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REGISTERS OF SEMIOTIC REPRESENTATIONS AIDING THE LEARNING OF COMBINATORIAL SITUATIONS

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Abstract: In order to analyze advances in the resolution of combinatorial situations, due to the identification, conversion and treatment of semiotic registers, two studies were carried out. In the first study, 5th grade students identified, from problems in natural language, registers in trees of possibilities, lists and numerical expressions. The second study, carried out with 5th, 7th and 9th grade students, was configured as an intervention study in which trees or lists were used as an intermediate representation of the departure register (natural language) to the arrival register (numerical expression). The results of the studies confirmed the hypothesis that the conversion to numerical expression is more complex than the conversion to trees or lists. It was also confirmed that trees are more congruent, than lists, with registers in numerical expression. It is concluded that the use of intermediate representations, such as trees or systematic lists, is a good teaching strategy for advances in the combinatorial reasoning of students in the early and middle years of schooling.

Keywords: Identification, Conversion, Treatment, Combinatorial situations, Intermediate representations.

Introduction

In the context of Mathematics Education, the importance of studying Combinatorics by students in the early and final years of Elementary School has been widely discussed and

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recommended. This has been a recommendation for some time in different countries, including Brazil, the country in which the present study was developed.

The early years working group of the National Council of Teachers of Mathematics (Working Group (K-4)) of the *Commission on Standards for School Mathematics* (NCTM, 1986) had already highlighted Combinatorics as an area of exploration within two of its themes for curriculum development. These themes were "Ways to build models of representations" and "Ways of counting / computation." On the current NCTM page, on the Data Analysis and Probability axis, the content is recommended using organized lists and tree diagrams for the survey of possibilities in simple probabilistic events.

In Brazil, according to the National Curriculum Parameters [PCN], a document officially in force until 2018 and still very present in school contexts, this content must be introduced at this level of education with the purpose of discussing "combinations, arrangements, permutations and, especially , the multiplicative principle of counting" (Brasil, 1997, p.40), through different types of representations. In the final years of Elementary School, it is expected that the discussion of this content will be expanded, so that the use of double-entry tables and tree diagrams (also known as trees of possibilities) favors the perception of a multiplicative calculation for solving problems involving combinatorial reasoning. (Brasil, 1998).

The National Common Curricular Base [BNCC] (Brasil, 2018), the current document that regulates essential learning in Brazilian schools of Early Childhood, Elementary and Secondary Education, also indicates work with counting problems from the early years of schooling. Learning is recommended through situations in which students become familiar with combinatorial reasoning, indicating work through personal registers, trees of possibilities and tables. In the final years of Elementary School, BNCC indicates that these problems must already be addressed through the Multiplicative Counting Principle. This principle, also known as the Fundamental Principle of Counting [4] [FPC], is a way of solving combinatorial situations

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[4] This principle is enunciated, according to Lima, Carvalho, Wagner and Morgado (2006, p. 125), as, "If a decision D1 can be made in p ways and, whatever this choice is, the decision D2

and the basis for the formulas used in the Combinatorial study, as it expresses the multiplicative nature of the different types of combinatorial problems (Lima, 2015, p .22).

As indicated in the aforementioned curriculum documents, symbolic representations play a very important role in mathematical learning, particularly in Combinatorics. Mathematical Education theorists have highlighted the influence of semiotic representations, among them Raymond Duval – who discusses the *identification*, *conversion* and *treatment* of semiotic registers – and Gérard Vergnaud – who highlights the triad (*situations*, *invariants* and *symbolic representations*) in mathematics conceptualization. Central issues of these theories are dealt with in the section that follows.

Thus, the present research is based on the Theory of Registers of Semiotic Representation (Duval, 2009) and the Theory of Conceptual Fields (Vergnaud, 1986), as well as authors who address both theories and the teaching of Combinatorics in elementary education. More specifically, the objective of this study was to analyze the role that the *identification* and transformations of *conversion* and *treatment* of representation registers play in the expansion of the knowledge of Combinatorics by Elementary School students.

For this, it is necessary to discuss the different *combinatorial situations* and their *invariants*, important elements of the Theory of Conceptual Fields, which will be discussed in more detail in the next section.

In the investigation, two studies were carried out. The first study is a survey of knowledge about the *identification of conversions* – from natural language to tree of possibilities or to list and from these to numerical expression – in different combinatorial situations (*arrangements, combinations, permutations* and *products of measures*). To this end, a test was applied to 5th grade students in which they were asked to identify which tree of possibilities and which list was the correct one in solving the different proposed situations. Then, students should identify the correct numerical expression to answer each problem. The second study was characterized by an intervention research in which the combinatorial situations were worked through

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can be made in q modes, then the number of ways to make the decisions D1 and D2 consecutively is equal to pxq". (Lima, 2015, p.24).

transitional auxiliary representations (tree of possibilities and systematic list), which are characterized as an *intermediate representation* between the departure register (natural language) and the arrival register (numerical expression). In this second study, 5th, 7th and 9th grade students took a pre-test, participated in two teaching sessions and, finally, took a post-test.

Next, the theories used to support this research – its methodological procedures and the analysis carried out – will be discussed, as well as previous studies on Combinatorics, the current research method and the main results obtained.

Theories regarding the role of representations in mathematical learning

As previously stated, the present research is based on two theories for its realization, being an innovative aspect of this study to consider them as complementary. Initially, assumptions of the Theory of Conceptual Fields (TCF), developed by Vergnaud (1986), are discussed, followed by the Theory of Registers of Semiotic Representation (TRSR), developed by Duval (2009). The complementarity of these two theories is highlighted, since both consider representations as essential in mathematical conceptualization, but they discuss different aspects of representations. For Vergnaud, three sets are essential for conceptualization: *representations*, *situations* and *invariants*. Despite deeply discussing the importance of language for the apprehension and operationalization of concepts, this author does not make a study about representations as semiotic systems, as Duval does with the TRSR. The latter author does not work with any kind of representations, but with those that obey characteristics that define them as registers of semiotic representation systems, that is, a register must be identifiable and allow transformation operations, both internal to the same register (*treatment*) as from one system to another (*conversion*), as will be shown in the data of this research.

Thus, in this research, the importance of investigating the *identification* and transformations (*conversion* and *treatment*) of registers of semiotic representations in different *combinatorial situations* is understood, taking into account their respective *invariants*.

Vergnaud (1986) states that the concepts are inserted in conceptual fields. For this author (1986, p.84), "A conceptual field can be defined as a set of situations whose domain requires a variety of concepts, procedures and symbolic representations in close connection". Therefore, for the formation of a concept, inserted in a *conceptual field*, Vergnaud (1986) highlights that a set of *situations* is necessary, which give it psychological meaning; a set of *invariants*, which are logical-operative properties; and a set of *symbols* used in the representation and operationalization of the concept. This theory, therefore, encompasses, in a single theoretical perspective, the development of progressively dominated situations, concepts and theorems necessary for the efficient operation in these situations and the symbols that can effectively represent these concepts and operations.

The study of the analysis of different categories of problems that can be worked with students, as proposed in the Theory of Conceptual Fields, also involves the study of procedures and symbolic representations that students use. Vergnaud (1994) states that it is an essential investigative task to understand why a symbolic representation is useful under certain conditions and when it can be replaced by a more abstract and general one.

The set of symbolic representations includes, among others, natural language, diagrams, graphs and numerical expressions. These are used to represent invariants and situations. In this sense, the teacher should help students develop their repertoires of representations and analyze which are the most appropriate for each situation worked on.

In the Theory of Registers of Semiotic Representation (TRSR), Duval (2011, 2017) emphasizes that the main characteristic of mathematical thinking is that one only has access to mathematical objects through representations. This characteristic of mathematical thinking leads Duval (2011, 2017) to define the following paradox: if access to the mathematical object is only possible through representations and never directly to it, how can one not confuse an object with its representation? This author's answer goes through the importance of working with a variety of semiotic representations, making transformations from semiotic representations into other semiotic representations. Thus, this theory highlights representations and their essential character in mathematics activities. It is through this reflection that Duval (2009) highlights that "there is no *noesis* without *semiosis*" (p.17), because it is not possible to apprehend the meaning (*noesis*) of a mathematical object without the use of a semiotic representation (*semiosis*).

