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Mathematics Textbooks as Subject of Study:
Producing Knowledge on the Presence of Geometry

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Abstract**:** Textbooks have been subject of national and international research and, in the Brazilian scenario, this thematic has been drawing researchers' attention since the beginning of the 21st century. Focusing on the scope of geometry, the objective of this paper is to the share results of a study about *the presence of Geometry in Mathematics Textbooks*, based on the map of the research conducted by the Theorem - Reflections on Geometry Mathematical Education research group, which take the mathematics textbook as subject of study. For this, the use of textbooks is contextualized and a literature review is subsequently presented, addressing studies that discuss geometry in general and in its particularities, and, lastly, we present the theoretical foundation of the investigations conducted by the group. The results of these analyses are divided into two fronts: investigations regarding geometric contents and concepts and development of other thematics associated with the presence of geometry in mathematics textbooks. Lastly, the section quilting and projecting revises the group's research fronts in a reflective way, interrelating them and relating them with other national and international fronts, in the light of a macro perspective. With this "patchwork", we hope to contribute with the discussions on mathematics textbooks and the production of knowledge, especially in the geometry scope.

*Keywords***:** Reading Geometrically, Opportunity-To-Learn. Cognitive Demand. Didactic Material. Brazilian Education.

Textbooks in Brazil

The development of technology has played a fundamental role in globalization, and, consequently, the educational system became more integrated with the use of digital tools. However, although we are surrounded by information mediated by computers, tablets and smartphones, in what regard teaching processes and methodologies, the textbook[3] is still the most widely used resource in the classroom routine (Matić, 2019).

Therefore, as the main supporting material for teaching and learning, the textbook directly influences the contents to be taught, the elaboration and planning of task and evaluations, the methodologies and the

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[3] The term 'textbook' refers to printed material. In Brazil, the digital version can not have different number of pages and content, only links can be added basically.

didactic procedures (Matić, 2019). In this sense, the textbook incorporates a curricular character, since it will propose, in a decisive way, *what* should be taught and *how* it should be taught (Lajolo, 1996, p. 4), being therefore regarded by the educational system as the closest resource to the intended curriculum[4] (Howson, 1995) and the one followed and practiced by educators (Silva Junior & Regnier, 2007). In the Brazilian scenario, the textbook has become the central resource in the contents repository and knowledge in the school environment (Azevedo, 2005). Given its importance and predominant use in the classroom routine, the textbook takes on a curricular nature (Silva Junior & Regnier, 2007), providing teachers and students with information and orientation (Carvalho & Silva, 2004). Therefore, the textbook constitutes the teacher's guiding material (often the only one), and is, in many cases, the only reading material to which the students have access (Brasil, 2011).

In addition to these characteristics, from a sociocultural point of view, the textbook can influence the development of critical reflection, since it contains and spreads ideas, identities, values, traditions and cultures from different social groups (Doğan & Torun, 2018; Matić, 2019).

In this context, and considering the international scope of the present study, it is fundamental to describe how the textbooks reach the classrooms and which Brazilian public policies are involved in this process. Firstly, it is important to note that a total of 32.2 million students of elementary and middle school and 7.9 million students of high school, respectively, 81.7% and 87.8%, are enrolled in public schools. The remaining students attend private schools, which are free to choose between textbooks provided by franchise systems or textbooks from other publishing companies, which are in most part evaluated by the Ministry of Education (MEC) and are used in the Brazilian public schools (Brasil, 2018).

Since 1938, the Brazilian government has been concerned about the textbooks used in the public schools. In this year, the Ministry of Education instituted the National Textbook Commission (Comissão

[4] The intended curriculum is an official document that establishes guidelines for the educational systems. In Brazil, this document is called Common Core National Curriculum (Base Nacional Comum Curricular – BNCC). For further information about the document access http://basenacionalcomum.mec.gov.br/. We understand that, in addition to the intended curriculum, other curriculums exist and permeate the educational system: planned, implemented and evaluated curriculums, among others (Kurz, 2011).

Nacional do Livro Didático – CNLD), which established conditions for the production, imports and use of textbooks (Mazzi, 2018).

Figure 1 shows some historical milestones that contributed for the maintenance of the public policies concerning Brazilian textbooks in the 20th century. One of the central points was the creation of the National Textbook Program (Programa Nacional do Livro Didático[5] – PNLD) in 1985. This program "constitutes a good example of a successful program" (Carvalho, 2018, p. 773), especially because it evaluates the textbooks and distributes them to around 40 million students[6].

