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A Theoretical Model of Mathematics for Teaching the Concept of Function

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Abstract: This paper presents a study using a discursive perspective to develop a theoretical model of Mathematics for Teaching of the function concept, employing the following sources: a systematic review of the research literature, two series of textbooks and a discussion study with a group of teachers. The model presents a descriptive language with a theoretical structure that relies fundamentally on the *realization and recognition rules* inspired in Basil Bernstein's theory. Also, the model is based on categories of *realizations* (landscapes) of the concept of function. The landscapes that make up the model are the tabular, diagram, algebraic, transformation machine, graphic, pattern generalization and formal landscapes. The model provides a discursive transparency for the communication about function, which may inform curriculum development and curriculum material design for students and teachers as well as planning strategies to address this topic in educational contexts.

Keywords: Mathematics for Education; Concept; Function; Realization; Recognition and Realization Rules.

Introduction

Investigations about the nature and the way Mathematics is developed, produced and used by the agents responsible for teaching it, have expanded considerably in recent decades (Barwell 2013; Chapman 2013; Davis and Renert 2009, 2013, 2014). The work done by Shulman (1987), that placed the knowledge of content and its integration with pedagogical knowledge at the forefront of education (Adler and Huillet 2008), is widely recognized as the theoretical starting point for research into fields that came to be known as Mathematical Knowledge for Teaching (MKT) and Mathematics for Teaching (MfT) (Adler and Davis 2006; Adler and Huillet 2008; Barwell 2013; Chapman 2013).

The MKT and MfT constructions have been developed using different theoretical and methodological structures as their foundation. Cognitivist perspectives pervade research on these

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constructs, but the approaches situated are growing and offer differentiated insights into such conceptualizations (Rhoads and Weber 2016).

In this study, we are going to assume a discursive conceptualization of MfT. Considering that mathematical communication in educational contexts is produced referring to mathematical concepts, we understand MfT in terms of a certain concept, which in this research is the concept of function.

The choice of the function concept emerges from the central role it plays in contemporary mathematics, permeating virtually many of its fields, and also being considered essential in other fields of science as a tool to model a wide range of phenomena (Güçler 2016; Steele at al. 2013).

According to Sierpinska (1992), the importance of this concept has reverberated in the school context, which is reflected in a substantial body of theoretical and/or empirical research on the teaching and learning of this content in the field of Mathematics Education (Ayalon et al. 2015; Dubinksy and Wilson 2013).

The function concept has a diversity of forms of communication (tables, algebraic expressions, graphs, etc. - usually called representations in the literature) and, consequently, of interpretations (Elia et al. 2007; Panaoura et al. 2017). Contrary to what occurs in scientific mathematics, where the introduction of a mathematical construct is done through its definition (Tabach; Nachlieli, 2015), studies have shown that the presentation of a formal definition[3] of the function concept, should be postponed in the teaching of this subject (Hansson 2006; Nachlieli, Tabach, 2012). Considering this, several alternatives and approaches have been presented to teach this concept (Callejo and Zapatera 2014; Hitt and González-Martin 2015; Wilkie 2016).

In light of the studies analyzed, we can infer there are varied communicative ways to *realize*[4] the teaching of the function concept. As a consequence, the scope of this study is to identify, characterize, outline and structure such diversity in communicating the function concept in teaching

[3] For example: "Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a functional relation in y if for all x∈E there exists a unique y∈F which is in the given relation with x" (Nachlieli; Tabach, 2012, p.14).
[4] Let us provisionally take the terms *realize* and *realization* as intuitive.

in terms of a conceptualization of the MfT of the Function Concept. We were inspired by concepts of Basil Bernstein's Theory of Codes (2000, 2003) to support and develop a conceptualization of MfT.

Mathematical Knowledge for Teaching and Mathematics for Teaching

The MKT model developed by Deborah Ball and colleagues stands out in the literature (for example, Ball at al. 2008). These authors built an MKT model that is composed of subdomains (Ball at al. 2008). According to Petrou and Goulding (2011), the MKT model proposed by Ball and colleagues is aligned with the cognitive tradition. Despite recognizing the context, therefore, the focus tends to be on the knowledge of an individual teacher (Petrou, Goulding, 2011).

For Chapman (2013, 2015) it is not clear how cultural variability is accounted for in these models, even though the approaches that describe MKT in categories are more visible in the literature and provide useful constructs for investigating a teachers' knowledge for mathematics teaching.

Hodgen (2011), who takes a situated perspective, argues that the mathematics teacher's knowledge is, like any other, "[...] *situated* within the complex and social world of the mathematics classrooms" (p. 27, emphasis added by the author). In spite of this position, however, according to Barwell (2013), "[...] it is difficult to shift entirely away from a discourse of knowledge as possessed by the individual teacher" (p. 599). As noted by Stylianides and Delaney (2011), it seems that acknowledging the cultural dimension of teachers' mathematical knowledge is a relatively recent phenomenon.

Adler and Huillet (2008) use the term MfT and, based on a social epistemological perspective, they assume that "[...] all mathematical activity is towards some purpose, and occurs within some or other (social) institution"(p. 22). Taking the same perspective, Kazima at al. (2008) argue that MfT is shaped both by the topic being taught and by the approach teachers use to introduce these concepts. Similarly, Andrews (2011) proposes the importance of recognizing not only the cultural context in which teaching and learning occurs, but also the topic under scrutiny.

Davis and Renert (2014) conceptualize Mathematics for Teaching[5] as the "[...] subject matter knowledge of mathematics teachers [...]" (p. 3). According to the authors, MfT "[...] enables a teacher to structure learning situations, interpret student actions mindfully, and respond flexibly, in ways that enable learners to extend understandings and expand the range of their interpretive possibilities through access to powerful connections and appropriate practice." (p. 4).

Summarily, we feel that the synthesis laid out allows us to corroborate the position taken by Rhoads and Weber (2016) that these constructs have been investigated based on the most varied epistemologies and, consequently, employing several methodological tools.

Assuming that different interpretations and characterizations of a certain phenomenon, and even its existence, depend on the theoretical lens used to construct and analyze it (Barbosa 2013), this study develops and structures a conceptualization for MfT, which will be characterized by outlining its specificities and discursive boundaries, and by making clear how its communication is possible through specific descriptions of the communicative rules that constitute it. In order to operationalize this objective, we take inspiration in concepts from the Codes Theory concepts by the educational sociologist Basil Bernstein (2000, 2003), adapting them for the purpose of the study according with explanations ahead.

The choice for the term MfT instead of MKT derives from the discursive theoretical framework used in the study. From this perspective, the communicative actions (discursive products) constitute the object of analysis itself; as such, no representations of cognitive categories will be attributed to them, which resonates with our discursive perspective.

A Perspective for a Theoretical Model of MfT of a Concept

According to Bernstein (2000, 2003), all communication is governed by inherent principles of the pedagogic practice in which it occurs. Pedagogic practice refers, for example, to the relationship between teachers and students in the teaching and learning of certain topics (Bernstein 2000). More broadly, Bernstein (2000) defines "[...] pedagogic practice as a fundamental social context through which cultural reproduction-production takes place" (p. 3).

[5] Davis and Renert (2014) use the Mathematics-for-Teaching or M_4T nomenclature.

Given that the communication realized in the teaching and learning of mathematics at the school setting is organized referring to mathematical concepts, we assume that such communication at that context is governed by their own principles. These are called *classification and framing* by Bernstein (2000, 2003). The principle of classification creates, regulates and legitimates boundaries between subjects, spaces, discourses, contents, practices, and objects, placing them into categories by isolating them; in short, the categories symbolize these boundaries (Bernstein 2000, 2003; Cause 2010). Isolation creates the space for a category to become specific (Bernstein 2000). The classification principle establishes *recognition rules*. These rules provide the means to distinguish the specificity of a category through the nature of its texts (Bernstein 2000, 2003). For Bernstein (2000), text is any communicative act expressed by someone, covering verbal, written, gestural or spatial texts. The relationships between the categories (the degree of isolation between them) are characterized by the variation in the classification values, and these values can vary from a stronger (C+) to a weaker (C-) classification. A C+ is said to exist when the categories are strongly isolated, that is, their borders are explicit; in this case, when the categories are more specialized. A C- occurs when the isolation is reduced (Bernstein 2000, 2003). For example, the gradation of the classification principle may be used to analyze the relationships intra-disciplinary in a given school. In this case, when there is a C+, contents are well insulated from each other by strong boundaries (Cause 2010; Morais and Neves 2007, 2011). Then there is a reduced or even absent relationship between their respective texts. Such a degree of classification generates a set of recognition rules that create a specific syntax for each content (Bernstein 2003; Cause, 2010). Morais and Neves (2011) suggest that one of the characteristics of the pedagogical practice that the research has shown to be fundamental for the scientific learning of the students is that with C- at the level of interdisciplinarity.

The framing principle deals with the nature of the control over the communicative rules, governing and legitimizing the communication forms by/between the categories of any pedagogic practice (Bernstein 2000). Analogously to the classification principle, there is variation in the gradation of the framing principle, these values can vary from a stronger (F +) to a weaker (F-) framing. The framing principle regulates the realization rules, which provide criteria for selecting and putting in relation the legitimate texts for each category, that is, for generating what counts as legitimate communication and, hence the range of possible texts (Bernstein, 2003). According to Bernstein (2000), "[...] different values of framing act selectively on realization rules and so on the production of different texts." (p. 18). Thus, "framing values shape the form of pedagogical communication in a given context", […] "conveying different rules for texts creation" (Morais and Neves 2009, p. 119).

At this point, we might say that classification and framing regulate the communication of the concept of function at school. Teachers and students are engaged in the process of recognizing rules to realize texts. It follows one may be able to find variations throughout different countries, regions, cities, schools, and classrooms. However, they are all part of which we call school mathematics, and it is possible to identify classification and framing rules that go through all those settings.

From this perspective, a MfT of a concept (the function concept, in this study) will be established identifying and characterizing its boundaries and communicational forms by revealing the recognition and realization rules generated from their potential classification and framing values, respectively, which might operate in the pedagogic relationships expressed (or to be expressed) in schooling contexts. We use classification, framing, recognition and realization rules as analytic tools to help us construct categories that express different ways of communicate the concept of function.

A mathematical concept is understood as a set of *realizations* (Davis and Renert 2014) (texts) that are associated or may be associated with the word that names it. So, the function concept is constituted of a set of realizations associated or potentially associated with the word "function". The realizations are considered texts, which can take the form of definitions, algorithms, metaphors, analogies, symbols, applications, gestures, drawings or concrete objects (Davis and Renert 2014).