It is emphasized that, for Duval (2012), a system of registers of semiotic representations must satisfy three essential conditions: be *identifiable,* possible to perform *conversion* and *treatment* transformations. Thus, when a register is identifiable it means that the individual is able to *identify* the concept represented in different forms of presentation.

Then, the individual must be able to transform this representation presented in a different representation of the same object, through a *conversion* transformation. Duval emphasizes that "to convert is to transform the representation of an object, a situation or information given in a register into a representation of that same object, that same situation or the same information in another register". (Duval, 2009, p.59). In addition, the individual also needs to make a transformation of *treatment*, a transformation internal to the register itself, in which an initial data of a representation is transformed within that same representation to obtain a terminal data. Thus, Duval (2009) exemplifies that "Calculation is an internal treatment to the registration of a symbolic writing of figures and letters [...]." (Duval, 2009, p.57).

On *conversion*, Duval deepens his discussion and concludes that the success of this task depends on the levels of congruence between the two representations used in the transformation process. For this, Duval lists three essential criteria to assess the level of congruence between two representations:

> The first is the *possibility of a 'semantic' correspondence* of the *significant elements*: to each simple significant unit of one of the representations, an elementary significant unit can be associated. [...] The second criterion is the *terminal 'semantic' univocity*: each elementary significant unit of the departure representation corresponds to a single elementary significant unit in the arrival representation register. [...] The third criterion is related to the organization of the significant units. The respective organizations of the significant units of two compared representations lead to apprehend in them the units in semantic correspondence in the same order in the two representations. (Duval, 2009, p.68-69).

Thus, when two representations meet the three criteria it means that they are congruent and the “success rate” in the conversion between these two representations is higher (DUVAL, 2009, p.19). When the two representations in question do not meet, or meet only one or two criteria, it means that the conversion will be influenced by the level of congruence, so that when they are totally non-congruent, that is, when they do not meet any criteria, one is less successful in converting one form of representation to another.

In addition to the congruence between representations, another important aspect to be considered is the role of auxiliary representations in the conversion activity, especially when the departure and arrival representations are strongly non-congruent. In these cases, Duval highlights the importance of a transitional representation that will assist in the conversion process between two representations.

With the example of addictive situations, Duval (2017, p. 93-94) highlights:

> It is necessary to resort to an auxiliary representation to understand the resolution of all additive problems, mainly those with non-congruent statements. [...] Such auxiliary representation is of course a transitional representation. The students abandon it as soon as they understand because its use seems to them a slow and costly procedure.

Thus, situations in which there is a conversion transformation between two strongly non-congruent representations, refer to the need to use an auxiliary representation, that is, when the departure representation and the arrival representation are not congruent, an auxiliary representation – more congruent, both with the departure representation and with the arrival representation – favors the understanding of the situation. When the formation of the concept is consolidated, this auxiliary representation can be abandoned, as, in general, it is a more detailed procedure, which demands more time, while the arrival representation is configured as a more objective, less costly procedure. In this way, this auxiliary representation also materializes as a transitional one, that is, temporary – while it is necessary. There are also cases in which these representations are always necessary for the resolution of the proposed activity.

Invariants and symbolic representations of combinatorial situations

Combinatorial situations, with their *invariants* of choice, ordering and exhaustion of possibilities, can be *represented* in several ways. Borba (2010) argues that the four combinatorial situations should be worked on at different levels of schooling, highlighting their invariants, that is, the logical-operative characteristics of these situations, as well as the analysis of the exhaustion of all possibilities. Thus, since the beginning of schooling, not only situations of *product of measures* can be discussed, but also those of *combination*, *arrangement* and *permutation*. In these situations, the *invariants of choice* and *ordering* are highlighted, to later discuss the *exhaustion* of possibilities. Thus, in a *product of measures*, there are two or more groups in which an element from each group is chosen to form different possibilities in which their ordering does not generate new possibilities. In the other situations, there is only one group and the choice depends on whether, or not, to select some elements for the formation of possibilities. In a *combination*, some elements are selected and the order of these elements does not generate new possibilities, unlike an *arrangement* situation in which the ordering generates new possibilities. In situations of *permutation*, all elements are used and the order generates new possibilities (Borba, 2010).

In **Figure 1** it is possible to see examples of the four *combinatorial situations*.

Product of measures:
Jane has four blouses (yellow, pink, orange and red) and two skirts (black and white). How many different ways will she be able to dress using one of her blouses and one of her skirts?

Combination:
Five people (Beatriz, Daniel, Joana, Carlos and Marina) shook hands. How many handshakes between different people were given?

Arrangement:
In how many possible ways can you write numbers with two different digits, using the five digits 1, 3, 5, 7 and 9?

Permutation:
How many different ways can three people (Maria, Luís and Carlos) position themselves in a row at the bank?

Figure 1: Examples of distinct combinatorial situations. Source: Adapted from Montenegro, 2018

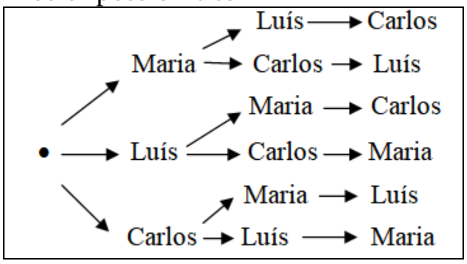
In general, the starting representation of a combinatorial situation is a problem expressed in natural language. This problem, therefore, can be *converted* into a numerical expression. Such a numerical expression can be *treated* internally, so that it is possible to arrive at a solution that indicates the total number of possibilities. However, between the departure register (enunciated in natural language) and the arrival register (numerical expression), different auxiliary or intermediate representations can be used, such as lists, trees of possibilities and tables (Pessoa &

Borba, 2009; Azevedo, 2013). These intermediary representations are more detailed, so that they indicate all the possibilities, which goes beyond indicating just how many are those possibilities, being, therefore, slower and more costly representations. However, similarly to what was suggested by Duval (2011, 2017) regarding other concepts, when students understand, in combinatorial situations, the relation between these intermediate representations and the numerical representation, they can leave aside the most costly representations and start using more economical representations.

Combinatorial situations are characterized by non-congruence in the conversion between the natural language register of the statement and the formal mathematical register of its resolution, since there is no semantic correspondence between the units of meaning of the departure and arrival representations. Thus, a transitional auxiliary representation is essential. In **Figure 2** it is possible to observe an example of a combinatorial situation of *permutation* in natural language, being solved by a list and a tree of possibilities as intermediate representations and their corresponding numerical expression.

How many different ways can three people (Maria, Luís and Carlos) position themselves in a row at the bank? Which numerical expression. solves this problem?

Tree of possibilities



Systematic list

Maria, Luís e Carlos Maria, Carlos e Luís	Luís, Maria e Carlos Luís, Carlos e Maria	Carlos, Maria e Luís Carlos, Luís e Maria

Numerical expression or Fundamental Principle of Counting (FPC)
3 x 2 x 1 = 6 possibilities

Multiplication: generalization of possibilities:
3 x 2 = 6 possibilities
(Each person being the first has two possibilities. Being three people, there are three times two).

Figure 2: Permutation situation solved by tree of possibilities, list and Fundamental Principle of Counting (FPC). Source: Adapted from Montenegro, 2018

With this, it is expected that systematized lists and trees of possibilities are configured as transitional auxiliary representations to the arrival of a formal mathematical register, in the case of the present study, the use of the Fundamental Principle of Counting. Thus, in this study, the role of lists and of the trees of possibilities will be analyzed as auxiliary representations from natural language to numerical expressions in combinatorial situations.

Previous studies on Combinatorics based on the Theory of Conceptual Fields and on the Theory of Registers of Semiotic Representation

Borba (2010) indicates that Combinatorics requires a type of reasoning that stimulates the hypothetical deductive thinking of students, that is, thinking about possibilities and not just what actually happened[5], which can be encouraged since the first years of basic schooling. In this sense, it is very important that students gradually develop their combinatorial reasoning, so that, when they reach High School - time for formal work with this content - they are already more familiar with this type of thinking. On combinatorial reasoning, this author emphasizes that it is

> [...] understood as a way of thinking present in the analysis of situations in which, given certain sets, the elements of them must be grouped, in order to meet specific criteria (of choice and / or ordering of the elements) and determine - directly or indirectly - the total number of possible groupings. (Borba, 2010, p. 3).