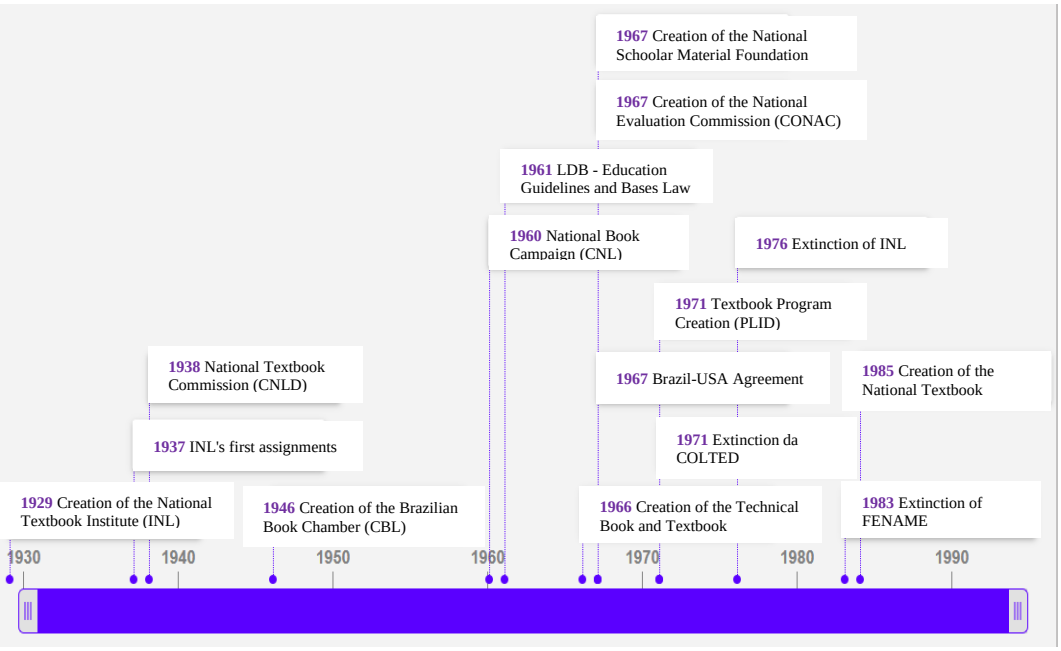

Figure 1: Timeline of the political milestones associated with textbooks
Source: Adapted from Mazzi (2018, p. 38)

In 1993 the Ministry of Education instituted a commission of specialists to analyze the quality of the most requested textbooks and establish general evaluation criteria, which were published in 1994. Thus, in 1996, the textbook pedagogical evaluation process started (PNLD/1997). Since then, the federal government has issued public edicts to regulate publishing companies' applications to provide textbooks

[5] The Decree 9.099, of 07/18/2017, unified the National Textbook Program (PNLD) and the School Library National Program (PNBE), creating the Textbook and Didactic Material National Program (PNLD) (Brasil, 2020).
[6] The material is funded by taxes paid by the population.

for different areas and levels. The textbooks are then analyzed and a list of the approved ones with their respective teacher's guide containing reviews and comments by the evaluators is released. Public school teachers meet to choose the most suitable textbook according to their teaching practices and the political-pedagogical projects[7]. The Ministry of Education acquires the textbooks and the copies are eventually distributed to the schools (Carvalho, 2018).

It is important to note that the material is evaluated on a four-year cycle, i.e., each year the textbooks of one of the educational levels - childhood education, elementary school, middle school and high school – are evaluated and distributed. The textbooks for the levels that are not being evaluated in a particular year are replaced; therefore, fewer copies are distributed. The PNLD textbook selection process has been the subject of rare[8] publications (e.g., Silva Junior, 2005 and Zambon & Terrazzan, 2013), as well as the contents and the way textbooks are used (Borba & Selva, 2013; Gonçalves & Bittar, 2017; Guimarães, Gitirana, Cavalcanti & Marques, 2007) as an educational resource.

Thus, this article is established based on this scenario, the aim is to present a summarized form, containing the results of studies about the presence of Geometry in Didactic Books of Mathematics, conducted by the Theorem group. Therefore, together, this text seeks to contribute to the research literature on Didactic Books, within the scope of Geometry.

Literature Review[9]

Given the relevance of the textbook in the Brazilian educational routine, it has become subject of several studies (Amaral & Hollebrands, 2017; Choppin, 2004; Fan, Zhu & Miao, 2013) and has drawn the attention of researchers from different investigation fronts. Oliveira, Guimarães and Bomény (1984) and

[7] The political-pedagogical project (PPP), elaborated by the school directory board, teachers and Community, provides guidelines for the teaching and learning process and contributes for the development of the society' critical and responsible awareness (Lopes, 2010).
[8] An in-depth search was carried out on these topics in journals and theses.
[9]For this section, it was considered the authors which Theorem – Reflection on Geometry and Mathematics Education research group, have studied. In addition, regards to the optimization of space (there are still 25 references), the main references are shared, which is according to the theme of the text, synthesizing results and research reflections that interweaving the research objective presented in this article.

Lajolo (1996) focused on discussions regarding the definition of textbook and its nature as supporting material to be used in the teaching and learning process. Choppin (2004) conceptualizes the functionalities of the textbook, characterized by its referential, instrumental, ideological/cultural and documental characters.

From a political and social perspective, Doğan and Torun (2018) and Matić (2019) addressed some aspects regarding the essential role that textbooks play in the educational system[10]. According to these authors, the textbook is still considered the main resource to convey the intended curriculum. Textbooks are used by the teachers to a greater extent when compared with other resources, either the traditional ones – board, concrete material or television – or the technological ones – computers, tablets and smartphones (Doğan & Torun, 2018; Matić, 2019).

In this sense, Doğan and Torun (2018) point to five aspects that have contributed to rank the textbooks as the favorite resource to be used in the classroom : 1. concise, clear and anticipated presentation of contents and concepts; 2. unicity of the contents presented by the textbooks of the same collection or similar, i.e., students from different places can use the same textbook and access common information. This content unicity consequently brings education unicity and is suitable for meeting the demands of a centralized system" (Doğan & Torun, 2018, p. 112); 3. the teachers fell safe and comfortable to use this resource, since it was elaborated and reviewed according to official documents that provide curricular guidelines; 4. matrix of primary and secondary sources of topics, information and knowledge and 5. allow the students to understand the development of contents throughout the teaching and learning process, since the textbooks are updated in each educational level.