Various realizations of the function concept known in the literature are usually referred to as representations, such as tables, algebraic expressions, and graphs. We chose to use the designation "realization" because the purpose is not to characterize a concept in a dualistic way, as if the mathematical object (function) had an autonomous existence, i. e., independent of its representations, (realizations, in our understanding). In short, a mathematical concept is constituted by its realizations, in such way that we can only speak of a concept in terms of its realizations themselves.

Based on these assumptions, we name Mathematics *in the* Teaching (MiT) of the Function Concept as the set of communicational acts (texts) properly being realized in the dynamics of the teaching of the function concept by the agents in charge of this task. It takes place according with the classification and framing principles that operate in a given pedagogic practice. So, to speak, MiT of the function concept refers to the way teachers participate in pedagogic practices carrying out their job of teaching the concept.

In its turn, we see Mathematics *for* Teaching (MfT) of a mathematical concept is a *re-presentation* of the MiT. We use the term *re-presentation*, separating the prefix with a hyphen because we want to suggest that the MfT of a concept refers to another communicative form (presented again) on the ways to realize the concept in pedagogic practices. Although MfT refers to MiT, the latter occurs only in the emergent dynamics of the pedagogic practice in the school context (i.e., in the pedagogical relations (to be) affected), whereas the former is only a re-presentation, that is, an idealization of the other.

As examples of MfT of a concept, that is, of *re-presentations* of MiT of a concept, we can mention: instructional materials addressing this concept and teachers investigating and presenting proposals to teach this concept. Among those and other possibilities, this study focuses on a characterization of MfT as a *theoretical model*. The purpose is to present it in a structured and systematic way, identifying its categories and properties descriptively.

In order to construct a model of MfT of a concept, we use recognition and realization rules as tool to form categories, which we call landscapes to employ Davis and Renert's (2014) terminology. A theoretical model of The MfT of a Concept can be built by using different sources. In the current study, we used a literature review, textbooks and a discussion of a group of teachers as sources of realizations of the function concept in such as way we are going to explain the reasons in the following paragraphs.

According to Davis and Renert (2014), there is an expressive body of research in the field of Mathematics Education investigating the variety of realizations (commonly named representations) in the understanding of a concept. The literature, therefore, appears to be a promising way to shed light on a wide range of realizations of the function concept.

The textbook is one of the main references of the pedagogic practice in the school context because it is a communication tool guiding and assisting teachers in their teaching tasks, providing support in the selection and sequencing of content, in the methodological strategies, in the assignment of tasks to students, and in the organization of evaluation activities (Alajmi 2012; Nicol and Crespo 2006; Reis 2014; Shield and Dole 2013). According to Mesa (2004) and Nicol and Crespo (2006), the textbook is an expression of the intended curriculum (objectives and aims for the teaching and learning of mathematics established by the regulatory bodies). In fact, from a Bernsteinian perspective, the textbook is the result of the selection and appropriation of texts arising in scientific fields and official documents established by the regulatory agencies in education, bringing all texts together in a special relation to one another, and transformed into texts for the purpose of teaching and learning. In Brazil, the textbook is legitimized by the educational system (Granville, 2008), which regulates, in its texts, the expression of the discourses from scientific fields and normative agencies in education through a textbook evaluation program.

Teachers play a central role in the teaching and learning process (Even and Ball 2009) since they are vital participants in the production of the mathematical communication carried out in the pedagogic practice. According to Davis and Renert (2014), teachers working together generate rich lists of realizations of a concept since they examine it in order to situate it in the context of their teaching experiences.

We understand that the three sources mentioned above provide a variability of realizations, which bring robustness to the theoretical model MfT of the Function Concept we aimed to construct. As we suggested above, we used the following sources for the construction of the theoretical model: analysis of studies investigating the teaching and/or learning of this concept (Santos and Barbosa 2019), textbooks (Santos and Barbosa 2017) and a collective study with teachers analyzing the teaching of the function concept (Santos and Barbosa 2016).

Methodological Aspects, Contexts, and Participants

In order to organize the realizations from the three sources into categories (landscapes) and to analyze their communicative implications, and, therefore, to construct a model, we get inspired not only the classification, framing, and recognition and realization rules concepts from Basil Bernstein's theory, but also part the organizational configuration of the Concept Study (CS) proposed by Davis and Renert (2009, 2013, 2014) as an *analytical tool* to structure the model.

CS is a strategy originally developed by Davis and Renert (2009, 2014) as a tool to discuss a MfT shared by a group of teachers. It is a participatory strategy carried out *with* teachers with the purpose of engaging them in analyzing the wide range of realizations, associations, and interpretations that constitute a mathematical concept and providing support to its teaching and learning. The Concept Study was structured in emphases. In the present research, we have chosen the following emphases to organize the sources: realizations, landscapes, and entailments (Davis and Renert 2014).

Based on the theoretical perspective that underlies this study, our way of using the entailments and landscapes emphases differs from the one originally given by those authors. Landscapes, here, are erected based on the convergence of recognition and realization rules. On the other hand, the entailments are seen as communicative potentialities and limitations arising from the different conceptual associations established by the realizations that make up each landscape, which reveal different understandings and communicative facets of a mathematical concept. In order to analyze the corpus of papers addressing the teaching and/or learning of the function concept, we employed in the systematic literature review, which is characterized as a method to identify, analyze, and synthesize large research bodies of acknowledged quality in a transparent, rigorous and integrative manner (Petticrew and Roberts 2006; Victor 2008). However, our approach on the *corpus* was to identify different realizations of the function concept.

The *corpus* of the systematic review consists of articles dealing with the teaching and/or learning of the function concept in the following journals: *Boletim de Educação Matemática* (BOLEMA), *Boletim do Grupo de Estudos e Pesquisas em Educação Matemática* (GEPEM), *Educação Matemática Pesquisa* (EMP), *Educational Studies in Mathematics* (*ESM*), *Journal of Mathematics Teacher Education* (*JMTE*), and Zetetiké. These journals, among others, are recognized for being responsible for publishing relevant studies in the field of Mathematics Education in Brazil and over the world, and all have been evaluated by the CAPES Brazilian Funding Agency as high reputation Since the current study was developed in Brazil, we sought to contemplate journals published in the country, in addition to journals that are considered international. We restricted the search period from 1990 to 2015[6], since we believe this timeframe is broad enough to compose a substantial and considerable *corpus* of studies to point out the realizations of the function concept circulating and being produced in the teaching of this concept. The selection was initially based on a reading of the title, abstract and keywords. As we recognized relevant data related to the research objective in the studies, these articles were fully read. This way, twenty-nine articles were selected, as shown in **Table 1**.

Journal	Authors
BOLEMA	Birgin (2012), Meneghetti and Redling (2012), Asghary, Shahvarani and Medghalchi (2013), Dazzi and Dullius (2013), Strapason and Bisognin (2013), Callejo and Zapatera (2014), Maciel and Cardoso (2014)
EMP	Rossini (2007), Beltrão and Igliori (2010)
GEPEM	Silva et al. (2001), Frant (2003), Maggio and Nehring (2012)
ESM	Even (1990), Confrey and Smith (1994), Schwarz and Dreyfus (1995), Slavit (1997), Yerushalmy (2000), Sajka (2003), Moschkovich (2004), Falcade, Laborde and Moriotti (2007), White (2009), Ayalon, Watson and Lerman (2015), Hitt, González-Martín (2015), Ronda (2015), Tabach and Nachlieli (2015).
JMTE	Sánchez and Llinares (2003), Steele, Hillen and Smith (2013), Wilkie (2014)
ZETETIKÉ	Brito and Almeida (2005)

Table 1 - List of articles selected per journal

Source: authors

The first textbook-selection step was carried out based on works recommended by the Brazilian Textbook Evaluation Program (BTEP) of 2014 (Brazil 2013a) and 2015 (Brazil 2014) for the final

[6] Some journals are not available online or started their activities after 1990: JMTE – 1998; BOLEMA – 2006; Zetetiké – 2001; EMP – 2004.

years of Middle School (*Ensino Fundamental II*) and High School[7]. The BTEP is run under the Ministry of Education in three-year cycles alternated for each education segment in order to provide teaching material to public basic education schools systematically, regularly and free of charge. The program selects the textbooks based on previously established criteria, which are both general and specific by area. The collections selected are endorsed in a written guide to teachers, which is composed of reviews, a brief description and an assessment of the characteristics of each textbook. Based on the analysis in the guides, the principal or the body of teachers at each school chooses the books that will be used in the three years following the publication of the Guide.

We carried out a complete reading of the guides of the years 2014 and 2015, analyzing them in detail, especially regarding which textbooks had clearer and simpler texts, more contextualized activities, a diversity and significant amount of exercises, and quality illustrations, bearing in mind that these are the criteria that teachers use in their selection of math textbooks approved by the guides, according to Trindade and Santos (2012) and Vieira (2013). As a result, we selected the collections *Matemática*, by the authors Luiz M. Imenes and Marcelo Lellis, for the 6th to 9th grade (Imenes and Lellis 2010a, 2010b, 2010c, 2010 d), and *Matemática*, by the author Manoel Paiva, for high school students (Paiva 2013a, 2013b, 2013c).

At last, the study with the group of teachers was implemented through an in-service teacher education program, organized and conducted by the first author, promoted by the Institute of Mathematics at the Federal University of Bahia (UFBA). The program took place between September and November 2015, and it had a total duration of 60 hours, thirty-two of which through face-to-face group discussions. All teachers who took part in the program had degrees in Mathematics and were teaching middle and/or high school at the time in the metropolitan region of the Brazilian city of Salvador of Bahia[8]. In **Table 2,** we present the profile of all participating teachers.

[7] In Brazil, the final years of basic education (Ensino Fundamental II, equivalent to middle school) lasts 4 years and teach students with an average age (standard) between 10 and 15 years. The following high school period lasts for 3 years.

[8] All participants signed the Informed Consent Form in compliance with Resolution 466/12, which governs research involving human subjects (Brazil, 2013b) and authorizes researchers to use all the information generated during a course on scientific research.

Name[9]	School level taught	Time teaching in years
Cibele	Middle and High School	4
Claudia	Middle School	4
Cledson	Middle School	5
Deise	High School	15
Elcio	Middle and High School	30
Prof. Eusébio	Middle and High School	15
Janice	Middle School	13
Luis	Middle School	3
Nadison	Middle and High School	15
Patrícia	Middle School	3
Regina	Middle School	20
Sampaio	Middle School	25
Talita	Middle and High School	1,5

Table 2 - Participant Profiles
Source: authors

The program was started with thirteen participants. However, after some were not able to show up regularly, in such a way there were seven participants left by the end of the fifth face-to-face meeting.

