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The constitution of mathematical knowledge with augmented reality

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Abstract: Digital Technologies are increasingly present in our activities. Many things we do we are not even able to imagine how they would be done, if we did not have the technological resources at hand. However, perhaps in the opposite direction of this, in the school context, or in teaching and learning, the discussion about the potential and the viability of these resources is still subject of a non-consensual discussion. When this context is Higher Education, specifically in undergraduate courses, the situation is even worse, as stated by research that we bring in this text. In disciplines such as Differential and Integral Calculus, Digital Technologies (DT) can contribute to a treatment in which aspects related to research and visualization are explored. Apps such as GeoGebra Augmented Reality, enhance the exploration of function graphs, for example, and, through movement, allow the analysis of invariants, favoring conceptual understanding. As we saw in the context of an activity[2] proposed for students of a Mathematics Degree course, the app allows for interaction between students and enables them to conduct explorations that allow them to assign meaning to the contents of the Calculus discipline. This, therefore, is the theme that we deal with in this article, using a phenomenological stance to expose the meaning of what constitutes knowledge for us with DT.

Keywords: Mathematical Education. Phenomenology. Digital Technologies. Teaching of Calculus.

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[2] The activity to which we refer is one of the actions linked to a project entitled *The constitution of mathematical knowledge with Augmented Reality*, approved and supported by São Paulo Research Foundation (FAPESP), under the number 2019/16799-4, whose objective is to investigate how students understands Differential and Integral Calculus contents when doing research with an Augmented Reality app. The students we are working with are from the undergraduate degree in Mathematics at São Paulo State University (Unesp), School of Engineering, Guaratinguetá, aged 18 to 25 years old.

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Introduction

In the Era of Digital Technology, the advent of the internet demands that schools rethink their role of teaching students how think and study, as stated by Nóvoa (2017) in a lecture organized by the Chapada Institute of Education and Research (ICEP), in Salvador, Bahia. It's senseless now to consider that the school's role should be to transmit information. It is necessary to *innovate*. However, what does it mean to innovate? In an article published on the Brazilian Society of Mathematical Education's (SBEM) Journal, Pereira, Mondini, Paulo and Mocrosky (2018) state that the school *innovates* as long as it stops being a simple transmitter of information and starts teaching the students to *do*, to *live* and to *be*. Building a teaching proposal that aims innovation demands questions related to the constitution of student's knowledge, as well as the meaning of innovation itself.

The work we have been developing in our research group[3] points that technologies provide opportunities to classroom situations where student's engagement is potentialized and their understanding of certain disciplinary contents are favored. However, when it comes to technologies, as highlighted by Lopes, Monteiro and Mill (2014), we may refer to chalk and blackboard, pen and paper, but also resources like software, the internet, smartphone, etc. This makes it so recently the term Digital Technologies (DT) or, more specifically, Digital Information and Communication Technologies (DICT) is used to emphasize technologies related to computers, tablets, digital games, software, among others gadgets from the so called digital universe, which allow for instantaneous communication and search of information.

To be among DICTs lead us to affirm we live in globalized world, where information access is made fast and easy. Digital resources are increasingly more familiar to new generations. However, this familiarity doesn't equal an update to pedagogical practices, that is, using digital resources in day to day situations doesn't imply knowing possibilities of teaching and learning with them. As stated by Paulo,

[3] The group to which we refer he is the FEM - Phenomenology in Mathematical Education group, which, since 2011, has been dedicated to the philosophical study in which it seeks to understand the educational practices available and carried out in cyberspace and through DICTs with projects approved by CNPq. More information about the group's work is available at http://fem.sepq.org.br/.

Firme and Tonéis (2010, p. 19), it is a matter of "technical competence" for use which must be "linked to a critical competence" to further learning, in a school context.

Thereby, considering DICTs for learning requires the constitution of an environment where there are conditions for production of knowledge, which demands a space of investigation where students may explore possibilities, expose conjectures and build arguments; a space of dialogue so that what is constituted within one's subjectivity may be made explicit and put to discussion inaugurating the intersubjective process, in which, through the means communication, the language is articulated, elaborated, refined and comprehended to convey what it was validated by all.

What is intended here is to go in the direction of this discussion, taking into consideration the starting of the lived experience with undergrad students from the Mathematics course of UNESP, in campus of Guaratinguetá. Through the project *The constitution of mathematical knowledge with Augmented Reality*, which is developed by us, we proposed a free course for the Math degree students, in which were explored contents from the Differential and Integral Calculus subject with DICTs or, more specifically, with an Augmented Reality app: the GeoGebra AR. To expose what we comprehended about the investigation done, the students involvement and the AR potentialities for learning, we will, firstly, present what led us to choose the teaching of Differential and Integral Calculus, bringing ways of understanding the treatment given to the content of this subject, native of not only our teaching practices but also emphasized in Mathematical Education researches. We will also shine light on discussing open possibilities to the work with Differential and Integral Calculus curricular content with Augmented Reality apps, as well as expose the stance we took on the research and on the conduction of the tasks proposed to and developed with the students, in a way that is made clear the soil in which we are situated, to cast the eyes we bring in this text.

