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An Analysis of the Influences of a Hybrid Learning Environment in the Solution of Vector Tasks according to the Anthropological Theory of the Didactic (ATD)

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ABSTRACT**:** This paper discusses a part of a doctoral study based on the theoretical pillars of the Anthropological Theory of the Didactic (ATD). Supported by this theory, our research interest followed a path that led us to the teaching of the mathematical object vector and its institutional configuration in a Mathematics Teaching course at Universidade Estadual de Feira de Santana (UEFS), in the state of Bahia, Brazil, centered on two dichotomous aspects. The first one refers to the possibility of application and institutional relevance of this mathematical object. The second aspect is related to the obstacles encountered in the didactical scope, which have impact on the teaching/learning of this mathematical object. In this discussion, the question that guided our investigation emerged: how to harness praxeological recombinations that can promote mediation between personal and institutional relationships in the scope of vector knowledge in the Mathematics Teaching course at UEFS? Based on this context, we aim at analyzing students' productions in terms of solution paths to a vector task in a Hybrid Learning Environment (HLE), from a developmental perspective based on the T4TEL didactical framework (Chaachoua, 2018). With regard to methodological support, Didactical Engineering provided directions that enabled us to make a comparison between a priori and a posteriori analyses, resulting in the identification of the reach of the techniques that were developed in contrast to the naturalized techniques that are part of this context.

*Keywords***:** Anthropological Theory of the Didactic – ATD, T4TEL, Vectors, Teaching and Learning, Hybrid Learning Environment – HLE.

Introduction

The mathematical object discussed in this study is the vector, which is geometrically conceived by the characteristics: magnitude, direction and sense. This conception is expanded in the domain of vector spaces, approached in the field of Linear Algebra, and developed through the institutional reach of the referred object. It acquires a sequential and recurring form in terms of curriculum component contents like Calculus, which comprises vectors: tangents, normals, gradients, rotation and others. As in Classical

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Mechanics, velocity, force and acceleration are based on the vector conception that integrates the curriculum corpus of Mathematics Teaching university courses in Brazil.

After this introduction, which briefly portrayed some elements of the institutional environment of vectors, we focus our attention on Vector Algebra, intrinsic to the Analytical Geometry that is part of the Mathematical Knowledge Nucleus, according to the university's Political-Pedagogical Project (PPP) (UEFS, 2018). The project belongs to the institutional constitution of the Mathematics Teaching course of *Universidade Estadual de Feira de Santana* (State University of Feira de Santana, in free translation), Bahia, Brazil, here referred to as CLM-UEFS.

At CLM-UEFS, vectors are located, in terms of *niche and habitat* [4], in the ecological scope of the discipline of Analytical Geometry. They circulate and function as anchors for several topics in this curriculum component, for example dot, cross and triple products, as well as lines and planes. This aspect emphasizes the presence of the object as an institutional prescription and constitutes the first encounter between students and the object at university.

However, in spite of its institutional relevance, there is a high failure rate among students who took curriculum disciplines that included the topic vectors in their teaching programs. Studies by Gomes (2015); Silva, Correa, Coelho & Ferraz (2016); Diogo (2016); Fornari, Cargnin, Gasparin & Araújo (2017) and others reported the same situation in several higher learning institutions in Brazil.

In order to stress or refute our hypothesis, we requested the Division of Academic Affairs (DAA) at *Universidade Estadual de Feira de Santana* (UEFS) to collect data on students who passed or failed disciplines that involve Analytical Geometry[5], that is, disciplines that use vector approach. Based on the

[4] These terms are related to the ecology of knowledge. They express the function that the object performs when it interacts with other objects and they locate the object in terms of institutional localization.

[5] EXA 129 – Analytical Geometry with 90 classroom hours per term (6 months), EXA 180 – Analytical Geometry and Linear Algebra I with 90 classroom hours per term (6 months) and currently EXA374 – Analytical Geometry with 60 classroom hours. The alterations, subdivisions and incorporations involving the referred disciplines stem from curriculum changes regulated by *Diretrizes Curriculares Nacionais* (National Curriculum Guidelines, in free translation) for Mathematics and Mathematics Teaching courses (CNE/CES 1.302/2001). Currently, CLM- UEFS also follows the guidelines provided by Resolution CNE/MEC nº 2 of July 1st, 2015.

data, we detected that, at CLM-UEFS, the situation is not different from other institutions, because the failure rate between 2004 and 2018 was 50% on average.

Based on this context, we introduce the structural path of the article, which is supported by the theoretical framework, by the presentation of some imbrications between epistemological and didactical obstacles related to the nature of vectors in the institutional scope, by an experimental design of the course and, finally, by the prelude to the conflict between *a priori* and *a posteriori* analyses.

2 A Dialogue Between the Theoretical Framework and the Research Question

In the structural universe of the Anthropological Theory of the Didactic (ATD) (Chevallard, 1996; 1998; 1999; 2001; 2006), we adopt the premise that everything can be characterized as an *object O*. Such breadth is so great that we will only discuss some of its elements, specifically the ones that can support our justification and model the discussion proposed in this text. In this sense, the theoretical foundation adopts hierarchical levels, starting with primitive terms like "people X and institutions I, as well as the remaining entities that I will introduce. They are, therefore, objects of a particular type" (Chevallard, 1996, p. 127), the "base material" for the theoretical construction referenced here.

Based on this foundation, the concepts objects O, people X and institutions I are in the origin of the ATD, and they organize the theoretical construction. Initially, the condition for the existence of objects depends on their adoption. In other words, they start to exist when they are acknowledged by the people X or the institution I. Then, there will be "personal" and "institutional" relationships called R(X,O) and R(I,O) (Chevallard, 1998, p.93), respectively, which highlights the interdependence aspects between these elements.

From an anthropological perspective, Chevallard (1997, p. 14) emphasizes that "every human activity stems from praxeology". This praxeological organization is structured

> [...] around a type of task T, which we initially find, a triplet formed by a technique (at least), τ, a technology of τ, θ, and a theory of θ, Θ. The set, called $[T \,/\, \tau / \,\theta\, /\, \Theta]$,

> constitutes a specific praxeology. This qualifier means that it is a praxeology related to a single type of task, T. (Chevallard, 1998, p. 5, our translation).[6]

This specific praxeology, based on a type of task T, can be described as follows: a practical block that composes "to know how to do", or *praxis,* called $\Pi = [T/\tau]$ and a theoretical-technological block, or *logos,* identified as "knowledge", in the usual sense of the term, expressed by $\Lambda = [\theta/\Theta]$. Both blocks are indissociable because "there are no *praxes* that are not accompanied by a *logos*" (Chevallard, 2018, p.34). Thus, the concept of the praxeology $\Pi \oplus \Lambda = [T/\ \tau\ /\theta/\Theta]$ will be one of the bases for our study.