Silva (2016) analyzed textbooks from a political and social view, investigating the construction of discourses based on a Foulcatian perspective, promoting a review of such discourses in the light of political culture (Silva, 2019), considering power relations (Santos & Silva, 2019) and gender issues

[10] Although the research by Doğan and Torun (2018) and Matić (2019) has been developed in Turkey and Croatia, respectively, both acknowledge the centrality of textbooks in their respective educational systems. This also applies to the Brazilian educational system, considering the complex scenario in which the textbook is inserted in the public policies scope.

(Souza & Silva, 2017). Textbooks and didactical materials have also been the focus of discussion on mathematics curriculum (Januario, 2018).

From an empirical view, Fan (2013) elaborated a conceptual structure for the classification of mathematics textbook research, focusing on the types of questions that guide the investigations and assuming that the textbook "can be viewed as an intermediate variable in the context of education and, consequently, textbook research can be defined as disciplined inquiry into issues about textbooks and the relationships between textbooks and other factors in education" (Fan, 2013, p. 776), the author relies on three macro areas of indagation: 1. Questions about the textbook itself; 2. Question on how different factors may affect the elaboration and publication of textbooks; and 3. Questions about how textbooks may influence other elements.

In this sense, Fan et al. (2013) conducted a literature review on mathematics textbook research and found that the studies mainly concentrated on the analysis of the textbook (63%) – the textbook itself (34%) or in comparison with other textbooks (29%) –, and its complement (37%), textbook use (25%) or in other areas of investigation (12%), such as digital textbook and its relation with student learning. In this scenario, "the development of research on mathematics textbooks has been unbalanced in different areas" (Fan et al., 2013, p. 633). The authors explain this imbalance, highlighting that the textbook is viewed by researchers as an isolated identity in the educational processes, while it should be viewed from a broader perspective.

As for the contribution of Brazilian research, the present study shares results of studies involving textbooks and geometry[11]. It is important to note that, often, studies tend to dissipate among the different mathematical contents/concepts (e.g., area of parallelogram and rectangle; volume; symmetry of reflection; hyperbola; line and circumference and trigonometric ratios of the rectangle triangle and among other investigations topic (e.g., geometric figures; mathematical demonstrations and justifications, technologies and historical perspectives).

[11] All the textbooks analyzed herein were approved by a PNLD.

As an example, one of the topics addressed in the investigations is the area (Santos & Bellemain, 2007; Teles & Sá, 2010) and the volume (Morais; Bellemain & Lima, 2014) of geometric figures. Concentrating on the parallelogram disposition in the elementary school textbooks, Santos e Bellemain (2007) sought to identify the existence of images in the tasks involving the parallelogram and how frequently they are displayed in a prototypical position[12]. Regarding the area of a parallelogram, the authors investigated the procedures and opportunities the students are given to interpret the data presented by the tasks. Additionally, the demand of situations addressing the area asking the measure of a surface or using the comparison of areas without asking for numerical measures was analyzed.

Teles and Sá (2010) mapped the questions that treated the area of the rectangle in elementary school. These questions were categorized into two perspectives: area of the rectangle as object of study and as a resource for other contents. The authors performed an analysis of the purely mathematical and real-life contexts involved in the area of the rectangle. Also, they sought to identify whether the position of the figures was prototypical and observed how the checkered grid is displayed in the tasks proposed by the textbook. Lastly, the authors discuss the presence or absence of figures and how they influence the resolution of the problems proposed.

Morais et al. (2014) focused on investigating how the concept of volume is taught in high school. The authors started by analyzing textbooks presenting the concept of volume, trying to understand how the tasks proposed addressed the three dimensions of volume assumed in the research – quantity, geometrical and numerical – and, thus, how the tasks are distributed and classified into measurement, comparison and production.

Pachêco, Pachêco and Silva (2017) conducted an analysis on some middle school textbooks addressing polygons, aiming to understand the sequence in which the theme is treated, how the geometric knowledge is fostered, what level of knowledge is required from the students to solve the tasks, and how these tasks are distributed throughout the textbooks used in this educational level.

[12] Affirming that an image is in this position means that the longer side of the parallelogram is horizontal and the figure is inclined to the right side.

Other mathematical concepts and contents found in textbook research are: reflection symmetry, hyperbola, line and circumference, and trigonometric ratios. Santos and Teles (2012) analyzed the concepts of reflection symmetry and translation associated with visual arts presented in mathematics textbooks used in elementary education. The authors investigated whether the concepts would comprise art modalities, such as drawing, folding, patterns, painting, architecture and pictures. Santos and Teles (2012) also analyzed the presence of textual reference regarding the concept of perpendicularity, congruence of figures and distance-conservation in symmetry. Habib (2013) focused on the analysis of the approaches adopted by high school textbooks to develop the concept of hyperbola, trying to understand if the representations explore situations where the origin is the center of the hyperbola and ones presenting translations and rotations as well. Additionally, Habib (2013) compared the treatment given to hyperbola by the analyzed textbooks, verifying the presence of a historical approach regarding the conics and the number of tasks associated with the construction of this curve. Finally, the author proposes five tasks to be developed with the students in the classroom as a way to complement and enrich the content presented by the textbooks.

Dallemole and Groenwald (2014) investigated high school textbooks addressing line and circumference and the types of registers used – idiomatic language, numerical, figural geometric, graphic and algebraic – and how these registers are present in the theoretical explanation of the content, in the solved exercises and in the tasks proposed. Additionally, the authors analyzed all the content, classifying it into other mathematics fields (functions, linear systems, plane geometry, among others) and in extra mathematics contextualization – geography, physics, arts, agriculture, social practices, technology, etc.