Differential and Integral Calculus in higher education

Differential and Integral Calculus or, simply, Calculus, as we will start referring to it here, is a subject present in most Brazilian higher education courses in the area of Exact Science due to its importance, and according to Igliori (2009, p. 13), that is because this where “fundamental notions for Advanced Mathematics such as: real numbers, function, infinity, etc” are worked on. But such subject importance is not restricted only to graduation courses in Brazil. In the United States, for instance, Bressoud (2020) highlights that its importance is so much that, in the 1950s, it was created a program by College Board denominated Advanced Placement (AP), whose objective was to promote the study of Calculus contents for high school students considered “mathematically strong”, contributing to their admission in the University, as well as their performance in a higher education course.

However, the author highlights the peculiar nature of the US educational system that has the financing of public high schools linked to the community where they are inserted, making it so the school curriculum and the qualification of its professionals obey only local criteria, which creates a huge inequality in the country. Still, due to the university not acknowledging the Calculus taken on high school and virtually repeating its contents on the first year of graduation courses, many high schools choose not to work with AP, which makes it that among the university students there are those completely familiarized with the subject contents and those who are getting in contact with them for the first time.

Regardless, Bressoud (2020) states that the higher education courses’ own structure, as it allows for students to opt for a specific area by the end of the second year, makes it so the Calculus course is the same for both students who will take it as their only Math course and students for whom Calculus is the base to proceed in advanced math studies. Such peculiarity of Calculus teaching in the US leads Bressoud (2020) to conclude that, as for public, ideological and financing matters, programs turned to high school students preparation (AP) will hardly be extinct, and it is left to the departments of each higher education institution to analyze the specific problems of their reality and promote interventions that may minimize, or even overcome, the inequalities perceived. Thereby, according to Bressoud (2020), what is seen in the Calculus course is the missing in students’ motivation which generates serious learning obstacles.

Bressoud, Ghedamsi, Martinez-Luaces & Törner (2016) exemplify these difficulties by citing the concept of limit of a function which, given its formal epsilon delta approach, leads students to focus on techniques and to make nonsensical combinations, with no understanding of what they are doing.

In Europe, Törner, Potari and Zachariades (2014) highlight that, although research on teaching of Calculus is limited, most focuses on student learning, emphasizing the difficulties they face towards important discipline concepts, as the idea of function, limit, infinity, continuity, derivative and integral. The authors, based on the literature review made, state that, although there is a proposal that the teaching of the discipline is done through the means of exploratory tasks, treating concepts in an intuitive way before systematization, the predominant practice is still the one of formal teaching with the exposition of contents, demonstration of theorems and practicing of exercises. They conclude stating that more research is necessary to expose the general way through which Calculus is taught in different European countries, in comparison to the USA and even other non-European nations.

In the Brazilian context, research such as that of Martins Junior (2015), which focuses on revising the literature on the teaching of calculus, shows that the learning difficulties in the discipline are diverse and stemming mainly from the way in which most teachers deal with content in the classroom, emphasizing demonstrative aspects and solving repetitive exercises, applying rules and memorization. Some causes for students' difficulties, pointed out by the author, are considered ordinary, such as the lack of previous knowledge of those entering higher education, and others have characteristics closely related to methodological aspects, such as the lack of connection between the different areas of mathematical knowledge and the little work that may involve a geometric interpretation of the concepts of Calculus. According to the author, a teaching proposal that allows students to mobilize concepts through activities that involve images and conceptual definitions is pointed out by the research as a possibility for understanding the contents and reducing the difficulties presented.

Other Brazilian researches such as Wrobel, Zeferino and Carneiro's (2013), point out that these difficulties cause high rates of failure and dropout in courses within the Exact Science area. Igliori (2009) understands that they can be comprehended due to the existence of a change in the way in which learning is conceived in higher education. In other words, curricular contents become "teaching objects" – and no longer learning objects – to be presented by the teacher to the student, who must be responsible for their learning, success or failure. The contents are taught, that is, exposed to students, with the rigorous techniques required by the course and it is up to them – the students – to understand and reproduce them correctly.

Corroborating what is highlighted by Igliori (2009), Pagani and Allevato (2014), as they carry out a mapping of theses and dissertations that address the teaching of Calculus in Brazil, they indicate that the main motivation for the work on this theme is precisely the high dropout rates and failure in the discipline, which lead to the search for ways to identify its causes or to propose means to overcome them.

Nascimento (2001) develops a study with students entering courses in the Exact Science area, mainly in engineering, and with the teachers responsible for the discipline of Calculus. It exposes the main beliefs – used as justifications by teachers – about the causes of difficulties: lack of basis for students entering higher education, the methodological differences between high school and higher education and difficulties intrinsic to the discipline. Nascimento's goal (2001) was to identify the correlation between students' performance in Calculus and the causes pointed out for their failure. He highlights that, even students entering the exact science courses, demonstrate great difficulty in dealing with algebra content, related to factorization, remarkable product, simplification, distributive property and exponentiation and with the trigonometric functions, especially with the function argument, leading them to make mistakes when working with limits and derivatives. Their results show that the methodological question and the way the previous concepts required in Calculus are treated in higher education courses are the most important factors and the main cause of the students' bad results and consequent failure and dropout. Based on the study he did with students on a chemical engineering course, he stresses that the work in pairs and study sessions are important to overcome the difficulties and give the teacher the opportunity to detect the errors and misunderstandings that students usually do not have the courage to speak out to the whole class.