In this context, characterized by antagonistic aspects which highlight divergences between the institutional relevance of vectors in Mathematics Teaching courses, specifically in CLM-UEFS, and obstacles of epistemological[7] and/or didactical[8] nature, our proposal for investigation arises: how can we harness praxeological[9] recombinations in order to promote mediation between personal and institutional relationships in terms of vector knowledge in CLM-UEFS? It is safe to say that we will discuss a "transposed praxeology $(\Pi \oplus \Lambda)^*$ that might be written as $(\Pi^*) \oplus \Lambda$, with a preserved *logos*, but with a modified *praxis* (…)" (Chevallard, 2018, p. 35), supported by the implementation of distinct teaching and learning organizations based on different forms of technique production.

The *praxes* of the subjects (teachers / students) will be modified based on the Hybrid Learning Environment (HLE), configured by us, as a space constituted by several didactic resources (media[10]). We consider the hypothesis that when we make different media available to students (Goulart e Farias, 2018, p.308), that is, different media from the conventional types, it is possible to integrate a flexible approach (Dreyfus, 1991) rather than mathematical rigor (Silva, Veloso & Abrantes, 1999). In line with that, our

[6] "Autour d'un type de tâches T, on trouve ainsi, en principe, un triplet formé d'une technique (au moins), τ, d'une technologie deτ, θ, et d'une théorie de θ, Θ. Le tout, noté [T/τ/θ/Θ], constitue une praxéologie ponctuelle, ce qualificatif signifiant qu'il s'agit d'une praxéologie relative à un unique type de tâches, T. Une telle praxéologie – ou organisation praxéologique – est donc constituée d'un bloc pratico-technique, [T/τ], et d'un bloc technologico-théorique, [θ/Θ]".

[7] See Bachelard (1996).

[8] See Brousseau (1976).

[9] See Chevallard (2018).

[10] See Chevallard (2007).

intention is to identify, implement and formalize a type of complement to teaching tools in the context of vectors, adding new tools to existing ones.

Moreover, it is possible to evaluate the gaps between personal relationships and institutional relationships around an object of knowledge, that is, vectors. As emphasized by Chevallard (1989, p. 219, our translation[11]),

> This relationship clearly includes everything that we usually think we can say – in terms of “knowledge”, “to know how to do”, “conceptions”, “abilities”, “domain”, “mental images”, “representations”, “attitude”, “phantasy”, etc… - of X about Os. Everything that can be affirmed – right or wrong, in conformity or not – must be considered (at best) for an aspect of the personal relationship between X and Os.

Such aspects will enable us to encounter new situations that are ready to be used as tools for the deconstruction and reconstruction of vector tasks as suggested by Casabò (2001, p. 16):

> The mathematics teaching and learning process corresponds to the activity of reconstructing mathematical organizations in order to use them in new situations and under different conditions […]. Thus, the objective of a teaching/learning process can be reformulated in terms of components of the mathematical organizations that will be reconstructed: the types of problems that can be solved, the types of techniques used, descriptive elements and justifications to base them on, the theoretical framework chosen, etc.

To mediate this praxeological dialogue, the theoretical framework T4TEL[12] (Chaachoua, 2018), inscribed within the domain of the ATD, presents a new formalization to the praxeological model stemming from two extensions: personal praxeologies (Croset & Chaachoua, 2016) based on the notion of praxeological equipment (Chevallard, 2009) and the notion of variables (Chaachoua & Bessot, 2016).

As our methodological path, we chose Didactical Engineering (DE), conceived by Brousseau (1982) and developed and promoted by Artigue (1988), because this tool provides ways of “extracting relationships between research and action […], allowing the development of an internal validation

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[11] De ce rapport personnel relève notamment tout ce qu'on croit ordinairement pouvoir dire - en termes de "savoir", de "savoir-faire", de "conceptions", de "compétences", de "maîtrise", d' "images mentales", de "représentations", d' "attitudes", de "fantasmes", etc…- de X à propos de Os. Tout ce qui peut être énoncé - à tort ou à raison, pertinemment ou non - doit être tenu (au mieux) pour un aspect du rapport personnel de X à Os.

[12] T4TEL: T4 refers to the praxeological quartet (Types of Tasks, Technique, Technology, Theory) and TEL means Technology-Enhanced Learning.

focused on the conflicts between the analyses of pre-established didactical knowledge (*a priori)* and productions by students (*a posteriori*) involved in this investigation.

3 Some Interconnections between the Nature of Vectors in Institutions and Epistemological and Didactical Obstacles

Based on methodological directions provided by Didactical Engineering (DE), previous analyses focused on studying obstacles located in Didactic Systems (DS), which – as highlighted by Chevallard (1996) – depend on the *milieu* to function, especially on the teaching system in which they participate. This fact reveals ecological correlations with other types of Didactic Systems (DS)[13], which, at the core of the institution, is disclosed as the *sine qua non* of learning: "it is important to notice that the formation of a didactic system – whatever it is – assumes a systemic environment whose role is essentially to create a whole set of necessary conditions for the existence of the didactic system." (Chevallard, 1996, p. 138).

It is important to stress that, based on the context described in this article, our approach will focus on Auxiliary Didactic Systems (ADS) or Induced Didactic Systems (IDS), represented by S $(X;\ \emptyset; O)$ or S $(X_1,\ X_2,\ X_3; \emptyset; O)$, which means that the position that should be occupied by a professor or study coordinator will remain unoccupied in order to encourage students to adopt a proactive attitude, as indicated by the Paradigm of Questioning the World (Chevallard, 2009; 2010; 2012).

The work O, specified in the Didactic Systems, represents the vectors, which are elements of a vector space from a mathematical point of view. In other words, they carry a certain generality that is characteristic of Linear Algebra. In these terms, researchers propose mathematical organizations taught at institutions of higher learning which consist of interconnections between Geometry and Linear Algebra, as described by Schneider:

> The lines and planes are immediately defined as linear varieties or the like. Vectors are elements of a vector space and several vectors are defined according to the concept of

[13] Chevallard (2011) states the existence of other Didactic Systems in addition to the Main Didactic Systems (MDS), which he calls auxiliary, induced and furtive.

> related parties. Therefore, the subject fits the vector record, and parametric and cartesian records "flow stemming from it". (Schneider, 2012, p.222, author's emphasis, our translation[14]).

Yet, in the specificities of the geometric field, vectors appear as elements belonging to Euclidean vector spaces: $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$. It leads us to paraphrase Dorier (1995) when we consider that the development of the concept of vector happened in search of understanding algebraic results, because it contributes to geometry through direction and sense, not only through length (scalar quantity), thus enabling the idea of motion.