Silva and Fonseca (2015) analyzed the process of teaching and learning trigonometric ratios in the rectangle triangle associating it with conceptions of cognitive neuroscience, "more specifically, [the authors sought to] investigate the neurochemical mechanisms involved in the detection, processing and interpretation of environment signals throughout the central nervous system" (Silva & Fonseca, 2015, p. 117). Thus, considering the basic systems of neural physiology the authors investigated the relations between the content of trigonometric ratios in the rectangle triangle – sine, cosine and tangent – and

textual representations of this content presented by mathematics textbooks used in the 9th year of fundamental school.

In addition to the studies mentioned hereinbefore, which focus on a particular mathematical content/concept, textbooks addressing geometric content have also been analyzed with attention to other different aspects, as is the case of the study conducted by Moretti and Brandt (2015), who investigated the tasks presented by basic education textbooks that approach geometric figures through the relations between different semiotic systems. In the same direction Oliveira and Bittar (2015) studied the mathematical justifications and analyzed some middle school textbooks to find what treatment they give to demonstrations and justifications in tasks involving geometric constructions.

The historical character of mathematics and geometry textbooks has also been subject of research. Camara and Pinto (2016); D'Esquivel (2016) and Matos and Silva (2011), researchers in the field of Mathematics Education History, analyzed textbooks from a historical view to understand in what ways a particular geometric topic was present over a particular period.

In this brief review, it is important to mention other theoretical referentials, as the French Mathematics Didactics (Santos & Bellemain, 2007); the Conceptual Field Theory (Morais et al., 2014; Teles & Sá, 2010); the Semiotic Representation Records (Dallemole & Groenwald, 2014), referentials that rely on conceptualizations and definitions about apprehensions and the types of approach on geometry developed by Duval (Moretti & Brandt, 2015), and the levels of geometric thinking development by Van Hiele (Pachêco et al.,2017).

In this scenario, and considering the possibilities of investigations offered by textbooks, the Theorem - Reflections on Geometry and Mathematical Education research group, of which the authors or the present study are members, has focused on analyses that involve teachers' choices regarding textbooks and the way these teachers select the tasks to be developed with the students to understand how geometry is treated in this context by investigating concepts, contents and approaches. The aim of the present study is to contribute with discussions about the textbook, the data presented herein is part of a knowledge

production process, and, along with the results provided by other interrelated studies, will constitute a tapestry[13] of information (D'Ambrosio & Borba, 2010) regarding textbooks.

Theoretical Framework

Despite the particularities of the studies developed by the Theorem, the group relies on theoretical referentials that ground the discussions and reflections of the group as a whole, and such referentials are present in great part of the research. The present study was developed based on the concepts of ***Geometric Reading of Diagrams*** by Dietiker & Brakoniecki (2014), who focused on interpretations and meanings given to the diagram by the reader and on the ***Opportunity-To-Learn – OTL*** by Wijaya; van den Heuvel-Panhuizen, & Doorman (2015), who addressed three perspectives regarding textbook tasks[14]: the type of context used and its objectives; the type of information given; and the level of cognitive demand of the tasks.

Reading diagrams geometrically

The research by Dietiker and Brakoniecki (2014) fostered reflection on what Rosenblatt (1988) regard as a reader: the one who composes (writes) about a previously interpreted meaning. In consonance with this view is the perspective proposed by De Freitas e Sinclair (2012), who consider the diagram a trace of its generalization – the reader's "gestures", rather than a static image. Pimm (2006) presents the notion of "seen as", the reader's interpretation that generates a mental construction, different for each individual, even when representing the same object.

In this context, Dietiker and Brakoniecki (2014) consider diagram reading a generative and creative act by the writer of meaning, i.e., the reader. From this conception, the authors identify and discuss eight

[13] The production of **tapestry** is a tradition, and quilting techniques have been passed on for generations (Ferraz, 2019). Of course, the beauty of the patchwork products is the ideal justification for their huge popularity. Tapestry is an art (Ferraz, 2019), it is created by sewing several pieces of fabric. We used this metaphor as D'Ambrosio & Borba (2010) relating the production of this artistic element with the production of knowledge in the scientific area, where different pieces of research compose a larger and harmonious piece.

[14] It is herein assumed that the term task regards any situation aimed at providing the students with conditions to develop processes to solve any kind of problem, associated with any mathematical concept or content.

dimensions of reading geometric diagrams. Some examples of these dimensions are: the one that involves recognizing the type of sign (icon, symbol or index) the diagram takes in the text; the one that involves the assumptions the reader is expected to make regarding the topological and metric aspects of the diagram; the one that involves understanding cultural conventions; and the reading dimension that involves reconstruing a particular diagram as a collection of geometric objects (by mentally drawing them).

According to Dietiker and Brakoniecki (2014), reading geometrically the diagram involves asking questions as: "What can be assumed?"; "Is it an icon or a symbol?"; " What more can I figure out about the geometric object?"; "Is it possible?"; "Is it representing a single object or multiple?"; and "In what ways is this diagram limiting my vision?." The authors affirm that none of these questions appear in the textbooks and would be reasonable to assume the inclusion of such questions in the curriculum would help students to develop sophisticated ways to interpret geometric diagrams.

Furthermore, based on this analysis, the authors define *reading geometrically* "as a reader's negotiation of meaning with what is, is not, and may be misleading in a diagram" (Dietiker & Brakoniecki, 2014, p. 6). They propose that reading geometrically "entails a re-drawing of the diagram in the reader's mind, taking into account its context (and thus, its purpose) while drawing upon prior experiences with conventions of drawing, marking, and labeling of diagrams" (Dietiker & Brakoniecki, 2014, p. 6). The authors emphasize that, "although it is tempting to make claims about "the" reading of a diagram, we question the ability to make such a claim.", proposing that reading diagrams is much more complex than originally acknowledged and suggesting a that we now stand back and wonder how students make sense of it all, considering the challenges of convention, context, categories of signs, deductive possibilities, assumptions that are deemed acceptable and others not acceptable, and misrepresentations that readers confront in textbooks For the authors, "it is not surprising that learning mathematics appears to many students as a game where the rules seem arbitrary and hidden" (Dietiker & Brakoniecki, 2014, p. 6).