Borba (2010) also points out that the different combinatorial situations (*product of measures, combination, arrangement* and *permutation*) must be worked on concurrently, so that the invariants involved in each of these situations are recognized.

Working with only one type of situation (such as the *product of measures* - usually the only situation explicitly worked on in the early years) does not allow for widespread recognition of the properties of the different types of combinatorial problems.

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[5] Inhelder and Piaget (1976, p. 241) emphasize that hypothetical deductive thinking is related to the “dissociation between the possible, the real and the necessary”.

Pessoa and Borba (2009), based on the Theory of Conceptual Fields (Vergnaud, 1991), carried out a survey with students from the 2nd grade of Elementary School to the 3rd grade of High School, in which they were asked to solve various combinatorial problems.

Through this research, the authors highlight that combinatorial reasoning is a type of thinking that is developed over a long period of time, and it is necessary to think about teaching strategies for each level of schooling. The children of early years showed signs of combinatorial reasoning, which gradually increased and was demonstrated in the performance of students in the final years of Elementary and High School. It was observed that performance is influenced by the order of magnitude of the numbers involved and also by the form of symbolic representation used in solving situations, among other factors.

Barreto and Borba (2010) analyzed, in the light of the Theory of Conceptual Fields – TCF, Mathematics textbooks from early years of schooling and concluded that combinatorial situations are present in different parts of the books, not only in the sections focused on working with multiplicative situations. There are activities of *arrangements*, *combinations*, *permutations* and *products of mesaures*, although only the latter type of problem is explicitly highlighted. In general, there is no information in the teacher's manual about the singular character of combinatorial thinking, nor about the invariants of each type of combinatorial situation, or about the different forms of representations that can be used for their development.

Borba, Montenegro and Bittar (2019) analyzed textbooks from the early years of schooling with respect to transformations of representations in solving combinatorial problems. It was observed that all the problems proposed in the books involved at least one conversion – in general, converting natural language and illustrative drawings into numerical expressions (multiplications that solved the problems). Other conversions observed in the analyzed books were from natural language and drawings to lists, to trees of possibilities or to tables – these conversions being more frequently requested in *product of measures* problems, but not common in other types of problems. The authors recommend that in order for conversions to be aids in students' cognitive development, they need to be requested more widely in different combinatorial situations.

Azerêdo (2013), based on the Theory of Registers of Semiotic Representation (TRSR), developed research in the early years of schooling, in which she argued that the semiotic representations of the multiplication operation constitute instruments of pedagogical mediation in the process of teaching and learning this subject. Specifically about the combinatorial situations, the author investigated the students' performance and the evaluation given by the class teacher in a problem of *product of measures*. The author points out that the students' difficulty in this type of situation was more evident when compared to the other multiplicative situations. In addition, it was a surprise, especially for the teachers of the classes who, despite the question presenting the representation of an input and output map, this map was little used for the resolution. The author also indicates that, for the use of different representations to influence the correct answers, it is necessary for the teacher to use them as a mediation instrument, assigning meaning to them, being the semiotic representation registers produced by children potentially effective instruments for this mediation.

Alves (2010), also based on the TRSR, after analyzing textbooks from the final years of Elementary School, developed an intervention project with a 9th grade class. In four modules, the author proposed solving activities in pairs, with the students having two lessons to respond to situations, and in the third lesson there was socialization of ideas and debate. The author found that, as the students were introduced to the different representation registers, they were able to better understand the different possibilities in Combinatorics calculations, as well as to discern about the importance, or not, of the order of the elements.

Based on the results of these and other previous studies, in the present study, it is intended, based on the two theories presented, to analyze the role that the *identification* and transformation of *treatment* and *conversion* of registers have in expanding Elementary School students’ knowledge of different *combinatorial situations*.

Method adopted in the two studies of the present research

From the study of the two theories proposed in this research, as well as the research already carried out on Combinatorics involving these theories, the first study of this research was elaborated, in which it was proposed that 5th grade students of Elementary School identify, in combinatorial situations , natural language conversions to lists or to trees of possibilities and from these to numerical expressions. In this direction, two types of test were applied to 16 students from a private school in Recife, Brazil.

In the tests, the same eight problems were proposed, two of each type of combinatorial situation, in which the students needed to identify between two alternatives which list or tree of possibilities responded correctly to the situation presented in natural language and, then, which of four alternatives corresponded to the numerical expression that would correctly answer the situation. The difference between the two types of tests was in the *combination* situations in which, one type of test had the repeated cases excluded and in the other type of test the repeated cases were not exposed, as it is possible to observe in **Figure 3**. After the students' solving of the problems, the data were analyzed according to errors, successes and justifications, with a primarily qualitative focus.

Márcia has four types of fruit at home (papaya, pineapple, orange and banana) and wants to make a salad using three of these fruits. How many different ways can she combine these fruits?

Disregarding the repeated cases:
Papaya, pineapple and orange
Papaya, pineapple and banana
Papaya orange and banana
Pineapple, orange and banana

Considering the repeated cases crossed out:
Papaya, pineapple and orange
Papaya, pineapple and banana
Papaya, orange and banana
~~Papaya, banana and pineapple~~
~~Papaya, banana and orange~~
~~Papaya, orange and pineapple~~

~~Orange, papaya and pineapple~~
~~Orange, pineapple and papaya~~
~~Orange, papaya and banana~~
~~Orange, banana and papaya~~
~~Orange, pineapple and banana~~
~~Orange, banana and pineapple~~

Pineapple, orange and banana
~~Pineapple, banana and orange~~
~~Pineapple, papaya and orange~~
~~Pineapple, orange and papaya~~
~~Pineapple, banana and papaya~~
~~Pineapple, papaya and banana~~

~~Banana, papaya and pineapple~~
~~Banana, pineapple and papaya~~
~~Banana, papaya and orange~~
~~Banana, orange and papaya~~
~~Banana, pineapple and orange~~
~~Banana, orange and pineapple~~

Figure 3: Situation of combination with the resolution through list, disregarding, or not, the repeated cases. Source: Adapted from Montenegro, 2018

From the results of this first study, it was possible to elaborate an intervention proposal for the second study of this research. Thus, different forms of intervention were carried out, with 121 students from the 5th, 7th and 9th grades of Elementary School, who took into account the TRSR, pointing out the conversions of representations as a strategy for the development of combinatorial reasoning, as well as the TCF, on the different combinatorial situations and their invariants. Thus, the study was carried out with two classes from each grade, with one class characterized by the group that worked with trees of possibilities as an intermediate representation from natural language to numerical expression and the second group used systematic lists as an auxiliary representation between the representation of departure (enunciated in natural language) and the representation of arrival (numerical expression).

The two classes of each school grade carried out the three stages of the research, in which in the first stage they answered a pre-test with eight combinatorial situations (two of each type) in which the number of possibilities was between 4 and 24. In the second stage they participated in two intervention sessions, each of one hour approximately, answering the pre-test questions using trees of possibilities or lists and the Fundamental Principle of Counting (numerical expression). In the third stage the students answered a post-test with eight other combinatorial situations where the number of possibilities varied; thus, in the first four problems the number of possibilities was between 6 and 30, in the last four problems the number of possibilities was between 56 and 120. With this, it was expected to analyze the use of the FPC in favorable situations, since the the use of a list and a tree of possibilities would not be sufficient to answer situations with a high number of possibilities.

The analysis of the results of the second study was carried out quantitatively using the SPSS (Statistical Package for the Social Sciences) software, comparing the performance of the three grades before and after the intervention, comparing each grades' performance in the pre and the post-test, and also comparing the three school grades amongst each other. Qualitative analysis of representations and strategies used by students before and after the intervention was also carried out.

For the first study, there were two hypotheses: the first related to the more difficult character of the *combination* problems, since to identify the correct numerical expression, as well as to justify it, it would be necessary to understand the need to disregard the repeated cases – once that the order of the elements in combinations does not indicate different possibilities. The second hypothesis indicated that identifying trees and lists would be easier than identifying numerical expressions, due to the greater congruence between natural language and lists or trees than between these registers and numerical expressions.