Due to the breadth and possibilities of use of this mathematical element, we can assume that the genesis of vectors carries in itself elements which destabilize concepts that are pre-established in the educational path, for example, scalar quantities, described by Poole (2006, p. 1) as "measurable quantities – such as length, area, volume, mass and temperature – that can be completely described by the specification of their magnitude." The same does not hold true for vector quantities, which depend on three inseparable components (sense, direction and magnitude).

This perspective is emphasized in studies by Bittar (2000), Lê Thi (2001), Ba and Dorier (2006), Affongon and Tossa (2014), who encouraged reflections focused on vectors in the educational domain. At the core of some of these investigations there are aspects that might cause learning problems, a fact that was highlighted by Affongon and Tossa (2014, p.2, our translation[15]):

> In natural language, the indication of direction is generally accompanied by a movement that is similar to the indication of place. This practice gives life to a cultural obstacle that must be removed in order to distinguish the sense of direction when learning geometric vectors.

Bittar (2000) reveals that French students find it difficult to think that a vector has several representations, besides the difficulty encountered in decomposing a vector on a basis. Similarly, the

[14] Les droites et plans y sont définis d'emblée comme variétés linéaires ou affines. Les vecteurs sont des éléments d'un espace vectoriel et des vecteurs multiples sont définis à partir de la notion de partie liée. L'entrée en matière relève donc du registre vectoriel et les registres paramétrique et cartésien en "découlent".

[15] Dans le langage naturel, l'indication de la direction est souvent accompagnée d'un geste semblable à l'indication d'un lieu. Cette pratique donne vie à un obstacle culturel qu'il faudra lever pour distinguer le sens de la direction au moment de l'apprentissage du vecteur géométrique.

obstacles to the appropriation of the two characteristics that are intrinsic to the vector sense are discussed in a study by Lê Thi (2001, p.159, our translation[16]):

> Since the introduction of the vector concept in the program, professors have stressed that their students have had considerable difficulty, especially in "distinguishing the vector from the segment, in distinguishing direction from sense, in taking the three characteristics of a vector in consideration, in considering true / false in vector equality, in searching for a number m such that $\vec{a} = m\vec{b}$, in decomposing a vector into two non-collinear vectors" and, finally, "in implementing vector knowledge in the solution to a geometry problem."

Such factors, such as resistance to learning, nurture the obstacles arising from the formation and structure of knowledge, which significantly affect institutional practices in which knowledge is developed, spread, taught and learned. This factor can be added to Brousseau's statement:

> Organizing the overcoming of an obstacle consists of proposing a situation that can evolve and through which students can evolve, according to convenient didactics. It does not mean transmitting the information that one wants to teach but finding a solution in which the information alone satisfies or enables them to reach a satisfactory result promoted by students' dedication (Brousseau, 1976, p. 115, our translation).

His statement assumes that every mathematical activity involves study activities performed in a specific situation and in a *milieu*. With that being said, our argumentative points are what is promoted and advocated in the structuring of *milieu*x derived from the Main Didactic Systems (MDS) - whose theme here is vectors –, ruled by a set of conditions and restrictions materialized in classrooms.

In this context, the first confrontation revealed itself when we analyzed some textbooks that approach the subject and that are included in the references and discipline programs for CLM-UEFS. Based on this analysis, we detected some discrepancies or variations in terms of how the concept of vector is introduced. For instance, Loreto & Loreto Junior (2014) stress that it is urgent to perform a study on the characteristics and properties of the oriented segments before formalizing a definition of this mathematical object.

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[16] Des les premieres annees d'introduction de la notion de vecteur dans le programme, les enseignants soulignent chez leurs eleves de nombreuses difficultes, notamment pour ≪ distinguer le vecteur du segment, distinguer la direction du sens, prendre en compte les trois caracteristiques du vecteur, envisager le vrai/faux d'une egalite vectorielle, chercher le nombre m tel que $\vec{a} = m\vec{b}$, decomposer un vecteur sur deux vecteurs non colineaires ≫ et enfin a ≪ mettre en oeuvre les savoirs vectoriels dans la resolution d'un probleme de geometrie.

Conde's (2004) and Corrêa's (2006) point of views initially suggest approaching vector representations in planes ($\mathbb{R}^2$) and in space ($\mathbb{R}^3$) before explaining the concept of vectors. Other authors prefer to start with an intuitive notion, which is directly linked to a differentiation between scalar and vector quantities (Camargo & Boulos, 2005). Vector representation as arrows, usually related to the introduction to the formalization of the concept of vectors, is an element that can limit the learning process. Arrows are characterized as segments in which an orientation is fixed, and fixing an orientation means choosing a sense.

In the case of **figure 1**, the oriented segment represented in it shows orientation from A to B. "Actually, we do not need an arrow for our purposes. Points A and B, and the order – first A, then B – are sufficient" (Camargo & Boulos, 2005, p. 4). The figured referenced in the quote presents the following representation:

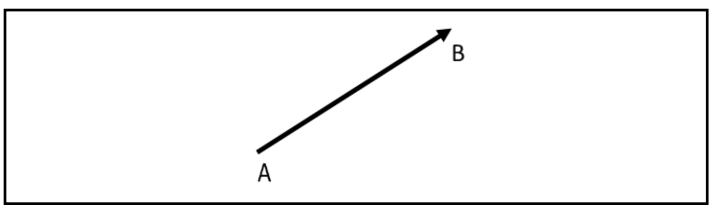


Figure 1: Vector Represented by an arrow

An association between the arrow and the oriented segment is formed. With that being said, it is important to highlight that vectors have some peculiarities that are different from the basic field of mathematics, as future mathematics teachers face a mathematical object that belongs to geometry, but also contains algebraic and arithmetic fundaments, in addition to being centered on the following characteristics: magnitude, direction and sense.

Studies by Heckler and Scaife (2015) emphasized the difficulties encountered by students when solving questions that considered vector representation by arrows, because: "[…] including the finding that different relative arrow orientations can prompt different solution paths and different kinds of mistakes, which suggests that students need to practice with a variety of relative orientations." (Heckler & Scaife, 2015, p. 1, our translation).

Another aspect that needs to be emphasized in the scope of an *a priori* analysis refers to the ostensive character of vector representation, which is described by Bosch (1994, p. 47) "as every object endowed with a sensitive nature, with a certain materiality and that is presented to the subject as a perceptible materiality." It means that it has a relatively concrete character, which is inscribed within the Hybrid Learning Environment (HLE).

4 A Summary of the Experimental Design

The experimental aspects of the research will not be fully described. However, we will include elements that can offer readers an overview of the direction of this investigation. The example included in this text contains a vector task solved by some CLM-UEFS students according to the techniques that are characteristic of didactic devices[17]: Manipulable Materials (MM), Paper/Pencil (PP). Clearly, due to space restrictions, we will not report productions developed by students through the GeoGebra (GG) software. We chose to offer comprehensive and detailed descriptions of the techniques used in each solution.