Opportunity-To-Learn

The concept of Opportunity-To-Learn (OTL) was introduced by Carroll (1963) and represent the amount of time allowed for learning by the school system and/or schedule. Other researchers subsequently provided contributions for the conceptualization of OTL viewing it from different perspectives (e.g., Berliner, 1978; Dougherty, 1996; Husén, 1967; Moore, DeStefano & Adelman, 2012). As for the dimension regarding the teacher, some authors focus on discussions concerning teaching practices (Stevens, 1996), teaching quality (Wang, 1998), the curriculum (Kurz, 2011) and teacher's knowledge (Walkowiak; Pinter & Berry 2017). The OTL model is therefore reconceptualized, presenting a new structure.

In addition to research aimed at (re)conceptualizing the model studies were conducted to foster reflections about teacher formation (Schmidt et al., 2008), public policies (Herman, Klein, & Abedi, 2000; McDonnell, 1995), students' learning (Stein; Engle; Smith & Hughes, 2008), students' errors (Wijaya et al., 2015); mathematical competences (Carrillo; Contreras & Zakaryan, 2014), the context of textbooks tasks (Wijaya et al., 2015, Wijaya; Van den Heuvel-Panhuizen; Doorman & Veldhuis 2018) and the cognitive demands of such tasks (Van Zanten & Van den Heuvel-Panhuizen, 2018).

From the OTL perspective, Wijaya et al. (2015, 2018) established a relation between the concept and textbooks. Based on the assumption that textbooks act as a conveyor of the curriculum and ground the teachers' planning and implementation decisions, they can be considered one of the variables that determine OTL offer (Wijaya et al., 2015). Therefore, different textbooks offer different OTL, and, in the long run, the students will produce different results in assessments (Wijaya et al., 2015).

Considering this, and concentrating on the context of the tasks offered by the mathematics textbooks, Wijaya et al. (2015) developed an analysis structure to investigate the characteristics of such teaks. The structure is divided into three central OTL aspects and analyze whether the tasks: 1- provide tasks that involve real-life questions and implicit mathematical procedures; 2- offer tasks with excessive and/or superfluous information; 3- offer experiences to address tasks of high Cognitive Demand (CD). **Chart 1** shows the parameters to analyze the tasks: context, information and cognitive demand

Task characteristic	Sub-category	Explanation
Type of context	No context	-Refers only to mathematical objects, symbols, or structures
	Camouflage context	-Experiences from everyday life or common-sense reasoning are not needed. -The mathematical operations needed to solve the problems are already obvious. -The solution can be found by combining all numbers given in the text.
	Relevant and essential context	-Common sense reasoning within the context is needed to understand and solve the problem. -The mathematical operation is not explicitly given. -Mathematical modeling is needed.
Purpose of context based task	Application	-The task is given after the explanation section.
	Modelling	-The task is given before the explanation section.
Type of information	Matching	-The tasks contain exactly the information needed to find the solution.
	Missing	-The tasks contain less information than needed, so students need to derive additional data.
	Superfluous	-The tasks contain more information than needed so students need to select information.
Type of cognitive demand	Reproduction	-Reproducing representations, definitions or facts -Interpreting simple and familiar representations -Memorization or performing explicit routine computations/ procedures
	Connection	-Integrating and connecting across content, situations or representations. -Non-routine problem solving. -Interpretation of problem situations and mathematical statements. - Engaging in simple mathematical reasoning.
	Reflection	-Reflecting on and gaining insight into mathematics. -Constructing original mathematical approaches. -Communicating complex arguments and complex reasoning. -Making generalizations.

Chart 1: Structure of textbook analysis
Source: Wijaya et al. (2015, p. 52)

Thus, through the analysis of context, information and cognitive demand, this structure subsidizes the categorization and comprehension of the tasks. Along with all the (re)conceptualizations, studies by Wijaya et al. (2015, 2018) represent an important contribution for this study, since their investigations are aligned with our objectives. Moreover, as it can be noted in the brief description given in this subsection, the concept of OTL is frequently (re)visited by the researchers, and, consequently, refined by the inclusion of new perspectives. In general, however, OTL can be defined as “opportunities which schools

[teachers, school managers and policies] provide students to learn what is expected of them." (Herman et al., 2000, p. 16). With specific regard to the present article, this definition is refined to the textbook, (re)defining OTL as opportunities to learn offered by the tasks presented by mathematics textbooks in the geometry scope.

Mapping Theorem research

The present study summarizes the results of a study on *the presence of Geometry in Mathematics Textbooks* carried out by the Theorem (Reflections on Geometry and Mathematical Education research group). Documentary research was the main type of research used (Lüdke & André, 1986). As the main methodological resource for the construction and analysis of data, the research was based on document analysis (Lüdke & André, 1986), content analysis (Bardin, 1996) and the design of horizontal, vertical and contextual analyzes (Charalambous; Delaney; Hsu & Mesa, 2010). In this context, studies have analyzed how geometry is treated in mathematics textbooks used in elementary school (Bardini; Amaral-Schio & Mazzi 2019; Santos, 2015) middle school (Gentil & Amaral-Schio, 2019; Godoy, 2016) and high school (Litoldo, 2019; Mazzi, 2018; Santos, 2019).