The research carried out by Martins Junior (2015) shows that failure in Calculus is not exclusive to students' basic mathematical training, but, above all, it is due to their little familiarity with the predominant mathematical practice in the discipline. The main problem, according to this author, is of an epistemological order, that is, it is related to how this knowledge occurs or can be constituted by the student. This problem, according to Rezende (2003), is not restricted to the Brazilian reality, given that works on this theme have been published and highlighted by international specialized literature, as can be seen, for example, in the Dossier published by ZDM Mathematics Education 46(4) of 2014 or in more recent texts, as in Bressoud's (2020) and Bressoud, Ghedamsi, Martinez-Luaces & Törner's (2016) works, already exposed in this text.

These studies, while not exhausting the topic, provide a panorama that allows us to see the situation of students in the Calculus subject, and this makes us resume the speech of Igliori (2009), about the way in which contents are treated in higher education – as teaching objects, assigning the student the responsibility for their learning. We questioned whether this "transition" from high school (or basic education) to higher education, requiring a more formal way of dealing with mathematical content, could be favored by a teaching approach that, through digital technologies, make the student understanding possible.

Researches such as those of Mamona-Downs (2001), Przenioslo (2005) and Tall (2009), mentioned by Törner, Potari and Zachariades (2014), highlight the importance of a Calculus teaching that values graphic exploration for the development of the visualization skill, which can be the basis for students to understand formal definitions and to overcome difficulties. The explorations, as the authors point out, can be done dynamically through technologies, giving the student the possibility to get involved with the proposed activity, which would allow them to assign meaning to what is done. It is important to understand what is enhanced in the teaching of Calculus with DICTs and our interest turns to an Augmented Reality app.

Understanding Augmented Reality in the teaching of Calculus

Scucuglia (2006), when discussing the potential of technologies for learning the Fundamental Theorem of Calculus, highlights the relevance of the investigative process and the way in which technologies contribute to the development of the *visualization* skill, through explorations that lead the student to inferences, raising conjectures and justifying them. Visualization, according to the author, is relevant for the study of graphs that, as stated by Alonso and Pinto (2017, p. 438), although they are significant to the learning of Calculus, they have been little emphasized in solving problems and, many times, they have not been considered as support for the understanding of a definition or for the study of theorems.

The fact that graphs are little explored in the teaching of Calculus may be related to the low appreciation of the visual ability, which, in turn, stems from a belief that visualization is something trivial and does not need to be taken into consideration as they can be misleading, and makes graphical explorations unnecessary or simply allegorical. However, according to Lemos and Bairral (2010, p. 65), visualization is complex, as it involves aspects such as “interpretation, drawing, forming mental images and visualizing movements and changes in forms”. In this sense, visualization is a process of interpreting that makes information more meaningful. That is, visualization is a process in which the subject's (individual’s) ability to *see* is developed. This view, as we understand it, does not just say about the visual ability inherent to the human being, to look at a graph of a certain function, for example, and identify whether it is a line or a parabola. It is, rather, to establish relationships between formulas, graphs, diagrams and verbal expressions that we use to make understanding possible. Seeing is, therefore, a way of getting involved in an investigative process.

The concept of Augmented Reality (AR) is related to the incorporation of virtual objects into the environment of our physical, concrete reality. In other words, AR technology refers to the reality of the physical world, which becomes a starting point and allows the subject (user) to experience virtual objects in another way, without losing contact with the environment in which they are. With the combination of virtual elements incorporated to elements of the physical world, the user may *feel* as if the real and virtual elements coexist in the same space, giving virtual objects the impression of reality. For the presence of these virtual objects to become possible, it is necessary to use devices compatible with this technology, such as computers, smartphones, tablets, special glasses, etc.

Augmented Reality (AR) apps make it possible to drag, deform, transform, enlarge or reduce images, and highlight ways of seeing, manipulating and experimenting, which can favor the development of the visualization process, starting from movement.

Gouveia (2010) reinforces that exploring visualization is important for the subject of Calculus, as it allows the student to analyze, for example, functions that generate three-dimensional graphs. This analysis requires an understanding of visual patterns, a very complex process, with a high level of abstraction. Orozco, Esteban and Trefftz (2006) present the results of a study in which AR technology was used in tutoring, for students in two classes of Calculus. One of the classes had the face-to-face teaching model and the other the teacher attended to the students remotely (the teacher was not physically present in the laboratory with the students). The app used was developed by Eafit University of Colombia aiming at teaching content in the subjects of Calculus, Physics, Chemistry and Biology. The NetMeeting platform was also used for communication between the teacher and the class in the moment of the experiment, allowing for remote communication.

The goal of the experiment was to identify *how the tool could expand the students' comprehension* of new concepts, since they had not yet been introduced to content involving neither equations with 3 unknowns, nor their three-dimensional representations.

In order to start working with students, functions of the type y = f (x) were considered, such as **y = x**, **y = x²**, **y = ln(x)** and **y =** e^x, these are identified by the authors as the most commonly considered function in Calculus courses and are likely to be difficult for students to understand.