The program format was inspired by the Concept Study groups carried out by Davis and Renert (2009, 2014), especially with respect to the sequential organization of activities. Only the first meeting was planned in advance, therefore, and the configuration of the other meetings emerged during the program of each previous session based on the discussions that took place.

To record the data from the program, we used a field diary, audiovisual recordings of all meetings and the written productions of the participants (records on paper and on the blackboard). Despite the wealth of data from the study with teachers, due to the objectives of the article we restrict ourselves to presenting the realizations of the concept of function and their entailments.

Summing up, we combined multiple sources: bibliographic research (Gil 2002), and two empirical studies - textbooks and a group of teachers[10]. In doing so, we intended to raise as many realizations of the concept of function as possible to build a rich model. Once the realizations had been collected, we read each of them, trying to identify which rules of recognition and realization we might derive from them. Then we were able to bring the realizations together by the convergence of those rules, allowing us to propose landscapes for the concept of function. Later, our interpretative

[9] Only the name of the teacher Talita is fictitious, the other participants disclosed their identification, using their first or last name.
[10] All analyzed sources focused on the function concept in middle and high school education.

work was to identify the entailments for each landscape. It allowed us to organize the landscapes into a structure, which we present below.

The Landscapes and their Entailments

The realizations identified as associated with the function concept in the three sources were grouped in the tabular, diagram, algebraic, transformation machine, graphic, pattern generalization, and formal landscapes.

Tabular landscape

The tabular landscape includes the realizations of a function as tables, which are realized by the organization of input data (elements of the domain of a functional relationship) and their corresponding output data (elements of the image of the functional relationship) in rows (or columns). Due to its nature, the realizations of this landscape have the communicative limitation that they can only be used for functional relationships that have domain and image sets with a finite number of elements.

Tabular realizations can be introduced even before the word *function* appears in communications for education, such as in situations to investigate the relationship of direct and inverse proportionality (Imenis and Lellis 2010b; Steele at al. 2013), as in the example described in **Part A** of **Table 3**. In this example, there are two functional relationships, namely, the one associating the side of a square to its perimeter and the other associating the side of a square to its area. In the first case, there is a direct proportionality and in the second there is not. In a note to the teachers, Imenis and Lellis (2010b) observed that the direct proportionality would be thereafter described by equations of the type $y = kx$, in which k is the proportionality constant.

Part A	**Part B**

Solve the questions related to the geometric figure of the square.

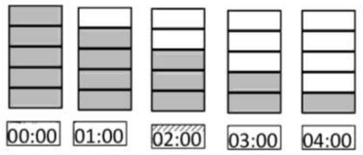
A) The table shows some measures concerning the squares. Complete it:

Side (cm)	Perimeter (cm)	Area (cm^2)
10	40	100
15		
20		
25		

B) Is the perimeter directly proportional to the length of the side?

C) Is there a direct

A water reservoir with a capacity of 1,000 liters is full. The meter is opened to empty it and a timer is triggered as soon as a constant flow starts, as shown in the figures below.



Fill out the table taking the above illustrations into account.

Time	0	0.5	1.0	1.5	2.0	2.5	3	4	5
Volume	1000	___	800	___	600	___	___	200	__

Does the volume of water observed in the reservoir depend on elapsed time?

Table 3 - Tabular

Source: Imenis and Lellis (2010b, p.146-147)

Source: Reproduced from Rossini (2007, p. 228 - 230)

The question reported in **Part B** of **Table 3** was suggested for the introduction of the function concept by a group of teachers in the study by Rossini (2007). The tabular realization is used to organize the data of the functional relationship and to characterize both the relationship of dependence between the variables (Rossini 2007; Silva et al. 2007) and the nature of these variables (Maggio and Nehring 2012; Strapason and Bisognin 2013) as notions of the function concept.

Teacher Cybele, a participant of the in-service program, also suggests the tabular realizations of functional situations in daily life to introduce the function concept, emphasizing the importance of making clear that "all values of *x* are associated with the values of *y* and that each value of *x* is associated with a single value of y" (2nd Meeting) - where *x* is the independent and *y* is the dependent variable. In this case, the purpose is to present the univalent nature of a functional relationship - each element of the input set (independent variable) is associated with a single element of the output set (the dependent variable) (Even, 1990; Steele at al. 2013) -, and, therefore, to establish a criteria for the recognition of a table as a realization of a functional relation, in addition to linking the notion of association between variables as a way of interpreting a functional relation.

Bloch (2003) and Schwarz and Dreyfus (1995) emphasize tabular realizations are generally partial, since these realizations only allow you to view some data of the functional relationship, which can lead to ambiguity, such as inferring that the functional relationship is linear or has an extreme value, even when this is not the case. In this sense, Prof. Eusébio stated in the 5th meeting: "If we have a phenomenon and focus on part of this phenomenon, then we'll have mathematical models (functional relationships) representing that fragment, but not the phenomenon as a whole." These considerations point to some communicative limitations (entailments) of the tabular realizations.

Diagram Landscape

This landscape is composed of function realizations as arrow diagrams, which are realized with all elements of the domain and range sets (indicated here by A and B, respectively) in two disjoint diagrams, matching each element of A with only one element of B (through an arrow). Based on those realizations it is possible to make explicit the arbitrary character of a functional relationship, indicating a communicative potentiality of the realizations of this landscape. For example, Paiva (2012a) and Meneghetti and Redling (2012) define a functional relationship as a correspondence between two non-empty sets A and B, in which each element of set A matches a single element of set B. The arbitrary nature of the functional relationship concerns both sets A and B, which need not be numeric, and the correspondence, which need not follow a pattern (Even 1990, Steele et al. 2013; Tabach and Nachlieli 2015). In **Part A** of **Table 4**, we present a realization of a functional relationship with a diagram.

Part A	**Part B**	**Part C**

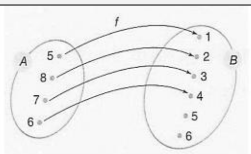
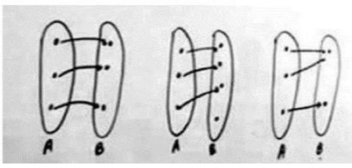
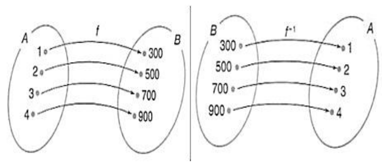

		

Table 4 - Diagram		
Source: Paiva (2013a, p. 121)	Source: Records from Teacher Luis Sergio - 7th Meeting	Source: Paiva (2013a, p. 143)

In the textbook (Paiva 2012a) and the discussion with the teachers, the diagram realizations were recommended for an introduction to the function definition, signalize that it is possible both to identify the domain, range, and image (as a subset of the range) sets of a functional relationship, and to present their respective symbolic notations. These elements, as Teacher Nadison emphasized, are part of the characterization of all types of functional relations, and as such, they compound the mathematical syntax of the function concept.

In the study with the teachers, the diagram realizations were used because of their iconic character to provide visibility to the definitions of injective, surjective and bijective (two-way correspondence) functional relationships (**Part B** of **Table 4**). With this characterization and recognition of a bijective functional relationship, Paiva (2012a) presents the definitions of an invertible functional relationship and its inverse relationship (**Part C** of **Table 4**).

As a communicative limitation of the diagram landscape, we mention the fact that they are restricted to functional relations with finite domain and range sets and a limited number of data, as well as hiding the notion of variation.

Algebraic Landscape

The algebraic landscape is made up of the function concept realizations establishing a functional relationship[11] as a correspondence, mapping, association or relationship between the independent and dependent variables in a unique way[12] through a law, formula or algebraic expression. When the independent variable is indicated by *x* and the dependent variable by *y*, the function realization as an algebraic expression is recognized and realized by the text $y = f(x)$.

For real functions with real variable, Paiva (2014 a) points out that when only the law of formation of the function $(y = f(x))$ is presented, one must consider that the domain of *f* is the broadest subset of R in which *f* can be defined and its range is *R*.

Imenis and Lellis (2010b, 2010c) introduce algebraic realizations even before the formal presentation of the function definition as *formulas* that express "[...] a relationship between quantities" (Imenis and Lellis 2010c, p.86). The authors suggest teachers explore the expressions: *it depends*, *varies* and *is a function of* because "[...] the use of these expressions helps to transmit ideas that develop the function concept" (Imenis and Lelis 2010b, p. 216). In Part A of Table 5, we show

[11] In this landscape, we address the algebraic realizations of functional relations whose domain and range are subsets of the set of real numbers *R*.
[12] With the exception of equivalent algebraic expressions.

an example of one of these algebraic realizations (formulas), in which it is possible to explore that: *Q depends on x*, *Q varies with x* or *Q is a function of x*. In addition, this formula allows someone to determine a unique value for *Q* based on any *x* given ($x \geq 0$[13]), establishing the criterion for the recognition of an algebraic formula, law or expression as an algebraic realization of a functional relationship, that is, a formula of the type $y = f(x)$ is the algebraic realization of a functional relationship f if, and only if, *y* is unique for each *x* (Confrey and Smith 1994).

Part A	**Part B**
In a certain town, the cost of the water consumed in a household is calculated in accordance with: The formula for $x \leq 20$ is $Q = 2{,}5x$ The formula for $x > 20$ is $Q = 4{,}7x - 44$, where x is consumption in m^3 and Q is the amount payable.	In some factory, the production cost *p*, in R$, of each chocolate depends on the quantity *q* of chocolates manufactured, and this quantity depends on the number *n* of machine hours. These dependencies are described by the following functions: $p = 3 + (500/q)$ and $q = 200n$ A) If this machine runs for only 5 hours, what will be the cost of production of each chocolate? (B) Express *p* as a function of *n*. (C) Express *n* as a function of *p*.
Table 5 - Algebraic	
Source: Imenis and Lellis (2010c, p.189-190)	Source: Paiva (2014a, p. 147)

Bertrand and Igliori (2010), Frant (2003), Maciel and Cardoso (2014), Rossini (2007), Prof. Nadison (2nd Meeting) recommend that the function concept should also be addressed in education as a mathematical model to describe natural, everyday life and other scientific phenomena, demonstrating its pragmatic nature, as well, in the first case bringing the academic texts closer to everyday texts. Corroborating this recommendation, the algebraic realizations of **Parts A** and **B** of **Table 5** are used to model phenomena mathematically, translating their behavior by clarifying the relation of dependence between the variables in a concise and compact manner, thereby providing the quantification of the phenomenon under investigation (Beltrão and Igliori 2010; Teacher Eusébio - 2nd Meeting; Slavit 1997).