Bessot & Chaachoua (2016, p. 3) consider that "every object of observation of a researcher is a task, and that a type of task is defined by an action verb and a fixed complement". Stemming from this definition, Task Type Generators (TG) emerge, determined by a type of task and a system of variables. According to the referred authors, the new object allows us to structure a set of types of tasks for research purposes. From this perspective, a TG has the following configuration: TG = [Action Verb; Fixed Complement; System of Variables] (Chaachoua & Bessot, 2016).

It is relevant to highlight that a TG is not a type of task. Its function consists of creating types of tasks according to the hierarchical structure defined by the variables (Chaachoua, 2018). Thus, we make the metaphorical interpretation that generators function as "machines" that produce tasks, to which the Hybrid Learning Environment (HLE) provides "fuel" for its operational functioning in interaction with

[17] Didactic devices concretize techniques (Chevallard, 2002, p.2)

the media, according to the Dialetics of Media and Milieu (Chevallard, 2007), regulated by the values of variables that are intrinsic to the nature of each didactic device.

Attached to this process, variables perform two functions that complement each other. The first one is to generate subtypes of tasks by adjusting the specificity level of each TG. The second function is to characterize the scope of techniques (Chaachoua & Bessot, 2018). The different values attributed to variables mediate the analysis of the praxeological recombination performed by some CLM-UEFS students.

4.1 Characteristic Features of the Course

The course *"Um Estudo sobre Vetores"* (A Study on Vectors, in free translation) had the objective of examining some of the crucial aspects of the discussion about vector teaching and learning at universities, supported by elements that were previously described in this text, and configured by a part of the experimentation stage that fueled the confrontation process between *a priori* and *a posteriori* analyses, which are characteristic of Didactical Engineering (Artigue, 1996, p.197).

CLM-UEFS second term students who were taking the discipline EXA 180 (Analytical Geometry and Linear Algebra) were the focus of the course, so they were invited to participate in this didactical enterprise. Twenty vacancies were offered, and 18 of them were filled. The meetings occurred outside of regular class hours, and after the end of the course, students received a certificate attesting 10 hours of extra activities, which can be considered as Complementary Activities (CA), according to the institution's Political-Pedagogical Project (PPP) (2018, p.43).

With that being said, the experimental background emerged when students were assigned vector tasks in Numerical, Algebraic and Geometric (NAG) domains (Farias, 2010), which were developed in the environments: Paper/Pencil (PP), Manipulable Materials (MM) and the GeoGebra software (GG), as exemplified in the structure of an organizational overview for the solution to task 1 (**figure 2**).

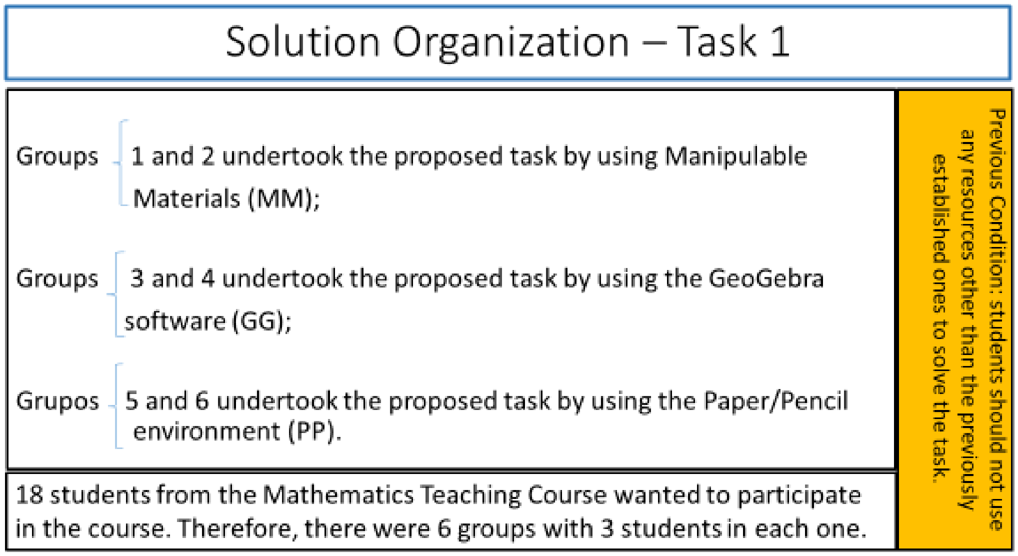


Figure 2: An example of organizational design

This perspective demanded effort from every subject[18] in the group in order to reach a joint response to the proposed task, which assumed the implementation of techniques, that is, "(…) students are given the possibility of understanding the specificity of the prescribed task (…) (Chaachoua, 2010, p.5, our translation[19])

Students acting in task solution adopted a cooperative attitude organized by techniques and gestures that are specific and intrinsic to each device. According to Chevallard (1998):

> Didactic tasks are, in fact, cooperative, in a certain number of contexts, because they must be performed collaboratively by several people x_1, ..., x_n, the performers of the task. In this case, each actor x_i must perform certain gestures, which constitute their roles in performing the collaborative task t, and their gestures are differentiated (according to the actors) and coordinated by the technique τ collaboratively implemented. (Chevallard, 1998, p.18, our translation[20])

[18] When individuals occupy such positions, they become subjects of the institutions – active subjects that contribute to give life to institutions because they are subjected to them. (Chevallard, 1999, p. 4, our translation)
[19] (...)soit on donne à l'élève la possibilité de reconnaître la spécificité de l'énoncé de la tâche prescrite (...).
[20] Les tâches didactiques, en effet, sont, dans un certain nombre de contextes, coopératives, en ce sens qu'elles doivent être accomplies de concert par plusieurs personnes x_1, ..., x_n, les acteurs de la tâche. On dira que chacun des acteurs xi doit en ce cas effectuer certains gestes, dont l'ensemble constitue alors son rôle dans l'accomplissement de la tâche coopérative t, ces gestes étant à la fois différenciés (selon les acteurs) et coordonnés entre eux par la technique τ mise en œuvre collectivement.

This perspective provided new paths to analyzing different sequences of actions embodied in gestures revealed through didactic devices (MM, PP, GG), which were added to praxeological recombinations and alterations. Thus, we emphasize the existence of the composition of *to know how to do* in the construction of solutions, with the descriptive design in the praxeological complex that is formed with each solution process.