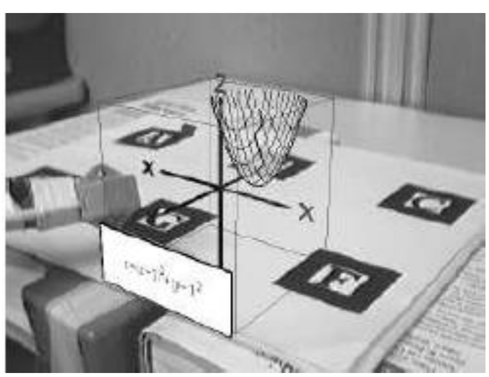

Figure 1: Virtual representation of a paraboloid with an AR app (Orozco, Esteban & Trefftz, 2006).

As a result, the authors point out that, through the explorations made, the students were able to infer generalizations, considering what they observed, and predict what would happen with the graphical representation of other functions. They conclude that the physical presence of the teacher did not cause a significant difference, but the dialogue was essential so that students could move forward with their explorations. The students' involvement in the activities was due to the AR app allow for them to concentrate on the teacher's guidance without losing contact with the virtual object and physical reality, unlike what happens with the Virtual Reality apps in which the user is immersed in the (totally virtual) environment and momentarily loses contact with the physical space. The environment named *micro-world* by the authors, which combines the virtual with the physical reality, made possible the interaction between the students, between them and the teacher, the execution of the explorations and the understanding of Calculus contents.

The participants of Orozco, Esteban and Trefftz's (2006) research, as they answered a questionnaire at the end of the experiment, nodded positively to the use of AR in the activities they developed. Their manifestations showed that, for them, the way they had contact with the content was interesting and described the feeling of having these objects "in their hands", liable to be moved as they pleased. They also showed greater ease in understanding the translation of functions, compared to other forms of representation.

For Quintero *et. al* (2015) studies involving AR in educational environments are a trend in the area of education that is still in its early stages. There are several researches in AR that present software and resources, but few that have possibilities for teaching Calculus, as well as bring a discussion about what differentiates it from other technologies. In this sense, the authors propose reflections that make it possible to understand the ways in which AR can provide learning experiences, especially about spatial visualization, involving the contents of the subject. The team which the authors of this study take part in developed an AR app that gives the user the possibility to explore graphs of real variable functions, some solids of revolution and two real variables functions.

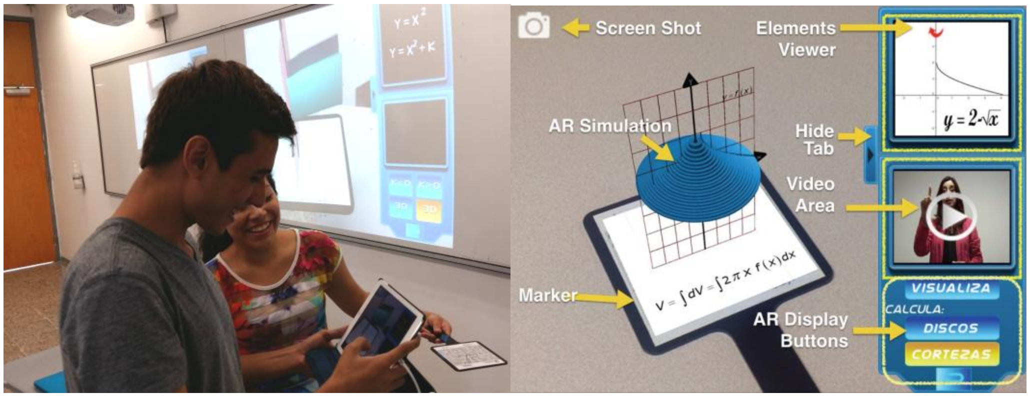

Figure 2: AR app developed by the team (Quintero *et al*, 2015).

The app brings some possibilities of interaction with mathematical functions of the parabolic, circular and sinusoidal type. In figure 2 it is possible to see that, from a marker and a tablet, the user has contact with the virtual object. On the device screen, you can watch a video about the AR simulator, there are buttons for interacting with digital content, you can hide information from the display to have a screen with less information to view the object.

As a result of a pilot study developed with students from an engineering course, the authors point out the importance of software for the development of diverse skills in mathematics and not just to deal with specific contents. Although they resumed their intention to present possibilities of using technologies for learning, the conclusions of this study were few.

Caballero and Duarte (2014) present a study about the use of a mobile platform with the resource of AR as complementary didactic material to the Calculus classes. In this platform, developed to be used through mobile devices, the content is presented in the form of text and images, also counting on the possibility of using AR.

To elaborate this platform, the authors used some devices, such as Sencha Touch and Vuforia for the development of the interface and use of AR, and Blender to create three-dimensional models. The contents presented were previously thought of when analyzing the program of the Calculus subject and they involved topics of limits, derivatives and integrals. The study was carried out with a group of

Calculus students, exploring different activities, and to analyze the potential of this resource the authors made one assessment at the beginning of the activity (diagnosis) and another after working with the AR.