The first hypothesis for the second study was that both methods of intervention – using trees or lists as intermediate representations – would be effective in expanding combinatorial reasoning. The second hypothesis was that there would be greater progress in the group that used trees of possibilities, mainly due to the perception of the multiplicative reasoning implicit in combinatorial situations. This is because this representation seems to indicate with greater clarity the one-to-many relations involved in combinatorial situations, since the organization in branches that indicate this multiplicative idea (**Figure 2**) is apparently more congruent with the mathematical operation necessary to solve problems in Combinatorics.

Results of Study 1: identification of conversions in combinatorial situations

In the first study, it was probed how 5th grade students identify conversions made in registers of different semiotic representations (natural language, tree of possibilities, list, numerical expression). The purpose was to analyze whether and how these students coordinated different representations of the same combinatorial situation. As there were 16 students and two problems of each type, the maximum score for each type of conversion was 32 and adding the two tests together, 64 was the maximum score.

In **Table 1**, it is possible to observe that, for the first conversion – from natural language (NL) to list (L) or to tree (T) – there are around 50% of correct identifications (NL – L: 36/64 and NL – T: 33/64), with a higher performance in Test 2 (with repeated cases crossed out). As for the second conversion, when the objective was to identify which numerical expression responds the situation, there are 25% of correct identifications (NL – L: 16/64 and NL – T: 16/64), also with higher performance in Test 2.

These results reinforce the greater difficulty of students in identifying numerical expressions in different combinatorial situations, than in identifying corresponding lists or trees of possibilities. They also seem to indicate greater congruence of natural language with the tree of possibilities and with the list and less congruence with the numerical expression. This is because the semantic units of the statement (in natural language) are in correspondence with the semantic units of the tree and the list, but not with the units of the numerical expression.

Test type	**Situation type**	**Conversion 1**		**Conversion 2**		**Total**
		NL → L	NL → T	L → NE	T → NE	
1 (Test without repeated cases)	PM	3	4	0	2	9
	C	4	1	1	0	6
	A	2	6	0	0	8
	P	5	3	1	1	10
Total Test 1		**14/32**	**14/32**	**2/32**	**3/32**	**33/128**
2 (Test with repeated crossed out cases)	PM	4	3	4	3	14
	C	5	5	3	2	15
	A	6	4	2	4	16
	P	7	7	5	4	23
Total Test 2		**22/32**	**19/32**	**14/32**	**13/32**	**68/128**
Total (Test 1 + Test 2)		**36/64**	**33/64**	**16/64**	**16/64**	**101/258**

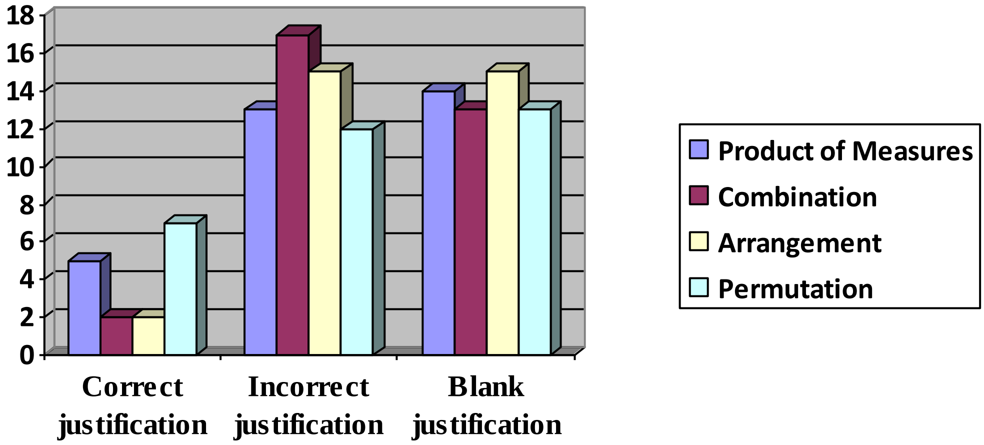
Table 1: Correct identification in each conversion by type of test and situation

NL: Natural language; L: List; T: Tree of possibilities; NE: Numerical Expressions; PM: Product of Measures; C: Combination; A: Arrangement; P: Permutation. Source: Research authors

According to Duval (2009), the identification, by the subject, of a mathematical object in different registers of semiotic representation, indicates conceptual apprehension of that object. Each register, be it in natural language, diagrams or mathematical symbols, is configured as "[...] systems of representation that are very different from each other and that each pose specific learning issues" (p. 38). Thus, it seems to us that students already indicated some apprehension of combinatorial situations, but they still needed learning focused on the use of numerical expressions that can be associated with each type of situation.

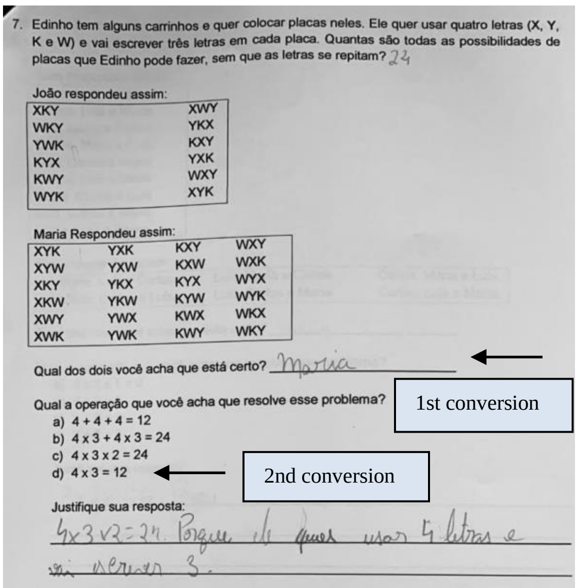
Regarding the types of test, it appears that, although the only difference between them, in presenting the problems, is in the *combination* situations, the second type of test showed better results, with little more than twice the correct answers (33 correct identifications in Test 1 and 68 in Test 2). To this fact, it is inferred that this difference in the *combination* problems may have resulted in a different analysis from the other types of problems, that is, having the repeated cases crossed out may have called attention (triggering a theorem-in-action, as called by Vergnuad) about when the ordering indicates, or not, different possibilities in the other types of situations, and this led students to think about other situations in which the order of the elements indicates different possibilities. From what the results indicate, there was in the test in which the repeated cases were crossed out a better performance in all combinatorial situations of the test, with emphasis on the correct identifications, including in the numerical expressions, because, of the 16 correct answers (in the total sets of the two tests), 14 (for conversion from list) and 13 (for conversion from tree) were in the second type of test.

In **Graph 1**, it can be seen that the number of correct justifications is smaller, when compared to the number of incorrect justifications and blank justifications, especially if the last two are added as both justifications that do not meet what was requested. In addition, it is clear that the situations of *combination* and *arrangement* have even less correct justifications. In *combination* situations, it is understood that the difficulty in finding a correct justification is due to the fact that repeated cases must be disconsidered, since in this type of situation, the order of the elements does not indicate different possibilities. In *arrangement* situations, students should consider ordering, justifying that the same elements in different orders constitute different possibilities, as well as that some elements will be selected, differently from the *permutations* in which all elements are used.



Graph 1: Quantitative of the type of response (correct, incorrect and blank justifications) according to each type of situation. Source: Research authors

In **Figure 4**, an *arrangement* situation can be observed in which the identifications of the presented conversions are correct, however, the justification presented is not consistent with the numerical expression, because when the student writes "Because he wants to use 4 letters and will write 3", does not explain the correct multiplication: 4 x 3 x 2. This multiplication indicates that for the choice of the 1st letter there are four possibilities, for the 2nd letter there are three possibilities and for the 3rd letter there are two possibilities.