In this line, the subjects of the institution CLM-UEFS occupied the position of proactive participants in the course "*Um Estudo Sobre Vetores"* and were invited to solve vector tasks by using techniques that were different from conventional ones, which caused a disruption to the general phenomenon of naturalization of tasks/techniques, because, according to Bosch and Chevallard (1999, p.6, our translation[21]): "The technique employed to perform them, although it was developed some day in the past, was so routinely used that it no longer appeared as such – the use of such technique to perform the task is now so evident that it no longer poses a problem."

The perspective assumed by the study breaks with the typical aspect of techniques because there is a willingness for the creation of techniques linked to the didactical device MM. Besides, as a direct consequence of this phenomenon, there are transformations in the Didactic System (DS), specifically in the relationships between an individual x or a group of individuals X and a specific work O, for "learning is a modification of the relationship between an individual X and O, that is to say, such relationship starts "to exist" (if it did not exist), or the relationship is modified (if it already existed)" (Chaachoua & Bittar, 2019, p. 31).

In the management of the path, due to a certain generative power that allows a TG to function as a device that produces types of tasks, we understand that the functioning of the TG caused transformations and imbalance in the didactic systems (ADS and IDS), leading to new praxeological organizations and reorganizations, because according to Chevallard et al (2001, p.253), "to organize is to create a praxeology. A new praxeology or a renewed one", stemming from the introduction of a repertoire of

[21] la technique utilisée pour les accomplir, bien qu'ayant été construite un jour, a été routinisée, au point de ne plus apparaître comme telle – utiliser cette technique pour accomplir une telle tâche va désormais de soi et ne pose plus aucun problème.

possibilities for solving tasks, which were guided by different techniques, in line with several variables ruled by research intentions and choices.

In addition to the characteristics of HLE, we consider it a *didactical effort* which provides an array of possibilities of study when focusing a given object O, based on the availability of resources and gathering of works in O_b. In other words, different devices are offered for the solution of tasks (T), in which the implementation of techniques (τ) is expected.

As a result of this conception, in task solving - even with the inclusion of the common environment Paper/Pencil (PP) - there is an emergence of interactions among didactical devices – GeoGebra (GG), Manipulable Materials (MM) or Paper/Pencil (PP). From this perspective, we describe the arrangement of variables $V_1 = \{GG, MM, PP\}$, in terms of devices, inscribed within the course with their derivations and ramifications. The **figure 3** shows solution path options for each task given to students.

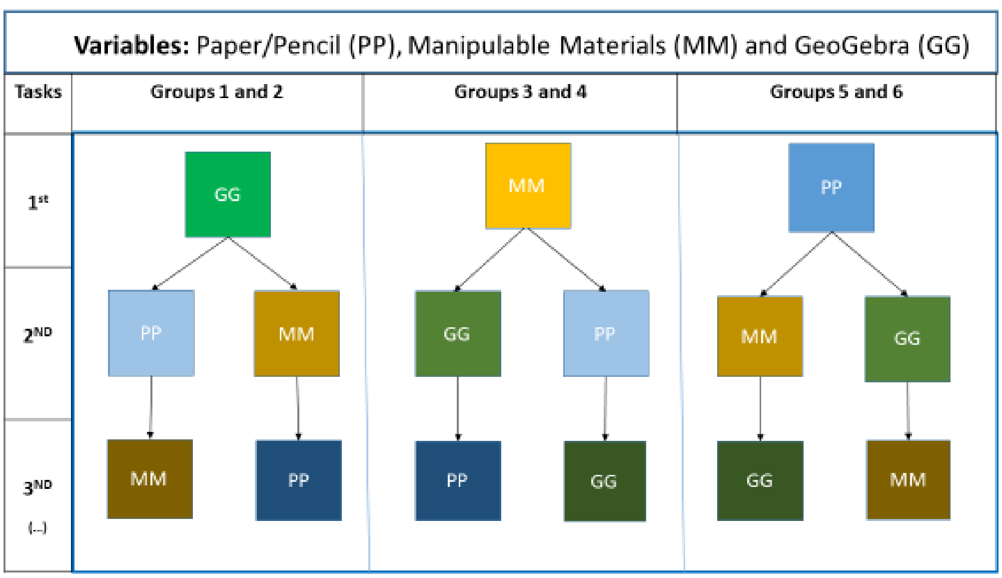


Figure 3: Variable arrangement design in terms of devices

When we interpret the first two lines in the first column, in reference to the first question, we find: groups 1 and 2 used a dynamic geometry device, the Geogebra software (GG). However, in the second question, they followed different paths, because group 1 preferred to solve the following question using the Paper/Pencil (PP) device, whereas the second group produced a solution using Manipulable Materials (MM).

The reference here is the reality of the settings constructed by CLM- UEFS students in the scope of vector task solutions guided by different didactical devices, which were directed by the System of Variables (VS) and integrated into the Task Type Generators (TG). Based on these indications, it is important to take into consideration the limited set of techniques, for working with different didactical devices paves the way for possible transformations and for a subsequent increase in the number of tasks, with repercussions on techniques, because, according to Chevallard (1989), we are generally limited to a small number of techniques:

> (...) in a given institution, about a given type of task T, there is generally a single technique, or, at least, a small number of techniques institutionally acknowledged, excluding possible alternative techniques – which may effectively exist, but in other institutions. (Chevallard, 1989, p. 93)

As a consequence of the emergence of alternative techniques originating from the tasks that were undertaken by students while circulating around the didactical settings characterized by elements that are intrinsic to each device, it is possible to suppose a break with what circulates in institutional terms, because "rather than making students face a type of task to which there are several techniques, the institution organizes the study through several types of tasks to which there is only one technique" (Chaachoua, 2010, p. 6, our translation[22]). That is to say, they must understand to which kind of task it belongs and which techniques can be used in order to achieve a solution that will be or will not be accepted by the institution. This reflection is complemented with the description of two cases, which exemplify how objectives, principles and methodology were implemented in concrete (MM) and non-concrete (PP) didactic processes. Supported by this perspective, we will briefly describe the work performed in the course with a task and the devices MM and PP.

[22] Ainsi, plutôt que de confronter l'élève à un type de tâches pour lequel il existe plusieurs techniques, l'institution organise l'étude à travers plusieurs types de tâche pour lesquels il y a une seule technique.