In their conclusions, Caballero and Duarte (2014) show an improvement in the autonomy of students who, after a certain point, were already willing to carry out investigations and sought validation among each other, without resorting to the teacher. They also point out that the use of the platform changes the learning space and time, which are no longer restricted to the classroom as in the case of conventional teaching. They emphasize the improvement in the understanding of the concepts of Calculus by the students demonstrated in the resolution of the exercises and in the exposure of the strategies used. However, despite these findings, the authors are not specific as to which contents were explored, nor do they describe the way in which AR favored their understanding, saying only that the improvement was perceived.

Research shows us that AR apps enable an investigative environment in which the generated objects are manipulated and from the subject's *movement* they come to be seen in different perspectives, favoring the identification of patterns and the search for regularities.

In this work, we emphasize the movement potentialized by AR apps that unlike what we interpret in 3D visualization software, make the object dynamic by subject's movement, the person who manipulates the smartphone or tablet. That is, the movement of the object in the app is not performed due to the user moving the mouse or touching the screen of the device. The user, the person holding the device in their hands, is the one who moves: raising his smartphone by raising his arm, climbing on the chair to be able to see through the smartphone screen the object projected on the table or on the floor, etc. The person moves and, as they move, they also move the object they are seeing through the screen, projected on the environment.

It intrigues us to know the difference between moving to see the movement of the projected object and moving the object with a mouse. Up to this point in our studies, we understand that the difference consists in considering the spatiality of the subject, of the person who moves and coexists with the objects in the environment, giving amplitude to the visual field.

We understand, with Merleau-Ponty (1994), that the visualization favored by the AR app allows me not to have to "fix" my gaze on a certain object, making the spectacle of what is around me disappear. In other words, I, the person who moves my device, experience the vision, adhering to my gaze the virtual object that is projected in the physical environment through the screen of my smartphone, without breaking the link with the world.

This means that visualization, in the case of three-dimensional graphs, for example, allows me to investigate their shape, which is not only given by their geometric contour, but is a shape that has a certain behavior, which is maintained or changed with movement, opening itself up to the understanding of those who question what they see. The object, the curve, the three-dimensional surface that can be seen through the smartphone screen, is visible, that is, it exists (it projects itself) and it moves in correlation with movement. For Merleau-Ponty (1994, p. 309) "the movement of visible objects is not the simple displacement of the color patches that correspond to them in the visual field", because they carry a sense that is perceived. For example, Merleau-Ponty (1994, p. 309) tells us: "In the movement of the branch that a bird has just abandoned, we read its flexibility or its elasticity, and this is how an apple branch and a birch branch immediately distinguish themselves". The movement gives us the possibility to identify the moved. The movement of the subject who holds the smartphone in their hands seeks the best "place" to see what is projected opening up possibilities to investigate it.

In this sense, to investigate the constructed graphs, the surfaces or curves generated by functions, the movement of the person's body – student who moves the smartphone – gives them the possibility to organize themselves and to orient themselves to explore what is seen in each one of the "positions" that they place their smartphone, it gives them the possibility to distinguish characteristic (or particular) aspects of each of the investigated situations making them unique, singular. Here, we dare say, we seek a way of seeing in which the relationships between formulas, graphs, verbal expressions have their origin, begin to establish themselves, and make sense to the student as the difficulties give way to understanding.

The constitution of knowledge in this research

The research we are developing, linked to the FEM (Phenomenology in Mathematical Education) group, and supported by São Paulo Research Foundation (FAPESP), focuses on the way in which Math Degree students understand Calculus content, when they are using an AR app – GeoGebra AR. We emphasize that this app was chosen due to research, even in other areas, highlight the possibility of interacting with virtual objects in the physical environment, without resorting to special equipment.

Although we are focusing on the understanding of the curricular content by the student, we understand that research can bring forward aspects that contribute to a didactic conception, in which it stands out that learning calculus implies knowing aspects of mathematics that, par excellence, is open to investigation. As emphasized by Fischbein (1994, p. 231), this investigation involves different moments such as: "enlightenment, hesitation, acceptance and refutation" and in the context of the classroom, such moments must be valued and encouraged for the constitution of knowledge by the student.

The option of working with students in the Mathematics Degree course is due to the importance of discussing the constitution of knowledge in teacher training, turning to the meaning and relevance of intuition and insight.

Ponte (2001), when highlighting the research practices by the mathematician community, brings the research carried out by Leone Burton. This author interviewed 70 mathematicians to find out the way in which they constitute their knowledge. Among several aspects highlighted, the author emphasizes that most of the mathematicians interviewed value intuition and insights in their investigation process, and regrets that in the context of teaching and learning these are not given due importance. That said,

> The author wonders, as intuition and insight are so important for mathematicians, why do they earn so little attention from teachers, and affirms her conviction in the value of a teaching style that does not encourage students to explore their intuitive reactions to a given situation. (Ponte, 2001, pp. 18-19).

This research by Burton (2001), described by Ponte (2001), as well as that by Ferreira (2019), which also highlights the knowledge production of the mathematician and the importance of visualization in the investigation process, including with computers, make us to see the relevance of our proposal, to consider the potential of investigating situations through an AR app in the context of teaching Calculus. As we understand it, in this way of investigating, intuitive exploration and visualization are valued for understanding the studied content.

