semióticas.* (1 ed). São Paulo: PROEM.

Fiorentini, D. (1995). Alguns modos de ver e conceber o ensino da Matemática no Brasil. *Zetetiké, 3*(4), 1-37.

Gascón, J. Las tres dimensiones fundamentales de um problema didáctico. el caso del álgebra elementar. Revista Latinoamericana de Investigación em Matémática Educativa, v.203 n.23, 2011. p.230-245.

Grenier, D. (1988). *Construction et étude du fonctionnement d'un processus d'enseignement sur la symétrie orthogonale en sixième.* 1988. Tese (doutorado em Didática da Matemática). Universidade Joseph Fourier Grenoble I. França.

Harel, G. & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in Collegiate Mathematics Education III* (pp. 234-283). Providence, RI: American Mathematical Society.

Harel, G. & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 805-842). Greenwich, CT: Information Age Pub Inc.

Houdement, C. & Kuzniak, A. (2006). Paradigmes Géométriques et enseignement de la géométrie. In *Annales de Didactique et de Sciences Cognitives*, 11, 175-193. halshs-00858709

Houdement, C. & Kuzniak, A. (2003). Elementary geometry split into different geometrical paradigms. In *Proceedings of European Research in Mathematics Education III.* Disponível em: http://ermeweb.free.fr/CERME3/Groups/TG7/TG7_Houdement_cerme3.pdf. Acesso em: maio de 2018.

Lastória, A. C.& Mizukami, M.G.N. (2002). Construção de material instrucional como ferramenta para aprendizagens docentes. In M.G.N. Mizukami, M.R. Reali (Org), *Aprendizagem Profissional da Docência: Saberes, Contextos e Práticas* (pp.187-207). São Carlos: EdUFSCar.

Lima, I. (2006). *De la modélisation de connaissances des élèves aux décisions didactiques des professeurs: Étude didactique dans le cas de la symétrie orthogonale.* 2006. Tese (Doutorado em Didática da Matemática). Universidade Joseph Fourier Grenoble I. França.

Margolinas, C. (2002). Situations, milieux, connaissances : analyse de l'activité du professeur. In J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot & R. Floris (Eds.), *Actes de la 11ème Ecole d'Eté de Didactique des Mathématiques* (pp. 141-156). Grenoble La Pensée Sauvage. Disponível em : <http://hal.archives-ouvertes.fr/docs/00/42/18/48/PDF/2002_T2-Cours2-Margolinas.pdf>. Acesso em: 06 jun. 2014.

Margolinas, C. (2004). *Points de vue de l'élève et du professeur: essai de développement de la théorie des situations didactiques.* Disponível em: <http://tel.archives-ouvertes.fr/docs/00/42/96/95/PDF/HDR_Margolinas.pdf>. Acesso em: 06 jun. 2014.

Nóvoa, A. (1992). Formação de professores e profissão docente. In A. Nóvoa (Org), *Os professores e a sua formação* (pp. 15-33). Lisboa: Dom Quixote.

Parzsyz, B. (2001). Articulation et déduction dans une démarche géométrique em PE1. In *Actes du 28eme colloque COPIREM (tours)* (pp. 99-110), Ed. Université d'Orléans.

Parzsyz, B. (2006). La géométrie dans l'enseignement secondaire et em formation de professeurs des écoles: de quoi s'agit-il? *Quaderni di Ricerca in Didática*, *17*, 128-151.

Perrin-Glorian, M. J. (2011). L'ingénierie didactique à l'interface de la recherche avec l'enseignement. Développement des ressources et formation des enseignants. In C. Margolinas, M. et al. (Éds). *En amont et en aval des ingénieries didactiques* (pp. 57- 78). Grenoble : La Pensée Sauvage.

Perrin-Glorian, M.J. (1994). Théorie des situations didactiques: naissance, développement, perspectives. In M. Artigue, R. Gras, C. Laborde & P. Tavignot (Org.), *Vingt ans de didactique des mathématiques en France. Recherches en Didactiques des Mathématiques* (pp. 96-147). Grenoble: La Pensée Sauvage Éditions.

Ponte, J. P. da (1992). Concepções dos Professores de Matemática e Processos de Formação. In *Educação Matemática: temas de investigação* (pp. 185-239). Lisboa: J. P. Ponte (Ed.).

Schön, D. A. (2000). *Educando o profissional reflexivo:* um novo design para o ensino e a aprendizagem. Porto Alegre: Artmed.

Schön (1995). Formar professores como profissionais reflexivos. In A. Nóvoa (Org.), *Os professores e a sua formação* (pp. 79-91). (2. Ed). Lisboa: Instituto de Inovação Educacional.

Silva, M. J. F. da & Lima, G. L. de (2015). Conhecimentos desenvolvidos em um curso de Licenciatura em Matemática na modalidade a distância. In *Anais do XIV CIAEM-IACME*, Chiapas, México.

Shulman, L.S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4 -14. https://doi.org/10.3102/0013189X015002004

Shulman, L. (1987). Conhecimento e Ensino: Fundamentos da Nova Reforma. *Harvard Educational Review*, *57*(1), 1-23.

Tardif, M. (2002). *Saberes Docentes e Formação Profissional*. Petrópolis: Editora Vozes.