7. Edinho tem alguns carrinhos e quer colocar placas neles. Ele quer usar quatro letras (X, Y, K e W) e vai escrever três letras em cada placa. Quantas são todas as possibilidades de placas que Edinho pode fazer, sem que as letras se repitam? 24

João respondeu assim:

XKY	XWY
WKY	YKX
YWK	KXY
KYX	YXK
KWY	WXY
WYK	XYK

Maria Respondeu assim:

XYK	YXK	KXY	WXY
XYW	YXW	KXW	WXK
XKY	YKX	KYX	WYX
XKW	YKW	KYW	WYK
XWY	YWX	KWX	WKX
XWK	YWK	KWY	WKY

Qual dos dois você acha que está certo? Maria

1st conversion

Qual a operação que você acha que resolve esse problema?

a) 4 + 4 + 4 = 12
b) 4 x 3 + 4 x 3 = 24
c) 4 x 3 x 2 = 24
d) 4 x 3 = 12

2nd conversion

Justifique sua resposta:

4x3x2=24. Porque ele quer usar 4 letras e vai escrever 3.

Figure 4: *Arrangement* situation in which the identification of conversions (from natural language to list and from this to numerical expression) are correct, but incorrect justification is given by Student 16. Source: The authors

Figure 5 shows the example of a correct answer in both the first and second conversions, as well as with a consistent justification. In this example, in which the child responds that it is "4 classes x 3 second places x 2 third places", it is understood that the child realized that any of the four classes can occupy the first place, leaving three classes for the second place and two classes for the third place, being necessary to make a multiplication between the factors. This adequate justification is a correct application of the Fundamental Principle of Counting – content not yet addressed in the classroom in the 5th grade of Elementary School, but an assumption (called *theorem-in-action* by Vergnaud) that the total number of possibilities can be obtained by multiplying the number of possibilities for each stage.

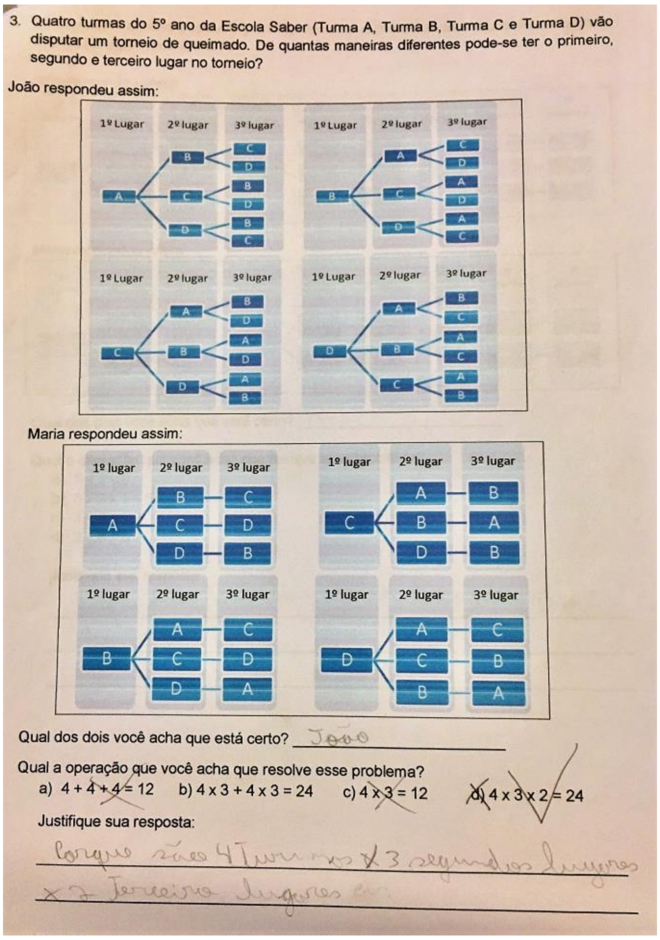
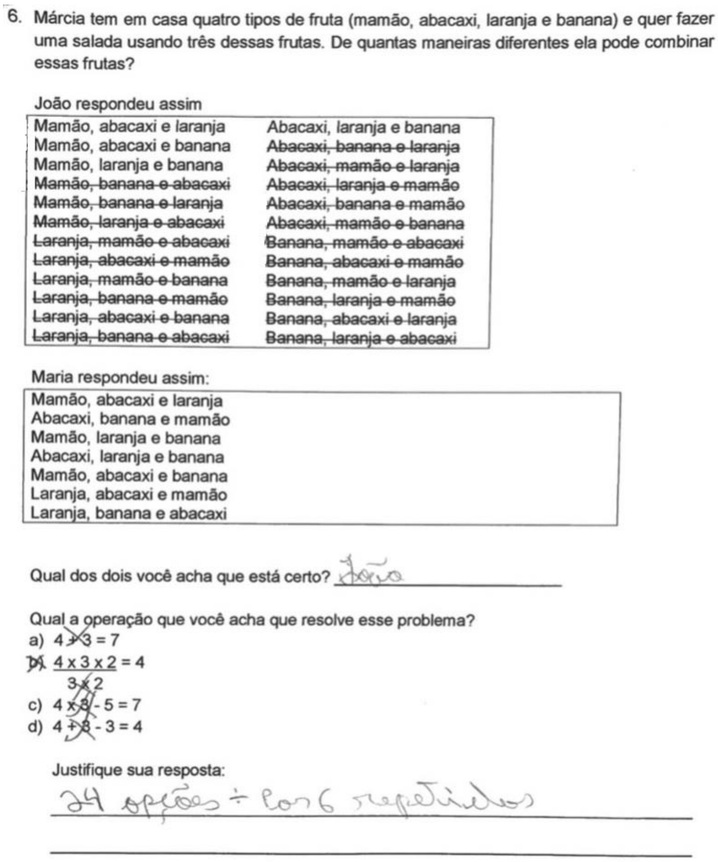


Figure 5: Arrangement situation answered correctly (with solution presented in a tree of possibilities) and consistent justification by Student 2. Source: Montenegro, 2018

Regarding the problems of *combination*, since the tests were different in presenting the resolution of this combinatorial situation, it is highlighted that, in the first conversion, from natural language to list or tree, students presented approximately half of the correct answers (15/32), indicating that, for the first conversion, the level of difficulty was lower. As for the second conversion, from list or tree to numerical expression, the number of correct answers decreased a large amount (6/32). When the justification for the marked numerical expression is analyzed, it is perceived that students had even more difficulty, as they presented only one correct justification for this type of situation. In this correct justification, the student points the division by six due to the repetition of the possibilities, as shown in **Figure 6**.


6. Márcia tem em casa quatro tipos de fruta (mamão, abacaxi, laranja e banana) e quer fazer uma salada usando três dessas frutas. De quantas maneiras diferentes ela pode combinar essas frutas?

João respondeu assim

Mamão, abacaxi e laranja	Abacaxi, laranja e banana
Mamão, abacaxi e banana	~~Abacaxi, banana e laranja~~
Mamão, laranja e banana	~~Abacaxi, mamão e laranja~~
~~Mamão, banana e abacaxi~~	~~Abacaxi, laranja e mamão~~
~~Mamão, banana e laranja~~	~~Abacaxi, banana e mamão~~
~~Mamão, laranja e abacaxi~~	~~Abacaxi, mamão e banana~~
~~Laranja, mamão e abacaxi~~	~~Banana, mamão e abacaxi~~
~~Laranja, abacaxi e mamão~~	~~Banana, abacaxi e mamão~~
~~Laranja, mamão e banana~~	~~Banana, mamão e laranja~~
~~Laranja, banana e mamão~~	~~Banana, laranja e mamão~~
~~Laranja, abacaxi e banana~~	~~Banana, abacaxi e laranja~~
~~Laranja, banana e abacaxi~~	~~Banana, laranja e abacaxi~~

Maria respondeu assim:

Mamão, abacaxi e laranja
Abacaxi, banana e mamão
Mamão, laranja e banana
Abacaxi, laranja e banana
Mamão, abacaxi e banana
Laranja, abacaxi e mamão
Laranja, banana e abacaxi

Qual dos dois você acha que está certo? João

Qual a operação que você acha que resolve esse problema?
a) 4 + 3 = 7
b) $\frac{4 \times 3 \times 2}{3 \times 2} = 4$
c) 4 x 3 - 5 = 7
d) 4 + 3 - 3 = 4

Justifique sua resposta:
24 opções ÷ por 6 repetidos

Figure 6: *Combination* situation, with the solution in a list, answered correctly by Student 2 and with a consistent justification. Source: Montenegro, 2018

Taking into account the results of this first study, it is understood that the higher success rate of the first conversion may indicate a greater congruence between natural language and lists or tree representations. The lower success rate of the second conversion suggests less congruence between lists or trees and numerical expressions. The results also point out that it is necessary to consider the particularities of these conversions in the different combinatorial situations. In addition, the better performance in the second type of test indicates that making explicit the exclusion of repeated cases in a *combination* situation seems to be a good way to discuss the invariants of this and other combinatorial situations – *arrangements*, *permutations* and *products of measures* –, a since it can draw attention to the ordering of elements, generating or not different possibilities.