It is worth mentioning that the study of AR apps is being developed in several areas of knowledge. Most of them use bookmarks available in specific libraries – like the ARToolkit. These markers are like codes previously generated (usually printed) that when captured by the device's camera (computer, smartphone, tablet, etc.) bring the virtual object in sync with the physical space in which the user is. The GeoGebra AR app does not require the use of markers, as the device itself recognizes flat surfaces in the physical environment and presents them to the user as a possible place to position virtual objects. This is an important aspect for working with Calculus, since the studies by Moussa, Ymai and Camargo (2017), show that apps developed with markers – for teaching Calculus – still present several problems.

In GeoGebra AR, as we enter equations, graphs are generated and projected in 3D in the physical environment, being possible to handle them (rotating, translating, enlarging or reducing them) through the body movement of the person (user) who manipulates (moves) the equipment. When editing the equations, the graphs show the changes in real time, allowing for visual exploration and, from there, an algebraic analysis that favors the articulation of information and leads the student to understand some more abstract concepts of Calculus.

This way of comprehending the relevance of visual information for learning Calculus, bring us to understand that the AR app allows for the possibility of exploring it through movement, through the subject's actions, who moves the object displayed on the screen of their smartphone, moving themselves around the virtual object, which is projected next to other objects in the physical space. This exploration allows students to raise hypotheses and encourage them to seek arguments that validate or refute them, initiating the investigative process and the constitution of knowledge, as we understand it in a phenomenological perspective.

Phenomenology as a stance taken in the research

Phenomenology wants to /.../ elucidate possibilities, possibilities of knowledge /.../ from its essential foundation /.../ in the sphere of origins.
(Husserl, 1990, p. 79)

Merleau-Ponty (1994, p. 1) states that "phenomenology is the study of essences /.../ a philosophy that restores essences in existence, and does not think that both man and world can be understood in any other way than from their 'facticity', from their way of being in the world with others.

This speech by the author allows us to say that, when a phenomenological stance is taken, the intention is to understand what is shown, that is, the phenomenon, as it is experienced by the subject. In the research that we bring in this text, the phenomenon investigated is the constitution of mathematical knowledge with Augmented Reality. To understand what is shown about this constitution, we propose to be with Math Degree students, investigating the contents of the Calculus discipline, as highlighted above. However, in order for what is shown to be understood in its way of appearing, it is necessary that we turn to the way in which the students get involved with the tasks and carry out the investigation, always paying attention to the sense that is made for them.

In order to explain what is understood in the research, the phenomenologist researcher describes the lived experience which, according to Bicudo (2011, p. 38), "becomes the key to the phenomenologically conducted qualitative research". However, describing the lived experience requires a reliable record of the actions and, therefore, in the research, we opted for filming the meetings with the students and recording the screen of the smartphones they used. The footage registers the students' speech exposing modes of investigation and understanding, and is transcribed by the researcher, becoming a text open to interpretation.

In other words, phenomenological research requires the researcher to transcribe the experience as faithfully as possible, but it requires that he transcend the description, interpreting it in light of what is questioned, what is significant in the experience, that is, what is relevant to the understanding of the phenomenon. Thus, with the description, the researcher begins the hermeneutic interpretation, engaging in a movement of coming and going from reading the text, seeking the meaning of the whole. He – the researcher – transcends the description by involving himself with the analysis that, in the context of phenomenological research, takes place in two distinct moments: that of ideographic analysis and that of nomothetic analysis.

In the ideographic analysis, we try to make it clear what is expressed in individual speeches and, for this, excerpts of the speeches that stand out are highlighted and called Units of Meaning. This is done for the speech of all the subjects of the research and afterwards another movement begins in which the invariants interpreted in these speeches are sought, which provides conditions to constitute the regions of generalities or open categories. This movement – the search for invariants – is that of nomothetic analysis. The open categories, when discussed, give the possibility to expose the understanding of the phenomenon.

For this text we bring an example of the situation experienced with the students, trying to expose the way we understand the constitution of knowledge of contents of Calculus with Augmented Reality.

The lived experience with AR

One of the tasks we proposed to the students is an example from the book *Calculus*, volume 2, by James Stewart (2013) and is part of the content of *Cylinders* and *Quadric Surfaces*. This example was proposed after the students had already explored other equations and their respective graphs. The intention of this task was for the students, by using software resources, to analyze some characteristics of the curve represented by the equation, including aspects that give the graphic representation its name.

It was proposed for them to choose a colleague to work in pairs[4], providing the possibility of dialogue and an expression of what they understand. The task suggested for the pairs is the one shown in Figure 3.

[4] The affinity among them made a few of them work in a trio, in which we chose not to interfere.

In GeoGebra, sketch the surface: $z = y^2 - x^2$ Make cuts for x = m, y = n and z = o, and observe the intersections of these planes against the surface. Male variations for the values of m, n and o.

Figure 3: Statement of the activity developed with the students.

Initially, students should type the equation into the software and observe the generated graph. Then, they had to create the x = m, y = n and z = o planes and, by using the Surface Intersection feature, observe the intersections generated between the equation graph and each of the planes. It was possible to modify the values of m, n and o, generated by the software in the form of Slider Controls, a GeoGebra feature, which allows for the variation of the values of these constants and, as the changes are made to the values, the user can simultaneously follow the changes in the graphs. With these variations, it was possible to modify the position of the planes that intercept the graph and generate the intersection curve, as we can see in Figure 4. In this figure we have the graph of the equation $z = y^2 - x^2$ (in orange) and the intersection curves (in blue) with the z = o plane, for the specific moment registered in the figure.