4.2 Description and analysis of a task within the scope of MM and PP

First, we will show readers the Task Type Generator which started the task discussed in this text. It is denoted as $TG_{t1} = [Write; vectors\ as\ a\ function\ of\ \vec{u}\ and\ \vec{v}; V_1, V_2, V_3, V_4, V_5, V_6, V_7]$, consisting of an action verb, a fixed complement and a system of variables, in this order. The Mathematical Organizations generated by TG will unleash associations among the values assumed by variables and, as a consequence, new praxeologies will be guided by didactical gestures characteristic of each technique. To better explain it, the **table 01** details the values that each variable may assume:

V_1- *Devices {GG, MM, PP};*
V_2- *Nature of MM (straws, rubber bands, string, EVA (ethylene-vinyl acetate) and others);*
V_3- *Supporting instruments connected with MM (scissors, glue, pins, styrofoam, etc);*
V_4- *Device menu (vector, vector from a point, segment, polygon, regular polygon, angle, length and others);*
V_5- *Mobile devices (laptop, cell phones, tablets);*
V_6- *PP instruments available (ruler, set-square, graph paper, etc);*
V_7- *Configuration (plane, space, rectangular, triangular, ...);*

Table 01: Values of Variables

From TG_{t1}, we considered the first question, which contained the following elements in its instructions: square properties, midpoint characteristics, vectors, vector operations and linear combinations (**table 02**). This task was defined in the field of mathematics, in which the didactical intention was to transpose it and solve it by admitting the values of V_1 in combination with other variables,

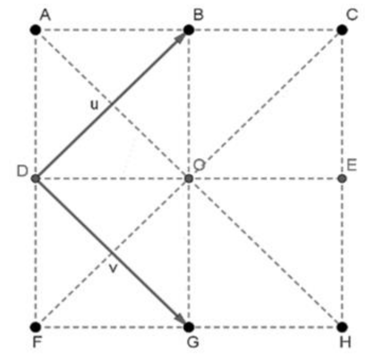
The figure is a square in which B, D, E and G are midpoints of its sides. $\vec{u} = \overrightarrow{DB}$ and $\vec{v} = \overrightarrow{DG}$. Write the following vectors as a function of $\vec{u}$ and $\vec{v}$: $\overrightarrow{GE}$, $\overrightarrow{EB}$, $\overrightarrow{DO}$, $\overrightarrow{BC}$, $\overrightarrow{FB}$, $\overrightarrow{AG}$ and $\overrightarrow{AH}$.	

Table 02: Instructions on Task 1 (t_1)

Considering the values of V_1 and TG_{t1}, two Task Type Subgenerators (Kaspary, 2020) arise. They are related to the MM and PP devices and are described in **table 03**:

(1) $T'G_{t1(MM)} = [Write; vectors\ as\ a\ function\ of\ \vec{u}\ and\ \vec{v}\ in\ MM; V_2, V_3\ ]$
(2) $T'G_{t1(PP)} = [Write; vectors\ as\ a\ function\ of\ \vec{u}\ and\ \vec{v}\ in\ PP;\ V_6, V_7]$

Table 03: Task Type Subgenerators

Students' productions will be presented following the order specified above. The first Task Type Generator produced tasks in a manipulable scope. From this perspective, we introduce group 1's production. It is important to highlight that the students in this group had the following materials available: rubber bands of different colors, pins, a Styrofoam board, a ruler and pencils (elements belonging to the Paper/Pencil environment) and glue. As a manipulable construct, this is the reproduction of the figure that is offered with the instructions on the task (**figure 4**):

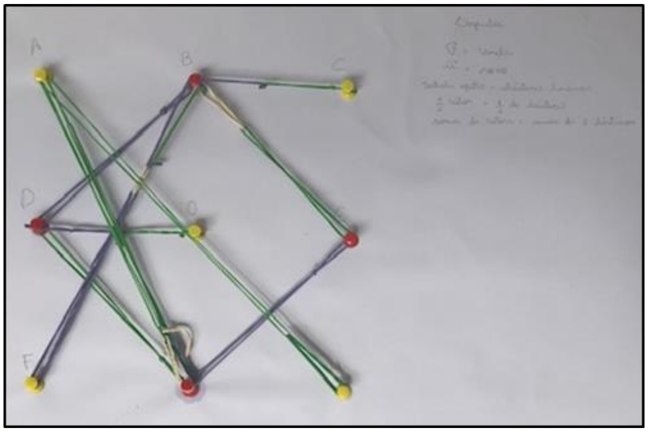


Figure 4: reproduction of the image from **table 02** (group 1)

We will now discuss this production, which embodies[23] the graphical representation that integrates the exercise. The praxeological recombinations involved "a part of a system that comprehends material or symbolic elements and manipulation rules, depending on the context of the action" (Mathé, 2012, p.197). A first impression emerged, leading us to understand that the techniques used in the solution were connected to a previous concept of disorder. Nevertheless, when we focused on each aspect of the production, that sensation decreased, and other elements arose. For instance, the vector representations were built according to the instructions given. Another important element is the key created by the group of students:

- $\vec{v}$ - green;
- $\vec{u}$ - purple;
- opposite sense: white rubber band;
- $\frac{1}{2}$ vector corresponds to $\frac{1}{2}$ rubber band;
- Sum of vectors: 2 or more rubber bands

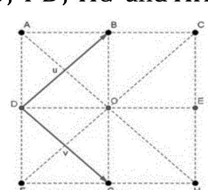
[23] Ver Watson, A.; Spyrou, P.; Tall, D. (2003)

The key led us to assume that, due to the great deal of information, they felt the need to specify the path they followed in their constructions. Such aspect was confirmed when we interrelated the construction in MM and the key, which allowed us to identify that the individuals in the group produced techniques aimed at solving the proposed task, because

> (...) it is possible to produce techniques that allow us to provide answers to questions initially asked. A new *know-how-to-do* is constructed, which still needs to be organized so as to ensure a regular functioning in the institution. (Bosh & Chevallard, 1999, p. 6, our translation[24]).

It is safe to say that it had an impact on the implementation of techniques resulting from the manipulation of ostensive objects (manipulative *praxis*), in this case MM, guided by non-ostensive objects (vector properties and definitions). This conception was identified in the several task solving techniques and reflected in the praxeological analysis.

Thus, we describe the praxeological analysis in terms of tasks T, techniques τ belonging to the practical block, with a justification stemming from the theoretical block in terms of technologies θ and theories Θ, as shown in **table 04** and **table 05**:

Praxeological analysis – practical block (T/τ) – 1st task	
Task (t)	The figure is a square in which B, D, E and G are midpoints of its sides. $\vec{u} = \overrightarrow{DB}$ and $\vec{v} = \overrightarrow{DG}$. Write the following vectors as a function of $\vec{u}$ and $\vec{v}$: $\overrightarrow{GE}$, $\overrightarrow{EB}$, $\overrightarrow{DO}$, $\overrightarrow{BC}$, $\overrightarrow{FB}$, $\overrightarrow{AG}$ and $\overrightarrow{AH}$. 

[24] Le cas échéant, on arrive, après un processus d'étude plus ou moins long, à produire des techniques permettant de fournir des réponses aux questions initialement posées. Un nouveau « savoir-faire » est construit, que l'on doit encore organiser pour lui assurer un fonctionnement régulier dans l'institution.