With specific regard to geometry contents, this study aimed to constitute a mosaic with the findings reported in the literature, since our objective is to amplify and deepen the discussion on the concepts of similarity (Amaral & Hollebrands, 2017; Amaral-Schio & Hollebrands, in press), polygons (Godoy & Amaral-Schio, 2017), area and volume (Sturion & Amaral-Schio, 2019) and symmetry (Sousa; Guimarães & Amaral-Schio, in press).

The Theorem has focused on other thematics in the geometry scope, having presented discussions regarding teaching practice and teacher formation. Among these discussions are studies concerning proofs and demonstrations (Mazzi, 2018); the role of diagrams in the teaching and learning process (Godoy, 2016; Santos, 2019); the role of tasks in the perspective of contextualization analysis (Amaral & Hollebrands, 2017; Litoldo, 2019) and in cognitive demand (Litoldo, 2019) and the role of geometry in nongeometric fields (Gentil, 2020). Investigations on digital learning objects have provided relevant

results in what regard their integration with Brazilian textbooks (Amaral-Schio & Ribeiro, 2018; Mazzi, Amaral-Schio & Moura, 2020; Prampolim & Amaral-Schio, 2018) and the little use of calculators demanded by the tasks included in high school textbooks (Amaral-Schio, 2018).

Periodic meetings of Theorem are held and, in this space, central issues are debated, which led to the constitution of the map below (**figure 2**). In it, one can observe the main themes of interest to the group. For the elaboration of this text, which summarizes their main research results, the authors worked on the elaboration of a joint analysis of these results, considering the procedures adopted, the theoretical references that supported their analyzes, among other fundamental points of each research.

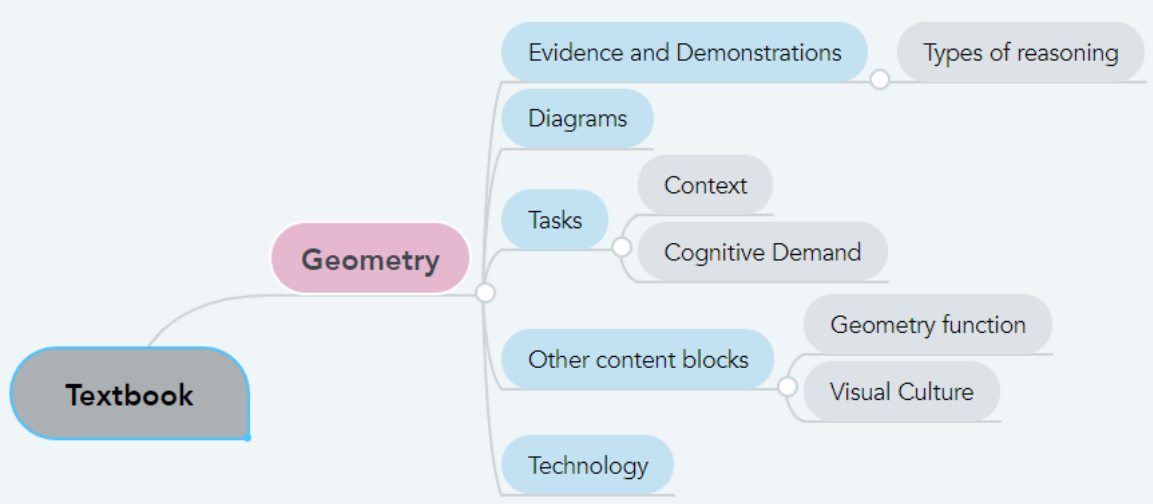

Figure 2: Organogram of Theorem research
Source: Elaborated by the authors (2020)

In addition to these investigations, an analysis on how teachers select the textbooks to be used in their respective states was performed (Andrade, 2019; Perovano, 2019), more specifically, how these teachers select the tasks to be used in their classrooms (Gonçalves, 2019). The following subsections present more details on the aspects associated with the studies mentioned before.

Geometry contents and concepts

Expanding the research by Godoy (2016), Godoy and Amaral (2017) carried out a study to analyze the concept of polygon included in elementary and middle school mathematics textbooks, and the results

demonstrated the need to pay close attention to the definitions given by the textbooks and how such definitions are used throughout the lessons in order to avoid misunderstandings regarding this content.

Regarding the contents of area and volume presented in the mathematics textbooks, Sturion and Amaral-Schio (2019) investigated how, if applicable, these contents were aligned to the BNCC (Brasil, 2018). The authors carried out a documental analysis of three high school mathematics textbook collections and found that, despite having been elaborated concomitantly to the BNCC (i.e., before the its official publication), most textbooks contemplated the competences and abilities proposed by the policy. Therefore, the new document that offers curricular guidelines basic education is not distant from the content of area and volume presented by the intended curriculum presented by the textbooks. On the other hand, the concept of symmetry, present in middle school, is strongly associated with the symmetry of reflection, which was modified by the BNCC guidelines (Sousa, Guimarães e Amaral-Schio, in press).

The content similarity was analyzed by Amaral and Hollebrands (in press), reviewed six textbooks, three from Brazil and three from the USA. The Brazilian textbooks consider the same definition for similar figures (based on angle measures and the proportion of the sides) and present the same sequence to present the content. In general, Brazilian textbooks present the idea of polygon similarity, especially triangle similarity (and the similarity cases), Thales's theorem and Egyptian pyramid problem, concluding with Thales's theorem. This content construction differs from the one found in the USA, which focuses on the algebraic definition, considering geometric transformations (especially homothety, which is briefly mentioned in the Brazilian textbooks). In summary, the Brazilian books make little use of algebra when treating geometry and use a static approach for treating the topic, while the American textbooks address the content in a dynamic way, essentially focused on algebra, even though they present the same definition used in Brazil.