Figure 4: Smartphone screen of on the pair in the moment of exploration.

After building the graph and the intersection plans, the students started to explore the task, dialoguing with their partners. The proposal was to explore the different concavities parabolas from the cuts made with the vertical planes x = m and y = n, and from the horizontal cuts z = o recognizing them as a family of hyperboles. The intention was for the students to understand that the strokes made would form a hyperbolic paraboloid.

Below, we bring an excerpt of a dialogue[5] between the members of a trio.

Participant 1: It will only give us a parabola by varying x = m.
Participant 2: Let me check something (*the student leaves his place and moves around the object to observe the intersection from a different place so that he could see if they were parabolas*). So there will be more and more parabolas.
Researcher: What about y = n? What will it give? Have you seen it?
Participant 1: No. Put y = n there. It will give a parable too, right?! Only downwards. But it is better to check, right?! [...]
Participant 3: [...] Here's what I did, look: z = o. Generating the intersection ...
Researcher: Right. Noe you move the slider for **o** and watch... Oh, you zoomed in too much. Zoom out to get a better look.
Participant 3: Oh, now it worked. It really is a hyperbole.
Researcher: You managed to put z = o.
Participant 1: z = o gives two lines and then a hyperbole. Yeah, they seem to form 90º, right?! (*the lines generated by the intersection when z = 0*). But they don't always have to. As they are, **a** would equal **b**, get it?!
Participant 2: Isn't the right term 'symmetrical'? Look there, depending on the values of a and b, the opening of the parabola will vary. (*the student moves the slider in z, and observes the variation in the intersection of the surface with the plane, drawing the colleague's attention*).
Participant 1: Yes, this is going to be the hyperbole like this and then it changes.
Participant 2: In this case, there are two lines when z = 0.

[5] We built some hyperlinks that refer to a short video in which it is possible to see the exploration that is being done with the app so that one can see what produces slider and the subject's (student that moves around the projected object) movement.

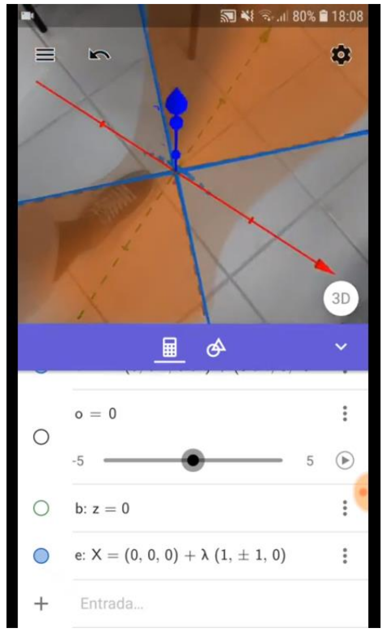

Figure 5: Smartphone screen when the pair makes the observations and fix plane z = 0.

The dialogue between the students and the researcher shows that by using the Slider it was possible to see the variations in the intersection of the graph with the planes. As they moved the slider, the students identified the curves produced. But in addition to moving the slider, they also moved themselves, as they looked for a place where they could see the graph from another perspective. A moment of exploration recorded by the students' device can be observed by clicking here, or through the QR Code reader in Figure 6.

Figure 6 - QR Code that links to a video of the students' exploration.

As they continued the exploration, the students saw that at a given moment the curves formed by the intersection were hyperboles. However, they were not identical hyperboles, and depending on the movement, the curves "changed direction" (axis) and, at a certain point, were shown as two intersecting lines, appearing to be perpendicular. This exploration can be seen here. Opportunities for students to raise hypotheses and look for ways to validate or refute them are shown. In some situations, it is seen that the participants use the software to validate conjectures previously formulated, as in this excerpt: "**Researcher**: What about y = n? What will it give? Have you seen it? **Participant 1**: No. Put y = n there. It will give a parable too, right?! Only downwards. But it is better to check, right?! […]".

We asked them how the curve generated by the intersection of the y = n plane with the graph would be, and Participant 1 by making an analogy with the intersection of the graph with the x = m plane proposes that this intersection will also be a parabola, but with a change only to the concavity. However, he says it is better to design the intersection graphically to validate his reasoning.

The search for understanding makes it possible for the student to *see* what shows itself as variable according to the movement. By moving the slider, they analyze what appears on the smartphone screen and explores its variations. They identify the curve, establishes relationships and concludes considering what they see with the software. The way in which the intersection curve behaves under the control of the slider requires that the student also moves in order to be able to see it from another perspective and to get answers to the questions addressed, either by the researcher or by the colleague who is also willing to see.

It is seen in the graphic construction spontaneity of the sketch, a freedom of the spectacle that is shown to it. If students could freely interrogate the movement, it might be possible to expose the "sensitive quality /.../ particular product of an attitude of curiosity" (Merleau-Ponty, 1994, p. 305). However, through the questions that are addressed to them and the context in which they find themselves - studying tasks of the Calculus discipline - they fix their gaze and seek to interpret the curve.