The technique depends on the chosen V_d. For example, the one described in item 2. **Techniques (τ)**	1) Identify points A, B, C, D, E, F, G, H and O, with no specific order; **2) Set the pins that represent the points, and which will later support the rubber bands;** 3) Identify and represent vectors $\vec{u}$ (green rubber band) and $\vec{v}$ (purple rubber band), as in the key; 4) Vector $\overrightarrow{GE}$ was represented in the same color as vector $\vec{u}$, that is, $\overrightarrow{GE}$ is one of the representatives of $\vec{u}$; 5) In the case of vector $\overrightarrow{EB}$, the group used ***the white rubber band attached to the extremity of the vector.*** According to the key created by the students, it means that $\overrightarrow{EB}$ has opposite sense to $\vec{v}$***;*** 6) $\overrightarrow{DO} = \frac{1}{2}\overrightarrow{DE} = \frac{1}{2}(\overrightarrow{DB} + \overrightarrow{DG}) = \frac{1}{2}(\overrightarrow{DB} + \overrightarrow{BE}) = \frac{1}{2}(\overrightarrow{DG} + \overrightarrow{GE}) = \frac{1}{2}(\vec{u} + \vec{v})$, **as specified in the key, $\frac{1}{2}$ vector corresponds to $\frac{1}{2}$ rubber band;** 7) $\overrightarrow{BC} = \overrightarrow{DO}$, as in the technique used in the previous item; 8) $\overrightarrow{FB}$ is a vector that does not correspond to one of the diagonals of the square or to one of its sides, and it is not parallel to them. This aspect demanded an enhanced repertoire of techniques from the subjects, resulting in $\overrightarrow{FB} = \frac{3}{2}\vec{u} - \frac{1}{2}\vec{v}$ 9) Vector $\overrightarrow{AG}$ also has the same characteristic as the previous item, resulting in $\overrightarrow{AG} = \frac{3}{2}\vec{v} - \frac{1}{2}\vec{u}$. **Here, the use of MM was not sufficient to solve the task. The students migrated to the PP environment. From this perspective, MM functioned as an auxiliary or supporting variable to solve this item;** 10) $\overrightarrow{AH}$ was seen as a multiple of $\vec{v}$, that is, $\overrightarrow{AH} = 2\vec{v}$. In this case, the manipulable material (MM) fulfilled its function as a variable.

Table 04: Praxeological description related to the practical block in MM

Praxeological analysis – theoretical block (θ/Θ) – 1st task	
Technologies (θ)	The characteristics of the square (regular polygon) are: θ_1: • Four congruent sides and parallel opposite sides; • Four right angles; • Diagonals are perpendicular; they are congruent and intersect in their midpoint; • The diagonal lines are bisectors of the corresponding right angles. θ_2: • Equivalence relation (equipollence) between segments and equality between vectors; • Operations between vectors: addition, subtraction and scalar multiplication; θ_3: • Linear combination.
Theories (Θ_1, Θ_2, Θ_3)	Intersections of Plane Geometry, Analytical Geometry and Linear Algebra.

Table 05: Praxeological description related to the theoretical block

In one more example of a subdivision of TG_{t1}, we introduce $T'G_{t1(PP)} = [Write; vectors\ as\ a\ function\ of\ \vec{u}\ and\ \vec{v}\ in\ PP;\ V_6, V_7]$. The variables that directed $T'G_{t1(PP)}$ were the instruments related to the Paper/Pencil (PP) device, available to the students that chose to work in this environment: sheets of paper, pencils, erasers and rulers.

We continue our discussion by deducing that the students who preferred the PP device - which is linked to the classical techniques of the Didactic Systems (DS), characteristic of the mathematical context – first worked individually, not collectively. In other words, they initially thought and recorded their work separately, and only then they discussed their thoughts. Thus, we identified traces of the Dialectics of the

Collective and the Individual (Costa, Arlego & Otero, 2015). This focus is revealed in the individual productions presented in the **figure 5**:

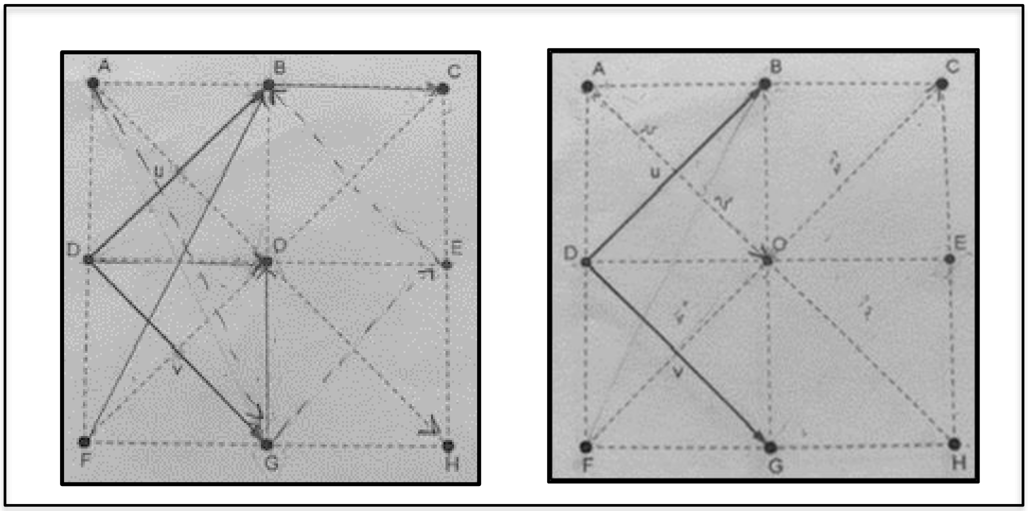


Figure 5: Individual productions based on the image in **table 02** (group 5)

In the image on the left, it was possible to identify links with **figure 4**, which allows us to infer the existence of praxeological connections[25] between the mentioned devices. For example, we could point the square formed by points D, G, E and B inscribed within the bigger square. This aspect was present in constructions developed in both devices and encouraged reflections on an equipollence relation, for a vector is an equipollence class of oriented segments (Loreto and Loreto Júnior, 2014, p.4) in which the following characteristics are present: magnitude, direction and sense.

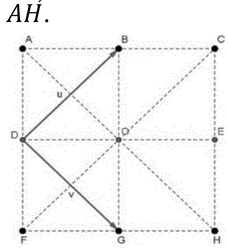
It is possible to notice that, in the work developed in the PP environment, the ostensive character has a prominent position in terms of a graphical representation of the square and its respective components, which was also detected in the constructions with manipulable materials. Therefore, the

[25] We understand that praxeological connections occur when we identify similarities between techniques used to solve a specific task.

tangible part of the types of interactions between the students and the material objects in the situation has two dimensions:

> a material dimension, related to the modalities of students' actions (the way they use the instruments, their methods of classification and construction of figures), and the language dimension, related to the way students designate objects and to the meaning that they attribute to the terms used." (Mathé, 2012, p. 199, our translation[26]).