In **Part B** of **Table 5**, we present a question proposed by Paiva (2014a), in which to solve item B, it is necessary to perform the composition *poq* algebraically based on the algebraic realizations of

[13] Because of the context of the problem.

p and q, and in item C its inverse, whose texts are $p = 3 + (5/(2n))$ and $n = 5/(2(p-3))$, respectively.

The algebraic realizations have concise texts that condensate information about the functional relationships into a single string of symbols (Schwarz and Dreyfus1995; Ronda 2015). This characteristic provides for both the recognition and characterization of types of functional relations (Wilkie 2014) regarding the execution of operations, such as adding, subtracting, multiplying, dividing, composing functional relations (when possible) and also determining the algebraic inverse of an invertible function (Sánchez and Llinares 2003; Ronda 2015; Yerushalmy 2000).

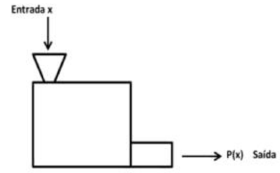
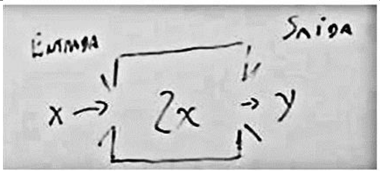
However, despite of communicative potentialities of these realizations previously mentioned, emphasizing algebraic realizations in the teaching of the function concept may make the function concept indistinguishable from other algebraic realizations (Sajka 2003). This predominance can have the following consequences, for example, (i) not considering other elements of a functional relationship, compromising the recognition, for example, that $f(x) = x + 3$ and $g(x) = (x^2 + x - 6)/(x-2)$ can set the same functional relationship depending on the domain (Schwarz; Dreyfus, 1995; Slavit, 1997); (ii) not taking into account that for a non-bijective algebraically feasible functional relationship, you can restrict your domain and/or range sets getting another functional relationship with the same algebraic realization, as long as it's bijective and, therefore, invertible[14]; (iii) preventing the recognition of functional relationships that can't realize algebraically (for example, the functional relationship that has a list of words as its domain, with each word matching its first vowel) (Steele; Hillen; Smith, 2013).

Transformation Machine Landscape

This landscape is composed of function concept realizations as a metaphor of a machine that transforms inputs (raw materials or input elements) into outputs (products or output elements). In

[14] For example, the square functional relationship $f: R \to R; f(x) = x^2$ is not bijective, but when its domain and range are restricted to the set of non-negative real numbers(R_+), we obtain the functional relationship $g: R_+ \to R_+; g(x) = x^2$, which is bijective and, therefore, invertible. Its inverse is the functional relationship $h: R_+ \to R_+; h(x) = \sqrt{x}$.

Table 6, we show two iconic texts of function concept realizations as a transformation machine, in which each input element is transformed/processed/modified into a (single) output element.

Part A	Part B
	

x	P(x)
0	1
1	2
2	4
3	8
5	32
8	256
10	1024

Table 6 - Transformation Machine

Source: Rossini (2007, p. 243)

Source: Records from Teacher Sampaio - 1st meeting

A transformation machine is more informal and related to the daily experience of students. For this reason, they are recommended by Asghary at al. (2013), Rossini (2007), Wilkie (2014) and by Teacher Sampaio to introduce the function concept in teaching.

Through the realizations as machine, it is possible to explore the relationship between the dependent and independent variables (Wilkie 2014), introduce the domain of a functional relationship as the set formed by the input elements and the image as the set consisting of the output elements (Rossini 2007; Teacher Sampaio - 1st Meeting), and also to incorporate the notions of process, change and transformation to the interpretative network of the function concept (Sánchez and Llinares, 2003; Teacher Sampaio - 1st meeting).

By revealing the notions of process, change, and transformation, the function concept realizations as transformation machine are only compatible with functional relations with numeric input (domain) and output (image) data, and which obey a law or formula, as in **Part A** of **Table 6**, in which the algebraic realization of the functional relationship is $P(x) = 2^x$, and in **Part B**, $y = 2x$. In addition, it is not possible to characterize the range of a functional relationship through such realizations. These considerations point to some communicative limitations of the realizations of this landscape.

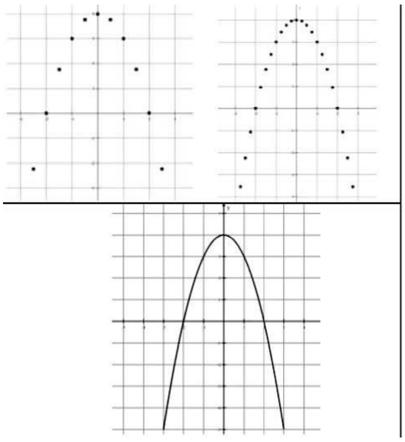
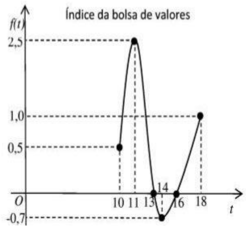
Graphic Landscape

The graphic landscape is formed of graphic realizations (graphs) of a functional relationship, in which the domain and range are subsets of the set of real numbers (R). The graph of a functional relationship $f: A \to B$,of this nature is a subset of $R \times R$, consisting of all ordered pairs (x, y), where $x$is a domain element of f (set A) and $y = f(x)$.

The recognition of a subset of the Cartesian coordinate system ($R \times R$) as the graphic realization of a functional relationship may be operationalized through the so-called vertical line test (Paiva 2014a; Teacher Sampaio - 7th Meeting; Slavit 1997; Steele at al. 2013). This test is based on the univalent nature of a functional relationship, and consists in drawing straight lines parallel to the vertical axis (of the dependent variables), passing through points of the abscissa x (independent variable), with x being a domain element of the relationship, so that this subset is a graphic realization of a functional relationship of this domain if, and only if, each one of these straight lines intersects the subset in a single point (Paiva 2014a; Teacher Sampaio - 7th Meeting; Steele at al. 2013).

The graphic realization of a functional relationship is presented in Imenis and Lellis (2010d) based on the algebraic realization. Considering the example of a functional relationship realized algebraically by $f(x) = -x^2 + 4$, the process shown by the authors to build a graphic realization of this functional relationship consists in organizing a tabular realization, marking some points ((x, $f(x)$)) in the Cartesian coordinate system, repeating the process considering more points, connecting these points, assuming that a curve called a parabola passes through them, so that "if we drew infinite points we would have a continuous curve without jumps or gaps" (Imenis and Lellis 2010d, p. 214). In **Part A** of **Table 7**, we reproduce the above example. The authors argue that this approach is an accessible way to explain to a student at this level of education why "[...] the points should be connected so as to form a **smooth curve**" (Imenis and Lellis 2010d, p. 213, emphasis by the authors).

Part A	Part B

	The following graph describes the index f(t) of a state's stock exchange in percentages, as a function of time t, in hours, since the beginning of trading at10 h, until its closing at 18 h on a given day. 
Table 7 - Graphic	
Source: Imenis and Lellis (2010b, p.214)	Source: Paiva (2014a, p. 126)

The adopted approach legitimizes not only the function realizations as graphs in the school context of basic education, but also the process of drawing them, which according to the authors is: "Formula→Table→Marking points→Joining points" (Imenis and Lellis 2010d, p. 214). We emphasize that this process is feasible[15] as long as it acknowledges what the expected graphic realization is, and, therefore, which points should be considered to realize the functional relationship graphically, with the support of the algebraic realization. Such a procedure to graphically realize a functional relationship based on the algebraic realization is also adopted in the high school collection (Paiva 2014a, 2014b, 2014c) under analysis. As specific types of functional relationships and their respective algebraic realizations are inserted, the realization of the corresponding graphs follows procedures in accordance with the functional relationship.

The aforementioned procedure establishes connections (*bridges*) between the algebraic and graphic landscapes. The use of digital technologies is recommended by Dazzi and Dullius (2013), Moschkovich (2003), and White (2009) to streamline and, therefore, encourage the establishment of *bridges* between the algebraic, graphic and/or tabular landscapes.

Through the graphical realizations it is possible to infer and analyze the properties and characteristics of the functional relationships, including: domain, image, signal, limits, growth and decline intervals, injectivity, and the existence of extremes and zeros (Paiva 2014a; Sánchez and

[15] Assuming that the functional relationship is realizable graphically and continuous.

Llinares 2003; Strapason and Bisognin 2012). As in the example reported in **Part B** of **Table 7**, which describes a stock market index on a given day. The global or local behavior of the phenomenon modeled by a functional relationship can, therefore, be viewed, analyzed, recognized (Prof. Eusébio - 5th Meeting; Teacher Sampaio - 3rd Meeting; Sánchez and Llinares 2003) and legitimized, in this context, based on the analysis of its graphic realization. This analysis of the graph presented in the previous paragraph enables the establishment, in our terms, of *bridges* between the algebraic and graphic landscapes through the recognition and legitimization of the equivalence between the procedures that are linked to the texts of each one of these landscapes (Bloch 2003; Moschkovich 2003; Slavit 1997).

Despite the communicative potential of the graphic realizations already mentioned, some studies consider that its predominance in teaching with a focus on continuous functional relations, mostly in linear and quadratic functional relationships, can hinder the recognition of functional relationships with graphic realizations that are not easily realizable, for example, its graphic realizations feature leaps), or even functional relationships that cannot be realized graphically such as the Dirichlet function $g(x) = \begin{cases} 0, \text{ if } x \text{ is rational} \\ 1, \text{ if } x \text{ is irrational} \end{cases}$ (Kleiner, 1993; Even, 1990; Steele; Hillen; Smith, 2013), which is discontinuous in all points of its domain.

Pattern Generalization Landscape

The pattern realization landscape of the function concept is composed of texts that can be used to determine the image of any element of the domain of a functional relationship (numerical sequences, sequences of geometrical shapes and functional phenomena[16] that can be realized algebraically), which are realized based on the recognition of a relationship between quantities and/or variables, through an informal inductive process, relying on some information or descriptions of the corresponding functional relationship (Carraher at al. 2008; Mavrikis et al. 2012; Wilkie 2014).