In view of the results already discussed, as well as the difficulty of 5th grade students in justifying their answers, for the second study, the need for an intervention study was considered. For the teaching of students, it was decided to use a tree of possibilities and systematic list as intermediate representations, since these representations can assist in the conversion of natural language to numerical expression. It was also decided to make explicit the exclusion of repeated cases from *combinations* to enable a greater discussion of the invariants of this and other combinatorial situations. In order to verify the feasibility of the intervention in different school grades, it was decided to carry it out with students from the 5th, 7th and 9th grades of Elementary School.

Results of Study 2: trees of possibilities and systematic lists as intermediate auxiliary representations in combinatorial situations

The second study was characterized by an experimental intervention research, in which a pre-test, two teaching sessions and a post-test were carried out with two classes of each of three school grades: 5th, 7th and 9th grade. The first class of each grade participated in the intervention using trees of possibilities as an intermediate representation (G1). The second class of each grade used systematic lists as an intermediate representation (G2).

During the intervention sessions, students were encouraged to discuss the combinatorial situations solved in the pre-test, taking into account the specific invariants, that is, the appropriate choice to solve each type of situation, as well as whether the order of choice generated different possibilities, and, finally, they were asked if there was any other possibility not yet considered in the answer given. What differentiated one intervention group from another was the intermediate representations used. In **Figure 7**, two examples of problem solving during the intervention can be seen with the example of a *combination* problem, using a tree of possibilities (G1) and a systematic list (G2).

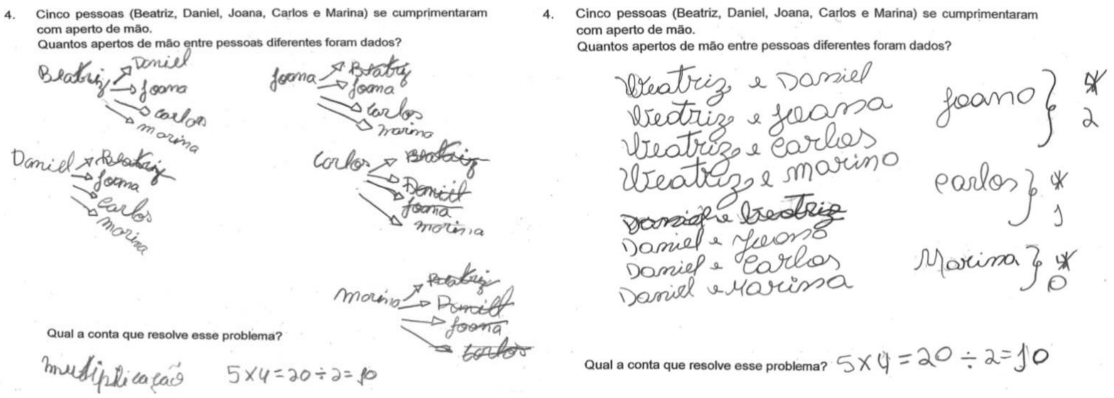


Figure 7: First intervention with Groups 1 and 2 of the 5th grade. Solving *combination* situations through auxiliary representations: tree of possibilities and systematic list.

Source: Montenegro, 2018

The tests were analyzed taking into account the students' performance in the survey of possibilities, as well as presenting a mathematical operation for their solution. In this direction, there were two questions in each test item. The first related to what were all the possibilities and the second concerning which operation (numeric expression) solved the problem.

In these two perspectives of analysis, were considered:

• 0 point for *error* - when the student answered incorrectly the survey of possibilities or the numerical expression of the situation;

• 1 point for *partial correctness 1* - when the student indicated less than half of the possibilities or indicated a correct mathematical operation, but the operation was wrong;

• 2 points for *partial correctness 2* - when the student indicated half or more of the number of possibilities or indicated a correct mathematical operation, but solved the operation incorrectly;

• 3 points for *total correctness* - when the student exhausted all possibilities and correctly indicated the operation with its correct resolution.

Table 2 shows the average performance of students from different school grades, both in the survey of possibilities and in the numerical expression that answers the problem, before and after the intervention sessions. It is noteworthy that there was a significant increase in the average performance in both intervention groups of all the school grades involved. Considering that the test had eight problems and each one could reach a maximum of 3 points, the total test score could be 24 points.

Grades	**Groups**	**Survey of possibilities**		**Numerical expression**	
		Pre-teste	**Post-teste**	**Pre-teste**	**Post-teste**
5th Grade	**G1 (Tree)**	1,89	6,11	0,31	4,57
	G2 (List)	2,85	5,15	1,20	3,30
7th Grade	**G1 (Tree)**	1,38	6,76	0,57	4,76
	G2 (List)	1,77	6,23	0,46	3,46
9th Grade	**G1 (Tree)**	6,74	9,52	3,68	7,89
	G2 (List)	6,25	8,43	2,56	5,93

Table 2: Average performance in the pre-test and in the post-test by grade and by intervention group, in which the total score could be 24 points.

Source: Research authors

Comparing the pre-test result separately with the post-test result of each group and each grade, significant differences were observed, both for the survey of possibilities ($p < 0.001$[6]) and for numerical expression ($p < 0.001$) that answers the problems. Thus, it appears that the interventions had very significant effects in both intervention groups and in all grades of schooling studied.

To analyze the difference between the intervention groups, the parametric T-test of independent samples was performed, comparing the G1 post-test (intermediate representation: tree of possibilities) with the G2 post-test (intermediate representation: list), in each grade. There were no significant differences between the intervention groups in any grade[7]. In this way, both the tree of possibilities and the list proved to be valid auxiliary representations that helped in the development of the combinatorial reasoning of the participants.

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[6] In statistical language 'p' indicates whether something is likely to be true and not the result of a random situation. In the statistic stating that a result is highly significant, it means that the hypothesis being tested is most likely true. In general, when $p < 0.05$ it is assumed that there is a probability of only 5% that the difference found is not true. Thus, the lower the p-value, the less likely it is that the difference is not true.

[7] 5th grade G1 x G2: survey of possibilities ($t(37) = 0.576$; $p = 0.568$), numerical expression ($t(37) = 0923$; $p = 0.362$); 7th grade G1 x G2: survey of possibilities ($t(45) = 0.440$; $p = 0.662$), numerical expression ($t(45) = 0.166$; $p = 0.300$); 9th grade G1 x G2: survey of possibilities ($t(33) = 0.650$; $p = 0.520$), numerical expression ($t(33) = 1.341$; $p = 0.189$).

Examining in more detail, questions with a greater number of possibilities were analyzed separately, in which a numerical calculation would be recommended, since the indication of all possibilities would be very costly. In this sense, a T-test of independent samples was performed to compare the performance of G1 (tree) and G2 (list) on these test items. The results indicated a significant difference between the groups in the comparison made with the students of the three grades ($t(119) = 3.162$; $p = 0.002$). Group 1, which had an intervention using the tree of possibilities, performed better in situations where the use of a numerical expression was recommended, indicating that this intermediate representation seems to have a greater degree of congruence with the numerical expression necessary to solve combinatorial problems, as students produced their numerical expressions (application of the Fundamental Principle of Counting) more easily from the trees of constructed possibilities.

An analysis by type of combinatorial situation was also carried out. **Table 3** shows the progress in the comparison between pre-test and post-test for each situation, in each grade and each experimental group. In only one case, in the 9th grade G2 arrangement situation, there was no evolution between pre and post-test. It should be noted that, for each of the averages, the maximum to be obtained was six points. Although some advances seem small, they are surprising, in particular considering the reduced teaching time (two sessions of an hour each, as highlighted above). Thus, it is noteworthy the low performance in the pre-test in all types of problems and that, with only two intervention sessions, the results were better in the post-test. It is to be expected that with more intervention sessions the results can be even better.