> Now, what is fixing? On the object side, it is to separate the fixed region from the rest of the field, it is to interrupt the total life of the spectacle /.../ on the subject side, it is to replace the global view, in which our gaze lends itself to the whole spectacle and lets itself be invaded by it, an observation, that is, a local vision that it governs in its own way. (Merleau-Ponty, 1994, p. 305).

The movement that takes place in the investigation when using the GeoGebra AR app, reveals two particularly interesting aspects. The first exposes the way in which the student is with the task in the context of the discipline, open to investigation, willing to seek to understand the content that is proposed to them: to understand the intersection of planes within a given surface. The second is the one that, by looking, lets you know what is shown. In both, the student's “self-move” is required. The student does not move as an object among others, he is sensitive to all others, he is able to understand them, to find the meaning of what he sees. But, what does he give himself to see? He gives himself to see the movement. What movement? The subject’s. He, a student of the Calculus discipline who has an Augmented Reality app seeking to understand the nature - or characteristics - of the intersection between planes and the surface generated by the equation z = y2 - x2. This movement is one of projecting oneself, of launching oneself into research, raising hypotheses, weaving analogies with what has already been seen in the discipline, anticipating the shape that the next curve will take, considering the one already seen. But it is also a movement of launching yourself to another place, in the physical space. Get on the chair to see if the curve that is projected on the table remains under such movement. The students, in dialogue, set themselves in motion to see, to understand what is exposed to them as a content of the subject of Calculus. They are not separate movements; they are movements of the subject who seeks to know.

Thus, if we may return to the question of the constitution of knowledge, what is significant in the research, for the time being, is the opening of the subject to exploration, the disposition that makes them launch themselves in the investigative process, trying to *see* what is shown, to understand if there is a similarity between the hyperboles seen or if the intersection can be lines.

Considerations about the lived experience

To close this text and to say what was possible to understand from lived experience with the students, we highlight the possibility of *seeing* favored by the movement. We start by asking, "what does it mean to *see* things"? With Husserl (1990) we understand that

> [...] there is no point in talking about things that exist and just need to be seen; but that this ‘merely existing’ are certain experiences of the specific and changing structure; that exists perception, fantasy, recollection, predication, etc., and that things are not in them as in a wrapper or in a container, but things are *constituted* in them, which in no way can be found as ingredients in those experiences. The ‘being given of things’ is to *be exhibited* (be represented) in such and such a way in such phenomena. (HUSSERL, 1990, pp. 32-33, emphasis added).

With that, it is understood that, when it is said that the primacy of knowledge is perception, we do not claim that visualization is an element (or ingredient) of perception that may give meaning to the graphics that are manipulated, for example. In the acts of perception, which are intentional acts, that is, consciously performed by the subject who turns to the AR app trying to move by moving, something is seen, something is shown and it is shown in a certain way for the subject to perceive. That is, something is shown to someone who wants to see, someone who seeks to see what is shown.

Thus, this way of seeing is constituted in perception, this way of visualizing what is shown by making itself known. Knowledge, thus understood, is not a psychic experience, an isolated act of the isolated subject, it is, rather, as Husserl (1990, p. 106, emphasis added) states an "*evident intuition* /.../ [in which] *the object constitutes itself*". But which object is constituted?

The research that we have brought about the subject of Calculus shows that there is a well-defined proposal of contents to be worked on: the study of functions, limits, derivatives, integrals and applications. Still, according to Alvarenga, Dorr & Vieira (2016, p. 55), "regardless of the scientific area, learning calculus is to understand the meaning of deriving and integrating a function". But, what does it mean to understand? The authors read show that a large part of the students' difficulties is related to the way in which the contents are treated in the classroom, with exposure of techniques by the teacher and repetition by the students. They also state that students reproduce integration and derivation techniques without knowing what they are doing. In this way of proceeding, understanding is not evident, if by understanding it is meant a certain way of seeing, of giving meaning to what is being considered. Therefore, in order to understand, what is being done must be clearly shown.

The tasks proposed to students, as we interpret them, are important for understanding the content of two-variable functions, which requires visualization in three-dimensional space, since their graphics are surfaces in space. The students' movement, to interpret the graphs generated in the AR app, gives them the possibility to identify the different parabolas and, by using the z = o slider, they can see the hyperbole being traced. This "seeing" is not a simple contemplative look at the object that is projected with the software. It is a look that establishes relationships, compares, identifies properties and favors the understanding of the behavior of the graph of a function in three-dimensional space.

There is student involvement with exploration activities, there is openness to dialogue. They seek what is shown and, for that, they move around the virtual object; they seek the "best place" to see, to assign meaning to the shape of the curve that draws the intersection of the planes and the surface. What are the conditions for this intersection to be a parabola? Under what circumstances will it be hyperbole? What sets them apart? The investigative movement establishes itself in this wanting to see and students move – literally – to know what these curves are.

The GeoGebra AR app allows for visualization, which makes them aware of the parabola as the intersection of the plane with a surface, and not that it presents it as an equation, for example. Its properties are not just stated, they are perceived, identified and systematized. By moving the slider, the axis of symmetry is known. In the student's movement, which changes the perspective of the gaze, the similarities and differences between the parabola and the hyperbole are being investigated and become clear.

Knowledge is then constituted for the subject in the way of seeing the curves that are displayed in such and such a way, becoming a phenomenon, that is, open to interrogation, with meaning.

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