The recognition and realization of pattern generalizations can be operationalized through two types of approaches: the relational approach through correspondence, or the explicit, recursive or

[16] By functional phenomena we mean those phenomena that can be modeled by a functional relationship.

covariation approach (Asghary at al. 2013; Aylon at al. 2015; Callejo and Zapatera 2014; Maciel and Cardoso 2014; Maggio and Nehring 2012; Rossini 2007; Wilkie 2014). The covariation approach is based on establishing how the independent and dependent variables vary together, while the relational approach consists in determining a pattern or rule that associates the independent variable directly with the dependent variable (Aylon at al. 2015; Callejo and Zapatera 2014; Cooney et al. 2013; Confrey and Smith 1994; Falcade at al. 2007; Hitt and González-Martin 2015; Slavit 1997; Wilkie, 2014).

In **Part A** of **Table 8**, we present a sequence of geometric figures whose pattern generalization was realized according to two approaches. In the recursive approach, a variation of the number q for squares is related to the variation in the number P for toothpicks. The recursive generalization in natural language described in **Part B** of **Table 8** can therefore also be done through such symbolic texts as: $P(1) = 4$; $P(q+1) = P(q) + 3, q \geq 1$, q a natural number. In the relational approach, the relationship of functional dependence between the number of toothpicks P and the number of squares q is made clear, which if realized through the symbolic texts becomes $P(q) = 4 + (q - 1)3 = 1 + 3q$, with $q \geq 1$, q a natural number, which corresponds to the algebraic realization of the functional relationship[17]. As can be seen, the pattern generalization realization of this sequence of geometric figures is based on an informal inductive process, which is recognized and legitimized as a form of argumentation in this context, working as a "permission" to determine any element of the sequence.

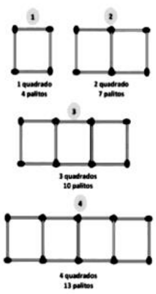
<table>
<tr><th>Part A</th><th>Part B</th></tr>
<tr><td>Observe the sequence of figures

</td><td>Recursive Pattern: 3 toothpicks suffice to form a new square, since 1 side of the last square can be used. As the number of squares varies (increases) from 1 in 1, the number of toothpicks varies (increases) from 3 in 3.
Relational Pattern:
(Figure 1) We started with 1 square and four (4) toothpicks.
(Figure 2): Number of toothpicks: $4 + 3 = 4 + 1 \cdot 3$</td></tr>
</table>

Table 8 - Pattern Generalizations
Source: Imenis and Lellis (2010a, p 260-261).

[17] Such a functional relationship is the restriction of an affine function to the set of natural numbers.

The function realizations as pattern generalizations of linear or affine functional relationships are recommended by papers in the *corpus* (Asghary at al. 2013; Callejo and Zapatera 2014; Maggio and Nehring 2012; Rossini 2007; Wilkie 2014)) and are presented in the textbooks under analysis (Imenis and Lellis 2010a, 2010b, 2010c) as an initial contact with texts that communicate this concept, even before explicitly addressing the formal content. The exploration of pattern generalizations may support the subsequent study of the function concept, considering that those realizations give visibility to the notions of variation and the relationship of dependence between the quantities/variables involved (Wilkie 2014), which subsequently can be recognized and legitimized as constituent notions of this concept's interpretative possibilities (Steele at al. 2013; Wilkie 2014), in addition to enabling a distinction between the independent and dependent variables (Study with the teachers - 7th meeting). Corroborating this understanding, Imenis and Lellis (2010a) suggest teachers should include the expressions: "[...] **depends** on [...]","[...] **varies** [...]","[...] **is a function of** [...]" (P. 255, emphasis by the authors) in the analysis of pattern generalizations since they consider that these texts contribute to the development of the function concept.

The covariation approach is intrinsically connected to the realization of a function as the rate of variation or rate of change (Confrey and Smith 1994; Aylon at al. 2015). The realization of a function as a rate of change expresses the relationship between the variation of outputs and their respective inputs (Aylon at al. 2015). For example, for the functional relationship described in **Table 8**, the rate of change is $\frac{\Delta P}{\Delta q} = \frac{P(q+1)-P(q)}{(q+1)-q} = \frac{P(q)+3-P(q)}{1} = 3$ (constant). A constant rate of change characterizes affine functional relationships (Birgin 2012). The functional relationship of the example is realized algebraically by$P(q) = 1 + 3q$. Note that the rate of change corresponds to the coefficient of the linear variable of the algebraic realization, which can also be interpreted as the gradient or slope of the line, which would be the graphic realization of this functional relationship (Birgin 2012; Steele at al. 2013). From this perspective, it is possible to establish *bridges* between the graphic, pattern generalization and algebraic landscapes.

Members of some families of functional relationships share the same rate of variation or change (Cooney et al. 2013). As a result, knowing such a realization of a function as the rate of change may enable the recognition of the type of functional relationship under study (Slavit 1997). These realizations can therefore work as a support to model functional phenomena (Aylon at al. 2015; Confrey and Smith 1994; Steele at al. 2013).

The realization of the function concept through pattern generalizations can also be used when developing specific types of studies of functional relationships (Brito and Almeida, 2005; Confrey and Smith 1994), in the modeling of phenomena or situations that are "mathematized" by these functional relations. The teachers who took part in the in-service program point out that texts with a more direct relationship with the local and specific context of the students, which we call non-school texts, lead to the recognition of the function concept as significant from the point of view of its applicability in everyday situations. From our perspective, there is suggests possibility of recognizing that such situations demand explanations, which can be realized legitimately through the school math texts on the function subject. In a study by Wilkie (2014), the teachers pointed out that organizing data in a tabular realization assists in the recognition of the type of regularity in the function realization as pattern generalization. That is, they established *bridges* between these landscapes.

For Ayalon at al. (2015), the two approaches for function realizations as pattern generalizations are complementary because they use distinctive interpretative perspectives for the function concept. Confrey and Smith (1995) consider the covariation approach to be more easily realizable. However, Callejo and Zapatera (2014) indicate that the emphasis on the recursive approach may prevent someone to obtain the (explicit) relational generalization, such as the choice of the linear model, although this is not the functional relationship that characterizes the phenomenon under analysis.

Formal Landscapes

This landscape consists of function concept realizations as formal definitions. We employ the adjective "formal" because these realizations are precise textual structures, similar to those characterizing legitimate definitions in the contemporary context of Academic Mathematics.

Function concept realizations as formal definitions, therefore, contain the necessary and sufficient conditions that assist in the recognition of functional relationships (Tabach and Nachlieli 2015) in their varied forms of realization.

In **Table 9** next, we reproduce three function realizations as formal definition extracted from the sources under analysis. The transcribed realization in **Part A** defines a functional relationship as a subset of a Cartesian product with special characteristics (it is based on set theory, therefore), and those in **Part B** and **C** define it as an association between variables with specific properties.

Part A	Part B	Part C
A function f is defined as any set of ordered pairs of elements such that if$(a, b) \in f, (c, d) \in f$ e $a = c$then $b = d$.	We say that a variable y is given as a function of one variable x if, and only if, for each value of x there is a single value of y. The condition that establishes the correspondence between the values of x and y is called the **law of association**, or simply the law between x and y. When possible, this law is expressed by an equation.	Given two non-empty sets (A and B). A relationship that associates to each$x \in A$ one $y \in B$, receives the name of function.
Table 9 - Formal Definition		
Source: Even (1990, p. 531).	Source: Paiva (2014a, p. 117, emphasis by the author)	Source: Transcript of the records from Prof. Sampaio - 7th meeting.

In the function realizations as formal definition, the univalent and arbitrary nature of the function concept is stated. Even (1990) and Steele at al. (2013) consider these two attributes as key characteristics of the function concept, since they allow to distinguish functional relationships (in any form of realization) from other relationships. The univalence characteristic is often used as a criterion for the recognition of functional relationships (Even 1990) realized by graphs (vertical line test) (Steele at al. 2013), tables and diagrams, as we outlined in the analysis of these landscapes.

Although the function realization as a formal definition is accurate, for Even (1990) it doesn't convey the interpretative possibilities of how the function concept is often used in mathematics, science or everyday life. Echoing this statement, Falcade, Labordi and Mariotti (2007) state that the function realizations as formal definition are devoid of the variable concept

According to Tabach and Nachlieli (2015), studies have shown that even students who are able to reproduce such realizations may contradict their texts when using them as a tool to recognize functional relationships, especially, according to Lambertus (2007), when faced with unfamiliar functional relationships, such as Dirichlet's function.

In the study we carried out with the teachers, Teacher Eusébio (7th Meeting) presented the function realization as formal definition reproduced in **Part C** of **Table 9** in conjunction with the diagram, algebraic and graphic realizations of a functional relationship. The teacher states that "[…] they are some possibilities we can use to illustrate the formal concept (realization as formal definition, from our perspective), let us say, with the representations (other realizations in our denomination) [...]" (7th meeting). Using a similar approach to first set out the function realization as formal definition (**Part B** of **Table 9**), Paiva (2014a) considers the functional relationship that correlates the average temperature of some days in a given month for a region, linking it to its diagram, table, graph, and algebraic realizations, highlighting the univalent and arbitrary nature. In these cases, we sought to establish connections (*bridges*) between those realizations in order to enable the recognition and realization of the texts of the function realizations as formal definition, from the logical structure perspective, considering the univalence and arbitrariness characteristics of different realizations.

Synthesis of a theoretical model of Mathematics for Teaching of the Function Concept

The present theoretical model of MfT of the function concept was organized in seven landscapes of the function concept identified in the three sources and was built using the recognition and realization rules as criteria to categorize the realizations.

In the analysis of landscapes and their entailments in the previous section, we sought to explain in detail the specific orientation of each landscape for the recognition, selection, and realization of the legitimate texts and interpretations constituting the function concept in educational contexts. The recognition rules enable the identification of each landscape, distinguishing it from other landscapes due to the specificity of its texts, and therefore regulate *what* texts are legitimate in each landscape.

The realization rules enable the selection and production of the legitimate texts composing each landscape, regulating *how* the texts of each landscape can be made public.

In **Table 10**, we present a synthesis of the analysis performed in the previous section, describing *what* texts characterize and constitute each landscape, and also *how* these texts may be realized in their different presentations. In addition, we summarize the entailments imposed by the realizations that are part of the landscapes.