The best results were in *product of measures* situations, followed by *permutations*. In general, minor advances were observed in *combination* situations – as also observed in previous studies (Pessoa & Borba, 2009) – in which the ordering of elements does not imply different possibilities.

It is noteworthy that, for the 5th grade of G1, the difference between the pre-test and post-test averages was significant in the situation of *product of measures* ($p = 0.003$) and *arrangement* ($p = 0.043$); in the 5th grade G2, only in the *permutation* situation ($p = 0.046$). In the G1 of the 7th grade, the difference was significant in the situations of *product of measures* (p <0.001) and *permutation* (p <0.001); in the 7th grade G2, the difference was significant in all types of problems (PM: p = 0.002) (C: p = 0.007) (A: p = 0.017) (P: p = 0.001). For the 9th grade, the difference was significant only in the *product of measures* in G2 (p = 0.025). Thus, it is indicated that advances were significant in different types of combinatorial situations, depending on the experimental group and the school grade.

	PM Pre	PM Post	C Pre	C Post	A Pre	A Post	P Pre	P Post
5th Grade G1 (Tree)	0,57	2,73	0,42	0,89	0,63	1,42	0,26	1,05
5 th Grade G2 (List)	1,50	2,30	0,20	0,60	0,80	1,10	0,35	1,15
7th Grade G1 (Tree)	0,57	2,76	0,38	0,71	0,28	0,95	0,14	2,33
7 th Grade G2 (List)	0,92	2,23	0,26	1,23	0,19	0,96	0,38	1,80
9 th Grade G1 (Tree)	3,52	4,26	0,89	1,89	1,0	1,63	1,31	1,73
9 th Grade G2 (List)	2,56	4,31	1,62	1,81	1,06	0,68	1,00	1,81

Table 3: Average performance in the survey of possibilities by type of problem, with 6 possible points in each problem. PM: Product of Measures; C: Combination; A: Arrangement; P: Permutation. Source: Research authors

In **Table 4**, it is possible to observe the performance averages in the indication of numerical expressions for the situations presented. With the exception of the *combination* situation in the 5th grade, in all other cases there was an advance in the averages.

In *combination* in the 5th grade, no student was able to indicate the corresponding numerical expression, even after the intervention, which corroborates the difficulty that students at this level have in operating this type of situation. In spite of this, it is noticed the progress of 5th grade students in the indication of numerical expressions in the other combinatorial situations, mainly because, when the statistical analysis was performed, G1 showed a significant difference between pre-test and post-test in all other types of combinatorial situations (PM: p =

0.003; C: p = 0.014; A: p = 0.044), and G2 showed a significant difference in the *permutation* situation (p = 0.045). The 7th grade G1 showed significant differences in the situation of *product of measures* (p = 0.002), *arrangement* (p = 0.021) and *permutation* (p = 0.009); G2 presented in situations of *product of measures* (p = 0.001) and *permutation* (p = 0.007). The 9th grade G1 showed a significant difference in the situations of *product measures* (p = 0.018), *combination* (p = 0.010) and *arrangement* (p = 0.037); G2 only in *product of measures* (p = 0.007). As in the survey of possibilities, advances in the indication of numerical expressions were significant in different types of combinatorial situations, for different experimental groups and different school grades.

	PM Pre	PM Post	C Pre	C Post	A Pre	A Post	P Pre	P Post
5th Grade G1 (Tree)	0,31	2,57	0,00	0,00	0,00	1,26	0,00	0,73
5 th Grade G2 (List)	1,05	2,05	0,00	0,00	0,15	0,70	0,00	0,55
7 th Grade G1 (Tree)	0,57	2,33	0,00	0,42	0,00	0,71	0,00	1,28
7 th Grade G2 (List)	0,46	1,80	0,00	0,23	0,00	0,53	0,00	0,88
9 th Grade G1 (Tree)	2,68	4,26	0,00	0,94	0,42	1,47	0,57	1,21
9 th Grade G2 (List)	1,81	3,87	0,00	0,18	0,37	0,68	0,37	1,18

Table 4: Average performance in indicating numerical expressions by type of problem, with 6 possible points in each problem. PM: Product of Measures; C: Combination; A: Arrangement; P: Permutation. Source: Research authors

The results presented indicate that both representations - trees of possibilities and lists - are effective in advancing performance to survey possibilities and the use of these representations can favor the development of combinatorial reasoning. It is noteworthy that the results show that the tree of possibilities seems to have a higher level of congruence with the numerical expression, when compared to the list, even if systematic. This is because in situations where the use of the numerical expression was recommended, with a high number of possibilities, G1 (tree) showed a significant difference in relation to G2 (list). In addition, in the analysis by type of combinatorial situation, it was also noticed that G1 presented better results in the use of numerical expressions.

Regarding the use of intermediate representations, in the pre-test some students already realized that the list was a good strategy for resolving situations, as can be seen in **Figures 8 and 9**. In **Figure 8** the 5th grade student performed a systematic list and then indicated a multiplication operation, thus presenting an intermediate representation in the list and the arrival representation, a mathematical operation. The list has also been observed as a spontaneous representation used by students from grades prior to the 5th grade in previous studies (Pessoa & Borba, 2009).

Figure 9 shows how a 9th grade student used the systematic list in a simplified way, so that he listed the six possibilities for words starting with the letter 'A' and then performed a multiplication by means of a generalization of possibilities, since if for the letter 'A' there are six possibilities, the other letters will also have six possibilities, being possible then, to multiply the number of possibilities per letter by the total number of letters.

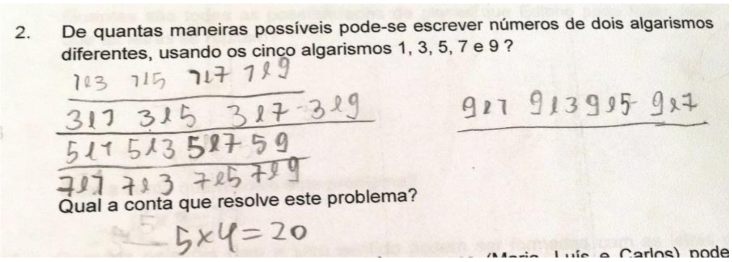


Figure 8: *Arrangement* situation with correct answer through the list of possibilities, with indication of the numerical expression that answers the problem, performed by a 5th grade student in the pre-test. Source: Montenegro, 2018

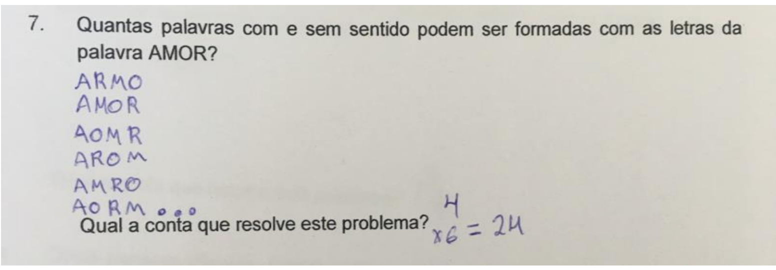


Figure 9: *Permutation* situation with correct answer by generalizing the possibilities, performed by a 9th grade student in the pre-test. Source: Montenegro, 2018

In the post-test, in addition to the list as an intermediate representation, there was also the use of the tree of possibilities by the group that used this representation in the intervention sessions, as shown in **Figures 10 and 11**. In **Figure 10** it is understood that the 5th grader started with a tree of possibilities and realized that the use of PFC would be sufficient to arrive at the desired answer.

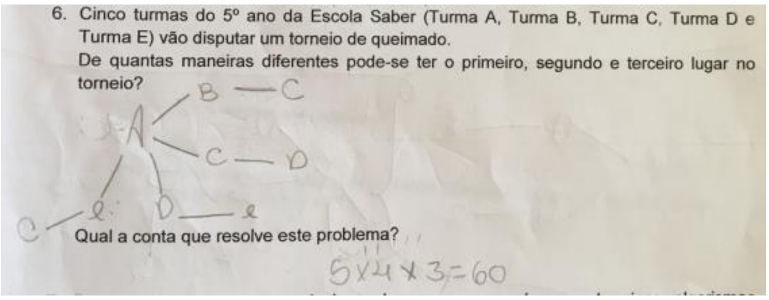


Figure 10: *Arrangement* situation with correct answer through the Fundamental Principle of Counting, performed by a 5th grade student in the post-test. Source: Montenegro, 2018

In **Figure 11**, the student also represented a tree of possibilities, but did not finish this representation, realizing that the use of the FCP is configured as a less expensive method. It is understood that the student also performed the count of repeated cases to divide, since in *combination* situations, the total number of cases must be divided by the number of repeated cases. In this example, it is clear that the student made explicit the treatment given to the numerical expression, that is, he presented the calculations that were performed to arrive at the response of the situation.