The following subsection describes other themes addressed by the Theorem, without focusing on specific contents.

The approaches in Geometry

Several thematics have been addressed by the Theorem aiming to investigate how geometry is treated in the mathematics textbooks. In this subsection we share the results of the research that focus on other aspects rather than on the concept of geometry. Regarding diagrams, Godoy (2016) identified different types of signs in the textbooks analyzed, as well as the study of metric and topological properties of some contents and the use of mental representations of geometric concepts. Through the analysis of the data concerning geometric reading (Dietiker & Brakoniecki, 2014), the author described the different types of sings according to the Theory of Signs by Peirce (1931) – icons, indicators or symbols - depending on the definition taken by the text and also on the possible understanding of the reader. The results showed that, in middle school, the textbook authors frequently use the indicators to introduce the topic in the chapters, then make use of icons to broaden the subject, and eventually form concepts through symbols. Based on Borges (2005), the author analyzed metric and topological properties, noting that there is a relation between the former and the quantitative part of diagrams; and between the latter with the qualitative part. Overall, the author used markers in the figures to exemplify ideas, concentrating on metric properties. Lastly, the study on mental representations was based on authors like Galperin (1989), who treats these representations as a process to constructed over the school years.

Santos (2019, 2018) develops a research that aims to analyze diagrams in content of Euclidean Geometry and Non-Euclidean Geometry, in three collections of high school mathematics textbooks approved by the PNLD. For that, it is based on the eight dimensions of the geometric reading of diagrams by Dietiker and Brakoniecki (2014), where it interconnects them with other theoretical references. Like Godoy (2016), it relies on Peirce (1931) on the relationship between sign and object, icon, index and symbol, and on Borges (2005), on the expansion of metric properties and topological notions. It also brings Dimmel and Herbst (2015) to explore the concept of geometric diacritics, and Laborde (2005), which defines theoretical and spatial-graphic domains. Santos (2019) is still based on the spiral of data analysis by Creswell (2014), regarding the data analysis and organization procedures. Thus, Santos (2018) concludes that the referred research has the potential to advance in new analyzes regarding the geometric reading of diagrams.

Regarding how demonstrations and proofs are treated by the available textbooks, Mazzi (2018) tried to understand if, and how, they were present in the geometry chapters of high school textbooks. Based on the assumption that proofs and demonstrations have distinct meanings in the mathematics education scope (Garnica, 1995; Hanna, 1983), the author relies on the different types of reasoning proposed by Reid and Knipping (2010): deductive, inductive, abductive and by analogy – to characterize these notions. In this sense, Mazzi (2018) discusses the idea of demonstration and associates it with deductive reasoning, the only type to ensure sureness reinforcing the rigorous character of mathematics. As for mathematical proofs, the author characterizes them based on inductive, abductive and by analogy reasonings, which, despite producing refutable conjectures, are relevant in teaching and learning processes. The author also investigated the presence of different demonstration methods – direct, contrapositive and ad absurdum – and proposed a discussion regarding the concepts permeating the notion of axiomatic system – thesis, hypothesis, theorem, etc. As a result, Mazzi (2018) found that deductive reasoning predominates in the textbook collections, followed by inductive reasoning. Abductive and by analogy reasonings were not frequently addressed by the collections, having been set aside by most textbook authors. In this sense, Mazzi & Amaral-Schio (2020) defend the use of different types of reasoning in mathematics teaching and learning, and emphasize the importance of tasks to articulate them, showing that each of these reasonings play a role in "doing mathematics."

Considering interactivity aspects (Cannito, 2009; Machado, Santos & Tanaka, 2013) and specialized knowledge of the teacher (Carrillo et al., 2018) as theoretical lenses, the initiative of integrating Digital Learning Objects (DLO) to the Brazilian textbook collections (PNLD 2014), was analyzed from different perspectives to understand the nature of the DLOs and how they are present in the textbooks and how they are described in the teacher's manuals. Amaral-Schio and Ribeiro (2018, p.182) analyzed middle school textbooks and found that the DLOs included were supported by a traditional view of education (transmissive mode), where interactivity consists in watching and listening passively, or in just clicking on something to obtain an answer, i.e., it merely consists in the domestication of the media.

Prampolim and Amaral-Schio (2018) analyzed middle school textbooks aiming to understand the learning environment and the level of interactions provided by the DLO. Based on theoretical contributions by Skovsmose (2011) and Lévy (1999), the authors observed the existence of an interactive approach in the use DLO by teachers and students. The importance of such interaction is revealed as it provides conditions for content exploration through ludic proposals. Regarding the learning environments, the results showed that the textbooks predominantly present pure mathematics environments, and very few ones concerning reality. Regarding improvements to DLO operationalization, Prampolim & Amaral-Schio (2018) emphasize the importance of a more factual integration with the textbook, making connections with the content to be conveyed. Moreover, the authors point to the lack of questions addressing DLO accessibility, since operationalizations by impaired user were not included.

Mazzi et al., (2020) approached the integration of DLO by the teacher. From a DLO proposed in a textbook, the authors used the Mathematics Teacher Specialized Knowledge model (MTSK) which is, at the same time, a theoretical and methodological contribution, to explore important aspects concerning the teaching practice involving technology. The authors note that, when choosing DLO integration in the school routine, the teacher should study not only the topics associated with DLO, but also look for possible relations between the contents and different curricular elements, in addition to reflecting about pedagogical matters involving such concepts.