Landscape	**"What"**	**"How"**	**Entailments**
Tabular landscape	Relationship between data (numeric or not) arranged in a table, provided that each data point in a row or column (input) is associated with one single data point in the row or column (output), respectively.	Organize data in a functional relationship in rows or columns so that the input data and the corresponding output data are on the same row or column.	*Potentials* -Highlights the notions of association and dependence. -Identifies dependent and independent variables. -Organizes data in a functional relationship -Recognizes proportional and non-proportional functions. *Limitations* - Doesn't infer correctly about the type of functional relationship and extreme value -Provides only a partial view of functional relationship.
Diagram Landscape	Correspondence between two arbitrary sets A and B arranged in separate diagrams, where each element of set A (input or domain) corresponds (through an arrow) with one element of set B (range or output).	Identify the domain and range of a functional relationship, arrange them into two separate diagrams, and associate each element of the domain to its image (with an arrow).	*Potentials* -Identifies the domain and range sets -Characterizes the image set. -Outlines the arbitrary and univalent nature of a functional relationship. -Presents the definitions injective, surjective and bijective functions. -Recognizes and defines invertible functional relationships. *Limitations* -Is restricted to functional relations with finite domain and range sets and a limited number of elements. -Hides the notion of variation.

Algebraic Landscape	A law, rule or formula in an algebraic text through which it is possible to explain in a unique way (with the exception of equivalent algebraic expressions) a (dependent) variable in terms of another (independent) variable.	Explain the relationship of dependence between the independent and dependent variables of a functional relationship through an algebraic law, rule or formula (using letters and symbols).	*Potentials* -Models phenomena. -Deals with quantitative aspects. -Demonstrates the relationship of dependence and variability. -Recognizes and defines functional relationship families. -Performs operations with functional relationships. -Composes and inverts functional relationships. *Limitations* -Makes it impossible to recognize functional relations that can't be realized algebraically. -Doesn't consider other elements of a functional relationship - domain and range.
Transformation Machine	Iconic text of a machine that transforms (obeying a rule) each data entry (input) into a single given of output (*Output*).	Realize an iconic text that characterizes a functional relationship (which obeys a rule) as a machine that transforms each element of the domain set into its corresponding image.	*Potentials* -Outlines the notions of process, change, transformation and relationship. -Introduces the domain and image set definitions of a functional relationship. *Limitations* -Subordinates the function concept to computational aspects. -Hinders the characterization of the range of a functional relationship.
Graphic Landscape	A subset of points: $G = \{(x, y), x \in A \text{ e } y \in B\}$, With A and B as subsets of R, so that if $(x, y_1) = (x, y_2)$then $y_1 = y_2$ (vertical line test). Notations: R is the set of real numbers; *x* is the independent variable and *y* is the dependent variable.	Plot the set of points (*x*, *y*) on the Cartesian coordinate system, such that *x* and *y* are in a functional relationship, considering *x* as the independent variable and *y* as the dependent variable. This data can be extracted from a tabular, diagram or algebraic realization.	*Potentials* -Identifies, characterizes and determines: domain, image, growth and decline intervals, signal, zeros and extremes. -Emphasizes the univalent nature. -Builds *bridges* with the algebraic landscape. -Recognizes functional relationship families. *Limitations* -Hinders the recognition of functional relations that can't or can't be easily realized graphically.

Pattern Generalization Landscape	declaratory or symbolic text which, based on some data or information about a functional relationship, explains the nature of the relationship (as a general or recursive rule), enabling the determination of the image of any element of the domain in a functional relationship.	Present a declaratory or symbolic text that expresses the general or recursive pattern of a functional relationship, based on some particular information.	*Potentials* -Gives visibility to the notions of variation and the relationship of dependence. -Enables the distinction between independent and dependent variables. -Recognizes functional relationship families. -Operates as a support in the modeling of functional phenomena. -Builds bridges between pattern generalization, algebraic and graphic landscapes. *Limitations* -Generates misunderstandings in the characterization of functional relationships, with a prevalence of the linear of affine model.
Formal Landscapes	Declarative text establishing a functional relationship as an arbitrary and univalent relationship between the elements of any two non-empty sets A and B or as a subset of the Cartesian product A x B	Realize a declaratory text defining a functional relationship explaining the characteristics of univalence and arbitrariness, with the use of quantifiers.	*Potentials* -Highlights the characteristics of univalence and arbitrariness. -Enables the recognition of functional relationships in different realizations. *Limitations* -Omits and limits the understanding of concepts and interpretations associated with the function concept, such as the notion of variation and dependence. -Requires familiarity with the terminology of quantifiers.

Table 10 - Synthesis of a theoretical MfT model of the function concept: the "what" and "how" of its texts and entailments

Source: authors

Bernstein's theory equipped us with a set of principles and a precise language to theoretically structure a *re-presentation* on the *what* and *how* of the Function Concept's realizations. We therefore focused both on the characteristics constituting and distinguishing the specialized form of the texts of each landscape, and their interpretative implications and limitations, as we summarized in **Table 10**. The model presents a micro overview of the nuances and multiple discursive formations of the communications that might be realized in the teaching of the function concept in the basic education context, in accordance with the regulation used (classification and framing) in this context.

For individuals to be able to produce legitimate texts in a given context (MiT in this study), they must be able to recognize (recognition rules) and produce context-appropriate texts (realization rules) (Ferreira at al. 2010). That is why we focused on the characteristics both constituting and distinguishing the specialized form of the texts of each landscape, and their interpretative and communicative implications and limitations.

In **Figure 1**, we present an iconic text to characterize the theoretical model of MfT of the function concept developed in this study. The landscapes were organized into separate rectangles with similar dimensions and arranged in a circular formation in order to outline that each landscape is characterized by specific texts with their own recognition and realization criteria. *From the point of view of the model,* it also indicates that the landscapes do not have hierarchical relationships, considering that these are categories of the function concept. We highlighted "*from the point of view of the model*" because the model is a *re-presentation* of the MiT of the function concept, which is dynamic and emerging, bearing in mind that this concerns the dimension of how the communicative participation might occur (discursive formations) of those who are responsible for teaching and learning the function concept in a pedagogic relationship.

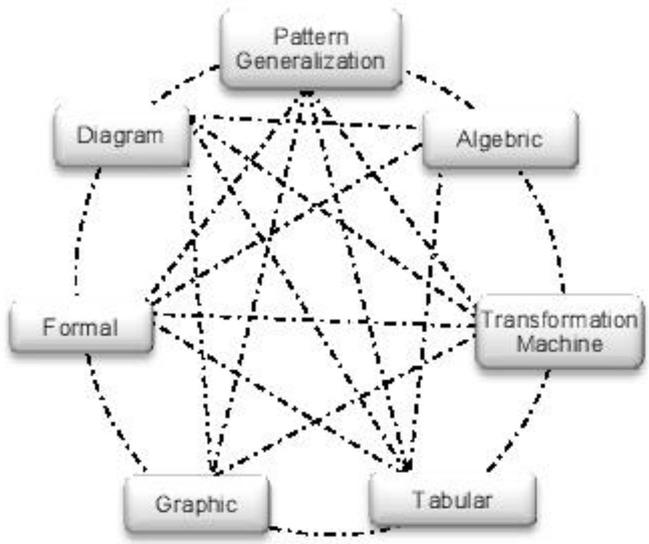


Figure 1 - A theoretical MfT model of the function
Source: authors

Finally, the dotted lines connecting all landscapes in pairs try to suggest the possibility of establishing (when possible) relationships (*bridges*) between them in the teaching process (that is, in MiT) of the function concept by the agents responsible for this task. The model has the potential to predict these *bridges*, but they only manifest themselves in pedagogical practice. Some of these *bridges* were identified in the analysis performed in the previous section, for example, in between pattern generalization, formal, graphic, tabular, and algebraic landscapes.

The classification principle can be used to analyze the relationships (*bridges*) between the landscapes (which are categories) of the function concept; we call such relationships *intra-concept relationships*. From this perspective, there is a weaker classification (C-) in the intra-concept relationships when *bridges* are established between the landscapes. In this case, there is a stronger link between their respective texts, and as mentioned in the previous section, it is possible to both develop and legitimize the equivalence between the procedures and interpretations of these landscapes, and to minimize the existing communicative difficulties and limitations of the realizations of each landscape.

Studies have pointed to the importance of establishing a weaker classification (C-) in the intra-concept relationships for the teaching of the function concept (it said in our words) referring to the algebraic, graphic and/or tabular landscapes (Ronda 2015; Slavit 2003). Such an approach enables characteristics and properties of the function concept to emerge in the different realizations (Ronda 2015), developing an integrated view of this concept instead of identifying it as one of its realizations (Elia et al. 2006; Nachlieli and Tabach 2012).

Since each landscape establishes aspects and particular realizations of the function concept with its own communicative rules, we believe there should also be a place for a C+ classification in the intra-concept relationships in the teaching of this concept, in such a way that the boundaries between landscapes are outlines. For the more, according to Bernstein (2000), a permanently C- classification may generate ambiguities in the communicative recognition and realization. Following a Bernsteinian point of view (Cause 2020; Morais and Neves 2007, 2011), the classification amongst different landscapes of the function concept should vary during the teaching of content and even during a class, making the intra-concept relationships sometimes more visible, sometimes less.

Bernstein (2000, 2003) uses the framing principle to analyze the nature of control over the communicative rules. When agents responsible for teaching impose a C+ on intra-concept relations, we can also consider the framing to be F+. Morais and Neves (2011), in a similar approach, propose to use the framing principle to analyze the relationship between school texts and everyday texts, even without referring to the relationship between people. With this understanding, for example, when the texts of the transformation machine landscape are used for the teaching of the concept of function, we can consider that F-, because there is a relationship between the school texts (function) and those of everyday life. For this reason, the realizations of this landscape were suggested in some of the analyzed sources to introduce this theme in teaching.

According to Bernstein (2000), the classification and framing values will define the pedagogic practice in the basic communication contexts, particularly in educational contexts. We believe that this analysis reveals the potential of the model to guide the planning of educational practices for the acquisition of the recognition and realization rules required to produce instructional texts about the function concept in accordance with the gradation in classification and framing values.

Concluding Remarks

In this study, we constructed a theoretical model of Mathematics for Teaching of the function concept. It uses a descriptive language for the context of textual production, which was developed by using inspiration in concepts of Bernstein's theory (adapted as descriptive), Concept Study (adapted as an analytical tool) and the sources we fore mentioned. The model aims to organize the characteristics in the function concept’s potential realizations, that identify, characterize, outline and structure such diversity in communicating the function concept in teaching. These characteristics can be analyzed in the micro and macro dimensions. The micro dimension is revealed in the summary presented in **Table 10**, where we focus on the textual indicators of the characteristics constituting and distinguishing the specialized forms of communication of each landscape, including their potentials and limitations. The macro dimension is represented in the iconic text of the model in **Figure 1**, which shows the multiple communicational instances of the function concept realizations, which show the diverse ways of realizing the function concept in basic education. In addition, the iconic text of Figure 1 also reflects the possible and different modalities of relations (*bridges*) that can be established between these communicational instances (landscapes) in the pedagogic practice (in MiT), depending on the gradation of the classification and framing principles operating on the communicative rules.