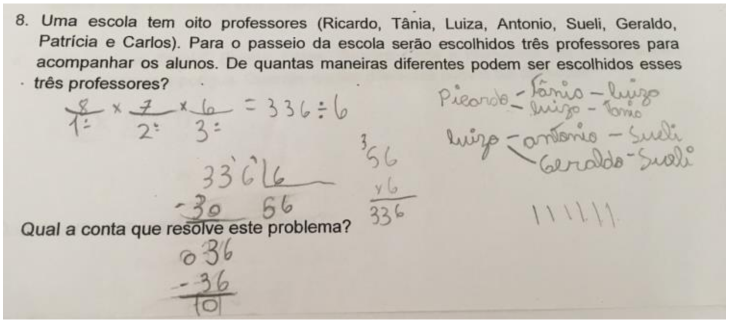
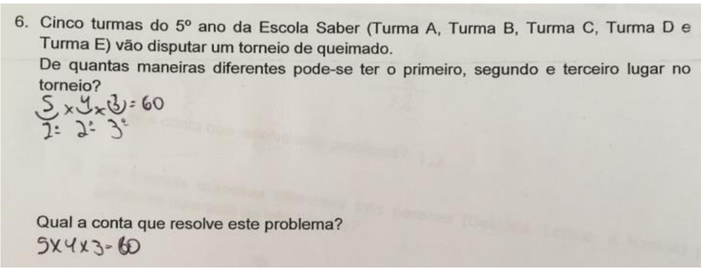


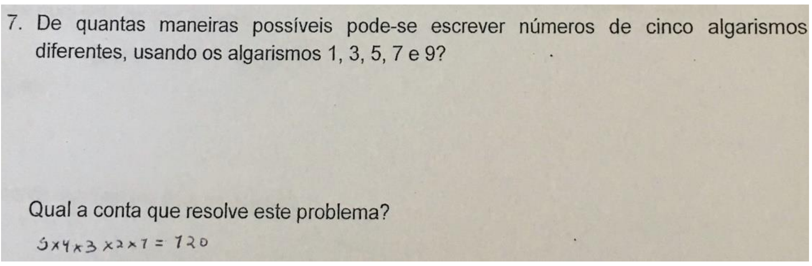
Figure 11: Combination situation with correct answer through the tree of possibilities and the Fundamental Principle of Counting, performed by a 7th grade student in the post-test. Source: Montenegro, 2018

In other cases, the students chose not to use an intermediate representation, possibly because, for that situation, its use was no longer necessary, being used the Fundamental Principle of Counting directly. This is because, on some occasions, the same student, when solving problems with fewer possibilities, used a list or tree as an intermediate representation, but when solving problems with a greater number of possibilities, he solved it directly with the FPC. **Figures 12 and 13** show examples in which students did not use an intermediate representation.


6. Cinco turmas do 5º ano da Escola Saber (Turma A, Turma B, Turma C, Turma D e Turma E) vão disputar um torneio de queimado.
De quantas maneiras diferentes pode-se ter o primeiro, segundo e terceiro lugar no torneio?
5 x 4 x 3 = 60
1º 2º 3º

Qual a conta que resolve este problema?
5x4x3=60

Figure 12: *Arrangement* situation with correct answer using the Fundamental Principle of Counting, performed by a 7th grade student in the post-test. Source: Montenegro, 2018


7. De quantas maneiras possíveis pode-se escrever números de cinco algarismos diferentes, usando os algarismos 1, 3, 5, 7 e 9?

Qual a conta que resolve este problema?
5x4x3x2x1 = 120

Figure 13: *Permutation* situation with correct answer through the PFC, performed by 9th grade student in the post-test. Source: Montenegro, 2018

Final considerations

In this research, the objective was to analyze the role that *identification* and transformation of *conversion* and *treatment* of registers have in advancing the knowledge of various *combinatorial situations.* For this, two studies were carried out.

In the first study, with students from the 5th grade of Elementary School, the hypotheses raised initially were confirmed, highlighting that the type of conversion carried out is relevant, with the conversion of tree of possibilities or list to the numerical expression being more difficult to identify than the conversion of natural language to a tree or list. The identifications are also influenced by the type of *situation* (Vergnaud, 1986), since the students showed greater difficulty with the combinatorial situations of *arrangement* and *combination*, mainly in the identification of the conversion to the numerical expression. This can also be seen in the justifications given for the resolution by numerical expression, since only one correct justification was found for each of these two problems of *combination*, given by the same student.

Depending on the results of the first study, highlighting the difficulty of 5th grade students in identifying conversions to numerical expression, in the second study, different interventions were proposed in 5th grade classes, as well as in 7th and 9th grade classes. Thus, the development of the combinatorial reasoning of children in the last year of the initial years (5th grade: 10-11 year old students) and youngsters in the middle of the final years (7th grade: 12-13 year old students) and in the last year of Elementary School (9th grade: 14-15 year old students) was investigated. The students were divided into two groups, so that in the three years surveyed there were interventions with a group that used trees of possibilities as intermediate representations – G1 – and with a group that used lists – G2 – between the starting register (language natural) and the arrival register (numerical expression).

In this second study, the results indicated that both *intermediate representations* – tree of possibilities and systematic list – are good resources for teaching Combinatorics, confirming the hypothesis that both intervention groups would advance in their combinatorial reasoning, since the two intervention groups advanced in performance, showing significant differences between the average obtained in the pre-test and the average obtained in the post-test. It is also noteworthy that there is additional evidence of advances in combinatorial reasoning, as students presented different analyses for each *combinatorial situation*, as well as, in each problem, thinking about distinct groupings and ordering and about the use of each representation in problems with less or greater number of possibilities. Despite this, when analyzing by type of *situation* in each grade, it is noticed that in the post-test the group that worked with trees (G1) performed better, mainly in the use of numerical expressions, compared to the group that worked with lists (G2). It is also clear that in G1 there was a greater number of correct answers in situations with a greater number of possibilities in the post-test, showing a significant difference with G2, only in those problems with a high number of possibilities. Thus, when the correctness of the problem was directly related to the use of a numerical expression, students who worked with trees of possibilities showed better performance than students who worked in the intervention with systematic lists, as a predicted hypotheses for this study, indicating a greater congruence of the tree of possibilities with numerical expressions.

Thus, it is concluded that it is possible to promote advances in the combinatorial reasoning of Elementary School students through the use of both intermediate representations used in this study, allowing, especially with the use of the tree of possibilities, a better performance in the presentation of expressions corresponding to the resolution of situations.

The discussions carried out show how necessary and important is a discussion articulating the Theory of Conceptual Fields and the Theory of Registers of Semiotic Representation. It was observed that the conversions have different levels of difficulty, depending on the type of register used, and the combinatorial situation treated. Thus, there is a need to analyze representation registers (Duval, 2009) in the light of different situations and their respective invariants (Vergnaud, 1986), since *identification*, *conversion* and *treatment* are important aspects, but it is necessary to consider that they are differentiated according to the *combinatorial situation* treated, be it an *arrangement,* a *combination*, a *permutation* or a *product of measures*.

In the first study, the students' difficulty in identifying the numerical expressions from trees of possibilities or lists, before a specific intervention process, gave space in the second study, after the intervention, to verify that it is possible to expand students’ combinatorial reasoning with a significant advance in the use of numerical expressions, through the use of these intermediate representations.

Thus, it is emphasized that work with different *combinatorial situations*, through the discussion of their *invariants*, and with the use of systematic *auxiliary representations*, in activities involving *identifications*, *conversions* and *treatments* of registers, must be taken into consideration for a more effective teaching and learning of Combinatorics in Elementary School.

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