Amaral-Schio (2018) analyzed the presence of technology in high school textbooks and observed that computers are hardly ever mentioned, either with regard to effective use or for curiosity's sake. The use of calculators is not very explored, and, in the geometry context, seems to be limited to the resolution of the four basic operations and volume calculations (Amaral-Schio, 2018).

Regarding tasks involving geometry from a contextualizing perspective, Amaral & Hollebrands (2017) and Litoldo (2019) analyzed the theoretical foundations, mostly based on the OTL conceptualization – in particular the one developed by Wijaya et al. (2015), on the Critical Mathematics Education (Skovsmose, 2001), and on the classifications of mathematical, artificial, semi-real and real contexts (Dekker & Querelle, 2002; Lana & Carrião, 2015; Ponte & Quaresma, 2012; Skovsmose, 2000).

When the teachers adopt a textbook as reference, they find tasks that include supposedly real problems that are actually not so close to the students' reality. How to deal with this situation? We believe that we can do more than just finding an answer or ignore such a task: it is possible to use this kind of proposal to discuss critical aspects of the contextualized tasks with the students and ensure a high CD (Amaral & Hollebrands, 2017).

In what regard the CD perspective, a low CD predominates in the geometric tasks offered by the textbooks (Litoldo, 2019), favoring experiences associated with memorization and procedural methods without any connection with and between the mathematical concepts. Thus, we observed that situations that provide opportunities for the student to develop higher reasoning levels, such as productions and justifications, are scarce in these subjects (Litoldo, 2019). As teachers, we can explore tasks thorough critical questions and adapt them to a high CD. Thus, the present study presents several points to reflect on how, as teachers, we can provide students with situations to develop criticism in relation to the tasks they are presented? Considering the contributions by Dietiker and Brakoniecki (2014), it is reasonable to assume that critical discussion /reflection could help students to develop more sophisticated ways to interpret tasks and think geometrically, making them good solvers of high CD problems.

Broadening the role of geometry to other mathematics fields, Gentil (2020) mapped the geometric aspects presented by the chapters concerning algebra, arithmetic, probability and statistic[15], characterizing the functionality of the geometric characters: illustrative, formative, explanatory, demonstrative, representative and mental image. Based on the concept of visual culture developed by Flores (2013), which concerns the conceptions that are inherent to culture and expressed in a visual form, the author found evidence of the use of geometry from a historical and cultural perspective, since it was shown to be a component of the visual discourse used in the classroom. In what regard the arithmetic and algebraic contents, the study points to the role of geometry as facilitator and provider in the learning process, since its insertion in these contents is made to contribute for the comprehension of abstract contents through geometric illustrations.

[15] These divisions are in accordance with the mathematical fields established by the BNCC.

The results shared herein are complemented with other studies that have been developed by the Theorem. These studies were not mentioned for being in the initial phase of investigation or for addressing themes other than the use of textbooks. In this sense, Peres (2016) analyzed the perspectives of the teachers in Euclidian geometry in mathematics licentiate courses in public institutions in the state of Sao Paulo, and Ferreira (2019) concentrated on the study of conics exploring different conceptual aspects. This is an overview of the group's current production, continuously contributing with a piece of knowledge to compose the patchwork.

Quilting and Projecting

To conclude the discussion proposed in the present research, we emphasize that the studies carried out by Theorem have the textbook as subject of study. The objective is to analyze the presence of geometry in this teaching resource, aiming to understand how this theme is treated with the use of different approaches and contexts (technology, diagrams, etc.) and how the contents are presented to the students. We concentrate on the analysis of how geometric concepts are addressed, constructed and constituted in the textbooks, the nature of the tasks and the relations established with real life. For this, we sought to compose a mosaic of topics, such as similarity, symmetry, area and volume, concepts that have been subject of research in other studies.

Diagram geometric reading and studies on OTL and CD grounded our investigations, where we discussed how textbooks can contribute with the students' OTL. Reflecting on CD can help teachers to create new possibilities regarding the use of textbooks. What is the teacher's role in making the proposed tasks effectively contribute for the students' learning, leading them to solve problems of high CD?

Have the available textbooks been providing students with experiences that foster the construction of mathematics, the argumentation through justifications and demonstrations? The literature has reported an increasing presence of geometry in the mathematics textbooks over the last years (Godoy, 2016); but how has geometry been associated with other mathematical areas? And what are the geometry tasks like? Do they bring situations were the students can identify problems in which geometry can contribute for the

solution? Do the diagrams presented in the tasks contribute for the comprehension of the topic to be studied? Has DLO integration to textbooks provided opportunities to explore these resources in a qualitatively different way in comparison with traditional media? Have the teachers reflected about these issues, especially while choosing the textbooks to be used in their lessons? Are the teachers aware of their role to foster a high CD level from the tasks? (Mazzi & Amaral-Schio, 2020). Are publishing companies following research results and reflecting about relevant aspects regarding the improvement of this teaching supporting material that is used in more than 80% of the Brazilian classrooms? What about the authors, is their identity present in the books they write? With the pasteurization that the PNLD has been imposing for the works to be approved, are the authors still able to write material that meets the commercial objectives of the publisher, the requirements of the PNLD and do that stand for their perspectives, ideals ("your face", "your identity")?

The topics involved in these questions need to be investigated and reflected upon; thus, the present study shares the results of research carried out by the Theorem and initiate the process for answering these questions and quilting the answers with others provided by the ongoing research on the topic, broadening reflections and promoting national and international interlocutions. We hope this information contributes to this growing body of research and brings new reflections on textbook analysis, especially in what regard the presence of Geometry.

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