Although studies indicate that the establishment of these *bridges* is not a simple task (Mousoulides and Gagatsis 2004; Doorman et al. 2012), we argue that the macro and micro views of the concept of function evidenced by the model might suggest pathways to build them.

According to Bernstein (2000), the legitimate textual production in a given context depends on dealing with both the recognition and realization rules (Morais and Neves 2007), and that such rules constitute a crucial factor for learning in educational contexts (Afonso and Neves 2000). As result we highly suggest that the constructed model can assist in the curriculum development and material production processes for students and teachers in basic education, by providing a discursive transparency regarding the recognition and realization rules for the communication of the function concept.

The results of the study suggest communicative transparency in terms of systematization, variability, and specificity for teaching function. It might, therefore, give insights and provide support, contributing to the designing of strategies and resources, for example, for teaching the function concept in the school context, the authors of teaching materials, or the professional programs for teachers. Also, the analytical and methodological framework developed to build a theoretical model of MfT of the Function Concept might provide reflections future research investigating this topic.

References

Adler, J. & Davis, Z. (2006). Opening another black box: researching mathematics for teaching mathematics teacher education. *Journal for Research in Mathematics Education 37*, p. 270-296.

Adler, J. & Huillet, D. (2008). The social production of mathematics for teaching. In Sullivan, P.; Wood, T. (eds). *International handbook of mathematics teacher education*: Vol 1. Knowledge and beliefs in mathematics teaching and learning development. Rotterdam, The Netherlands: Sense Publishers, p. 195-222.

Afonso, M. & Neves, I. P. (2000). Influência da prática pedagógica na mudança conceptual em ciências: um estudo sociológico. *Revista Portuguesa de Educação*, Universidade do Minho, Braga, Portugal, vol. 13, núm. 1, p. 247-282.

Alajmi, A. H. (2012). How do elementary textbooks address fractions? A review of mathematics textbooks in the USA, Japan, and Kuwait. *Educational Studies in Mathematics*, V. 79, I. 1, p. 239 – 261.

Andrews, R. (2005). The Place of Systematic Reviews in Education Research. *British Journal of Educational Studies*, V. 53, N. 4, p. 399-416.

Asghary, N.; Shahvarani, A. & Medghalchi, A. R. (2013). Sobre o Processo de Mudança de Professores das Séries Iniciais Relativo ao Desenvolvimento do Pensamento Funcional. *Bolema*, Rio Claro (SP), v. 27, n. 47, p. 1007-1026.

Ayalon, M., Watson, A. & Lerman, S. (2015). Functions represented as linear sequential data: relationships between presentation and student responses. *Educational Studies in Mathematics*, V.90, I. 3, p. 321 – 329.

Ball, D. L. & Bass, H. (2009). With an eye on the mathematical horizon: Knowing mathematics for teaching to learners' mathematical futures. Paper presented at *The 2009 Curtis Center Mathematics and Teaching Conference*. Retrieved online http://www.mathematik.tu-dortmund.de/ieem/cms/media/BzMU/BzMU2009/Beitraege/Hauptvortraege/BALL_Deborah_BASS_Hyman_2009_Horizon.pdf . Accessed 05 May. 2016.

Ball, D. L.; Thames, M. H. & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, p. 389-407.

Barbosa, J. C. (2013). Designing written tasks in the pedagogic recontextualising field: proposing a theoretical model. In: *7th International Mathematics Education and Society Conference*, 2013, Cape Town. Proceedings of the Seventh International mathematics Education and Society Conference. Cape Town: University of Cape Town, 2007. v. 1. p. 213-222.

Barwell, R. (2013). Discursive psychology as an alternative perspective on mathematics teacher Knowledge. *ZDM Mathematics Education*, V. 45, p.595-606.

Beltrão, M. E. P. & Igliori, S. B. C. (2010). Modelagem Matemática e Aplicações: Abordagens Para o Ensino de Funções**.** *Educação Matemática Pesquisa*, v.12, n.1, p.17-42.

Bernstein, B. (2000). *Pedagogy, symbolic control and identity***:** theory, research, critique. New York: Rowman & Littlefield.

Bernstein, B. (2003). *Class, codes and control*: the structuring of pedagogic discourse. New York: Routledge.

Birgin, O. (2012). Investigation of Eighth-Grade Students' Understanding of the Slope of the Linear Function. *Bolema*, v. 26, n. 42A, p. 139-162.

Brasil. (2013a). Ministério da Educação, Secretaria de Educação Básica. *Guia de livros didáticos*: PNLD 2014. Matemática. Ensino Fundamental – Anos finais. Brasília. 104 p.

Brasil. (2013b) Ministério da Saúde. Conselho Nacional de Saúde. *Comissão Nacional de Ética em Pesquisa*. Plataforma Brasil. Retrieved online: http://aplicacao.saude.gov.br/plataformabrasil/login.jsf Accessed 21 Aug. 2015.

Brasil. (2014). Ministério da Educação, Secretaria de Educação Básica. *Guia de livros didáticos*: PNLD 2015. Matemática. Ensino Médio. Brasília. 108 p.

Brito, D. S. & Almeida, L. M. W. (2005). O conceito de função em situações de modelagem matemática. *Zetetiké*, Cempem, Unicamp, v.13 , n. 23, p. 63 – 86.

Callejo, M. L. & Zapatera, A. (2014). Flexibilidad en la Resolución de Problemas de Identificación de Patrones Lineales en Estudiantes de Educación Secundaria. *Bolema*. v. 28, n.48, p. 64-88.

Carraher, D. W., Martinez, M. V. & Schliemann, A. D. (2008). Early algebra and mathematical generalization. *ZDM Mathematics Education*, V. 40, p. 3-22.

Cause, L. (2010). Bernstein's Code Theory and the educational Researcher. *Asian Social Science*, vol. 6, N. 5, p. 3-9.

Chapman, O. (2013). Investigating teachers' knowledge for teaching mathematics. *Journal for Research in Mathematics Education*, 16, p. 237-243.

Chapman, O. (2015). Understanding and supporting mathematics teachers'knowledge for Teaching. *Journal of Mathematics Teacher Education*. V. 18, I. 2, p. 101-103.

Confrey, J. & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics*, V. 26, I. 2-3, p. 135-164.

Cooney, T. J. et al. (2013). *Developing Essential Understanding of Functions*: Grades 9 – 12. Essential understanding series. The National Council of Teachers of Mathematics.

Davis, B. & Renert, M. (2009). Mathematisc-for-Teaching as shared dynamic participation. *For the Learning of Mathematics*. Vol. 29, N. 3, p. 37-43.

Davis, B. & Renert, M. (2013). Profound understanding of emergent mathematics: broadening the construct of teachers'disciplinary knowledge. *Educational Studies in Mathematics*, v. 82, , p. 245 – 265.

Davis, B. & Renert, M. (2014). *The Math Teachers Know*: Profund Understanding of Emergent Matematics. Routledge Taylor & Francis Group. 141 p.

Dazzi, G. J. & Dullius, M. M. (2013). Ensino de Funções Polinomiais de Grau Maior que Dois Através da Análise de seus Gráficos, com Auxílio do *Software* Graphmatica. *Bolema*. v. 27, n. 46, p. 381-398.

Doorman, M. et al. (2012). Tool use and development of the function concept: from repeated the function concept: from repeated caculations to functional thinking. *International Journal of Science and Mathematics Education*. V. 10, I. 6, p. 1243-1267.

Dubinksy, E. & Wilson, E. (2013). High school students' understanding of the function concept. *Journal of Mathematical Behavior*, 32, p. 83-101.

Elia, I. et al. (2006). Relations between secondary pupils' conceptions about functions and problem solving in different representations. *International Journal of Science and Mathematics Education*. p. 317-333.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics*, V. 21, p. 521-544.

Even, R. & Ball, D. (Eds.). (2009). The professional education and development of teachers of mathematics - *The 15th ICMI Study*. New York, NY: Springer.

Falcade, R.; Laborde, C. & Mariotti, M. A. (2007). Approaching functions: Cabri tools as instruments of semiotic mediation**.** *Educational Studies in Mathematics*, V. 66, I. 3, p. 317-333.

Ferreira, S., Morais, A. & Neves, I (2010). Concepção de Currículos de Ciências: análise dos princípios

ideológicos e pedagógicos dos autores. *Educação* & *Realidade*. 35(1), p. 283-309.Frant, B. J. (2003). As equações e conceito de função. *Boletim GEPEM*, n. 42, p. 71 – 80.

Gagatsis, A. & Shiakalli, M. (2004). Ability to translate from one representation of the concept of function to another and mathematical problem solving. *Educational Psychology*, 24(5), 645–657.

Gil, A. C. (2002). *Como elaborar projetos de pesquisa*. 4. ed. - São Paulo. Atlas.

Granville, M. A. (2008). O discurso pedagógico dos livros didáticos da década de sessenta: reflexos ou reproduções das "políticas públicas de educação" da época? *1ª JIED – Jornada Internacional de Estudos do Discurso*. Maringá – PR. Retrieved online <http://www.dle.uem.br/jied/trab3.html>, Accessed 17 June 2015.

Güçler, B. (2016). Making implicit metalevel rules of the discourse on function explicit topics of reflection in the classroom to foster student learning. *Educational Studies in Mathematics*. V. 91, p. 375 -393.

Hansson, O. (2006). *Studying the views of preservice teachers on the concept of function*. Doctoral Thesis. Luleå University of Technology, Suécia.

Hitt, F; González-Martín, A. S. (2015). Covariation between variables in a modelling process: The ACODESA (collaborative learning, scientific debate and self-reflection) method. *Educational Studies in Mathematics*. V. 88, I. 3, p. 201-219.

Hodgen, J. (2011). Knowing and Identity: A Situated Theory of Mathematics Knowledge in Teaching. In Rowland, T.; Ruthven, K. (Eds). *Mathematics Knowledge in Teaching*. Mathematics Education Library. Springer. V. 50, p. 27-42.

Imenes, L. M. & Lellis, M. (2010a). *Matemática* – Imenes & Lelis, 6º ano. Editora Moderna. São Paulo.

Imenes, L. M. & Lellis, M. (2010b). *Matemática* – Imenes & Lelis, 7º ano. Editora Moderna. São Paulo.

Imenes, L. M. & Lellis, M. (2010c). *Matemática* – Imenes & Lelis, 8º ano. Editora Moderna. São Paulo.

Imenes, L. M. & Lellis, M. (2010c). *Matemática* – Imenes & Lelis, 9º ano. Editora Moderna. São Paulo.

Kazima, M. & Pillay, V.; Adler, J. (2008). Mathematics for teaching: observations from two case studies**.**

Kleiner, I. (1993). Functions: Historical and Pedagogic Aspects. *Science & Education*. Vol 2, p. 183-209.

Mavrikis, M. et al. (2012). Sowing the seeds of algebraic generalization: designing epistemic affordances for an intelligent microworld. *Journal of Computer Assisted Learning*, Wiley.

Maciel, P. R. C. & Cardoso, T. F. L. (2014) A História do Conceito de Função em Vídeo: uma proposta para a aprendizagem. *Bolema*. Rio Claro (SP), v. 28, n. 50, p. 1348-1367.

Maggio, D. P. & Nehring, C. M. (2012) Saberes docentes acerca das representações semióticas do conceito de função: Atuais desafios à educação matemática, *Boletim Gepem*, n. 61, p. 95-108.

Meneghetti, R. C. G. & Redling, J. P. (2012). Tarefas Alternativas para o Ensino e a Aprendizagem de Funções: análise de uma intervenção no Ensino Médio. *Bolema*. Rio Claro (SP), v. 26, n. 42A, p. 193-229.

Mesa, V. (2004). Characterizing Practices Associated with Functions in Middle School Textbooks: An Empirical Approach. *Educational Studies in Mathematics*, V.56, I. 2-3, p. 255-286.

Moschkovich, J. (2004). Appropriating mathematical practices: a case study of learning to use and explore functions through Interaction with a tutor. *Educational Studies in Mathematics*, V. 55, I: 2-3, p. 49-80.

Morais, A. M. & Neves, I. P. (2001). Pedagogic social contexts: Studies for a sociology of learning. *In* A. Morais, A.; Neves, I; Davies, B. & Daniels, H. (Eds.). *Towards a sociology of pedagogy*: The contribuition of Basil Bernstein to research. New York: Peter Lang. Cap. 8, p. 185-221.

Morais, A. M. & Neves, I. P. (2007). A Teoria de Basil Bernstein- Alguns aspectos fundamentais. *Revista Práxis Educativa*, 2 (2), p.115-130.

Morais, A. M. & Neves, I. P. (2011). Educational texts and contexts that work: Discussing the optimization of a model of pedagogic practice. In Frandji, D. & Vitale, P. (Eds.). *Knowledge, pedagogy & society*: International perspectives on Basil Bernstein's sociology of education. London: Routledge. Cap. 12.

Mousoulides, N. & Gagatsis, A. (2004). Algebraic and Geometric approach in function problem solving.

Proceedings of the 28th Conference of the International, Group for the Psychology of Mathematics Education, Vol 3, p. 385–392.

Nachlieli, T. & Tabach, M. (2012). Growing mathematical objects in the classroom-the case of function. *International Journal of Educational Research*, 51/52, p. 10-27.

Nicol, C. C. & Crespo, S. M. (2006). Learning to teach with Mathematics textbooks: How preservice teachers interpret and use curriculum materials. *Educational Studies in Mathematics*, V.62, I. 3, p. 331-355.

Nyikahadzoyi, M. R. (2015). Teachers' Knowledge of the Concept of a Function: A theoretical framework. *International Journal of Science and Mathematics Education*. V. 13, I. 2 – Supplement, p. 261-283.

Panaoura, A., Michael-Chrysanthou, P., Gagatsis, A., Elia, I & Philippou, A. (2017). A Structural Model Related to the Understanding of the Concept of Function: Definition and Problem Solving. *International Journal of Science and Mathematics Education*. V. 15, p. 723-740.

Paiva, M. (2013a). *Matemática*: Paiva. Ensino Médio. Editora Moderna. Vol 1. 2ª edição. São Paulo.

Paiva, M. (2013b). *Matemática*: Paiva. Ensino Médio. Editora Moderna. Vol 2. 2ª edição. São Paulo.

Paiva, M. (2013c). *Matemática*: Paiva. Ensino Médio. Editora Moderna. Vol 3. 2ª edição. São Paulo.

Petrou, M. & Goulding, M. (2011). Conceptualising Teachers' Mathematical Knowledge in Teaching. The Cultural Dimension of Teachers' Mathematical Knowledge. In Rowland, T.; Ruthven, K. (Eds). *Mathematics Knowledge in Teaching*. Mathematics Education Library. Springer. V. 50, p. 9 -26.

Petticrew, M. & Roberts, H. (2006). *Systematic Reviews in the Social Sciences:* A Practical Guide. Oxford: Blackwell.

Prediger, S. (2010). How to develop mathematics-for-teaching and for understanding: the case of meanings of the equal sign. *Journal Mathematics Teacher Education*, N. 13, p. 73-93.

Rhoads, K. & Weber, K. (2016). Exemplary high school mathematics teacher's reflection on teaching: A situated cognition perspective on content knowledge. *International Journal of Education*, V. 78, p. 1-12.

Ronda, E. (2015). Growth points in linking representations of function: a research-based framework. *Educational Studies in Mathematics*, n. 90, p. 303-319.

Rossini, R. (2007). Evolução das organizações matemáticas e didáticas construídas em torno do conceito de função em uma formação de professores. *Educação Matemática Pesquisa*. São Paulo, v. 9, n. 2, p. 205-247.

Sánchez, V. & Llinares, S. (2003). Four student teachers' pedagogical reasoning on functions. *Journal of Mathematics Teacher Education*, v. 6, I. 1, p. 5-25.

Santos, G. L. D. & Barbosa, J. C. (2016). Um modelo teórico de matemática para o ensino do conceito de

função a partir de um estudo com professores. *Unión* (San Cristobal de La Laguna), v. 48, p. 143-167.

Santos, G. L. D. & Barbosa, J. C. (2017). Um modelo teórico de matemática para o ensino do conceito de

função a partir de realizações em livros didáticos. *Educação Matemática Pesquisa*, v. 19, p. 315-338.

Santos, G. L. D. & Barbosa, J. C. (2019). Um modelo teórico de Matemática para o Ensino do Conceito

de Função a partir Artigos científicos. *Boletim GEPEM*, v. 74, p. 127-143.

Sajka, M. (2003). A secondary school student's understanding of the Concept of function – a case study. *Educational Studies in Mathematics*, V. 53, p. 229-254.

Schwarz, B. & Dreyfus, T. (1995). New actions upon old objects: A new ontological perspective on functions. *Educational Studies in Mathematics*, V. 29, I-3, p. 259-291.

Shield, M. & Dole, S. (2013). Assessing the potential of mathematics textbooks to promote deep learning. *Educational Studies in Mathematics*, V. 82, I. 3, p.183-199.

Shulman, L. (1987). Knowledge and Teaching: Foundations of the new reform. *Harvard Educational Review*, 57(1), p.1-22.

Sierpinska, A. (1992). On understanding the notion of function. In E. Dubinsky & G. Harel (Eds.), The concept of function. *Aspects of epistemology and pedagogy*. United States: The Mathematical Association of America, p. 25-58.

Silva, A.; Coelho, A.; Machado, B.; Rubinstein, C.; Kindel, S.; Batisra, K.; Marques, M.; Bluner, N.; Alves, N. & Feigel, Z. L. et al. (2001). Pesquisando, discutindo, pensando e produzindo material sobre funções. *Boletim GEPEM,* n. 38, p. 55 -72.

Slavit, D. (1997). An alternate route to the reification of function. *Educational Studies in Mathematics*, V.33, I. 3, p. 259-281.

Speer, N. M.; King, K. D. & Howell, H. (2015). Definitions of mathematical knowledge for teaching: using these constructs in research on secondary and college mathematics teachers. *Journal of Mathematics Teacher Education*. V. 18, I. 2, p. 105-122.

Steele, M.; Hillen, A. F. & Smith, M. S. (2013). Developing mathematical knowledge for teaching in a methods course: the case of function. *Journal of Mathematics Teacher Education*, v. 16, I. 6, p. 451-483.

Strapason, L. P. R. & Bisognin, E. (2013). Jogos Pedagógicos para o Ensino de Funções no Primeiro Ano do Ensino Médio. *Bolema,* Rio Claro (SP), v. 27, n. 46, p. 579-595.

Stylianides, A. J. & Delaney, S. (2011). The Cultural Dimension of Teachers' Mathematical Knowledge. In Rowland, T. & Ruthven, K. (Eds). *Mathematics Knowledge in Teaching*. Mathematics Education Library. Springer. V. 50, p. 179 -191.

Tabach, M. & Nachlieli, T. (2015). Classroom engagement towards using definitionsfor developing mathematical objects: the case of function. *Educational Studies in Mathematics*, V.90, p. 163-187.

Trindade, D. A. & Santos, I. B. (2012). Critérios apontados por professores de matemática aracajuanos para seleção do livro didático. In: *VI Colóquio Internacional Educação e Contemporaneidade*, São Cristóvão- Sergipe-Br. Anais VI Educon.

Victor, L. (2008). Systematic reviewing. In: *Social Research Update*. Retrieved online: http://sru.soc.surrey.ac.uk/SRU54.pdf. Accessed 04 Aug. 2014.

Vieira, C. M. (2013). Professores dos anos iniciais do ensino fundamental e livros didáticos de matemática. *Tese de Doutorado*. Programa de Pós-graduação da Faculdade de Educação (FaE) da Universidade Federal de Minas Gerais (UFMG). Minas Gerais-Br.

White, T. (2009). Encrypted objects and decryption processes: Problem-solving with functions in a learning environment based on cryptography. *Educational Studies in Mathematics*, V. 72, I-1, p. 17-37.

Wilkie, K. J. (2014) Upper primary school teachers' mathematical knowledge for teaching functional thinking in algebra. *Journal of Mathematics Teacher Education*, n. 17, p. 397-428.

Wilkie, K. J. (2016). Students' use of variables and multiple representations in generalizing functional relationships prior to secondary school. *Educational Studies in Mathematics*, V.93, p. 333-361.

Yerushalmy, M. (2000). Problem solving strategies and mathematical Resources: a longitudinal view on problem solving in a function based approach to algebra. *Educational Studies in Mathematics*, V. 43, I- 2, p. 125-147.