It is relevant to emphasize that the elements harnessed for the action, interaction and recording of gestures differ and emerge from the prevalence of techniques that are centered on the representatives $\vec{u}$ and $\vec{v}$, characterized by arrows with defined length, direction and sense. From this perspective, it is possible to establish links between them, since we identified similar effects in the lines stemming from PP and in the tangible productions from MM. In line with our interpretations, there are connections between the use of the two devices, combined with the translation potential of equivalence class notion that is characteristic of the technological-theoretical sphere (θ/Θ) in accordance with the equivalent symbolic representations belonging to the perceptual and graphic spheres contained in the practical block (T/τ) (**table 06**).

Praxeological analysis – practical block (T/τ) – 1st task	
Task (t)	The figure is a square in which B, D, E and G are midpoints of its sides. $\vec{u} = \overrightarrow{DB}$ and $\vec{v} = \overrightarrow{DG}$. Write the following vectors as a function of $\vec{u}$ and $\vec{v}$: $\overrightarrow{GE}$, $\overrightarrow{EB}$, $\overrightarrow{DO}$, $\overrightarrow{BC}$, $\overrightarrow{FB}$, $\overrightarrow{AG}$ and $\overrightarrow{AH}$. 

[26] La partie tangible des modalités d'interaction des élèves avec les objets matériels de la situation admet, à mon sens, deux dimensions: une dimension matérielle, relative aux modalités d'action matérielle des élèves (leur mode d'usage des instruments, leur méthode de classement de solides, leur méthode de construction de figures), et une dimension langagière, relevant de la façon dont les élèves désignent les objets et la signification qu'ils assignent aux termes employés.

The gestures that accompanied the techniques in the PP environment are based on calculations and segment tracing from the graphic representation of the task. **Techniques (τ)**	1) Identify vectors $\vec{u}$ and $\vec{v}$; 2) Vector $\overrightarrow{GE}$ was represented by a traced line with the same vector $\vec{u}$, that is, $\overrightarrow{GE}$ is one of the representatives of $\vec{u}$; 3) In the case of vector $\overrightarrow{EB}$, a traced line was also used, but in an opposite sense to $\vec{v}$, thus the conclusion that $\overrightarrow{EB}=-\vec{v}$ 4) The following was defined: $\overrightarrow{DO} = \frac{1}{2}\overrightarrow{DE} = \frac{1}{2}(\overrightarrow{DB} + \overrightarrow{DG}) = \frac{1}{2}(\overrightarrow{DB} + \overrightarrow{BE}) = \frac{1}{2}(\overrightarrow{DG} + \overrightarrow{GE}) = \frac{1}{2}(\vec{u} + \vec{v})$; 5) $\overrightarrow{BC} = \overrightarrow{DO}$, as in the technique used in the previous item; 6) $\overrightarrow{FB} = \overrightarrow{FA} + \overrightarrow{AB}$ (Sum of vectors) **(I)**; $= 2\overrightarrow{DA} + \overrightarrow{AB}$ (Multiplication of a vector and a scalar) **(II)**; $= 2(\overrightarrow{DB} + \overrightarrow{BA}) + \overrightarrow{AB}$ (I and II) **(III)**; $= 2\overrightarrow{DB} - 2\overrightarrow{AB} + \overrightarrow{AB}$ (Property of (II) and inversion of sense) **(IV)**; $= 2\overrightarrow{DB} - \overrightarrow{AB}$ (II and a subtraction between vectors) **(V)**; $= 2\overrightarrow{DB} - \overrightarrow{DO}$ ($\overrightarrow{DO}$ represents $\overrightarrow{AB}$) **(VI)**; $= 2\vec{u} - \frac{1}{2}(\vec{u} + \vec{v})$ ($2\overrightarrow{DB}$ is equivalent to $2\vec{u}$ and $\overrightarrow{DO}$ results from $\frac{1}{2}(\vec{u} + \vec{v})$) **(VII);** $= 2\vec{u} - \frac{1}{2}\vec{u} - \frac{1}{2}\vec{v}$ (Property of (II)) **(VIII)**; $= \frac{3}{2}\vec{u} - \frac{1}{2}\vec{v}$ (Subtraction between vectors and II) **(IX)** 7) Vector $\overrightarrow{AG}$ has the same characteristic as in the previous item. From the same sequence of techniques, there is $\overrightarrow{AG} = \frac{3}{2}\vec{v} - \frac{1}{2}\vec{u}$. 8) $\overrightarrow{AH}$ was identified as a multiple of $\vec{v}$, that is, $\overrightarrow{AH} = 2\vec{v}$.

Table 06: Praxeological description related to the practical block in PP

Since we are discussing the same task, the theoretical block remains the same, directly linked to items I, II… IX. In this sense, Chevallard (2018, p. 35) highlights one of the outcomes of a transposed praxeology $(\Pi^*) \oplus \Lambda$, in which the *praxis* is modified, and the *logos* is preserved. In order to determine vectors $\overrightarrow{FB}, \overrightarrow{AG}$ *and* $\overrightarrow{AH}$ through linear combinations stemming from $\vec{u}$ and $\vec{v}$, the subjects had to make use of associations and *maneuvers*, since they could not find answers promptly. This fact led us to reflect based on Cirade's and Matheron's (1998) conception of decomposition of a task T into subtasks, that is, the technique is described by a sequence of types of tasks.

5 A Prelude to the Conflict Between *a Priori* and *a Posteriori* Analyses

In our *a priori* analyses, we decided to identify what we aimed at achieving in the domain of vector knowledge, in the institutional sphere of Mathematics Teaching courses. In other words, vector-related knowledge that is relevant and recommended, belonging to future mathematics teachers' praxeological equipment (Chevallard, 2009). To do so, in addition to what is contained in discipline programs, textbooks offered some direction in terms of knowledge to be taught and learned by students, as follows: the intuitive notion that is interconnected with the differentiation between vector and scalar quantities and representations by arrows, the definition of a vector according to the concept of equivalence class, countless representations of a same vector, operations, representations in plane or in space and vector decomposition.

These topics provide the foundation for the vector approach performed in Analytical Geometry classes. Based on these pillars, our attention was focused on the tasks developed in classes and on assessment questions that involved this content. Thus, we identified the techniques used by students and the didactic devices that supported and accompanied their solution processes. We detected that the classical Paper/Pencil (PP) is the only didactic device that is presented as an exclusive tool in problem solving. Yet, we understand that it is necessary to expand and conjugate this repertoire of didactical investment, that is, to make interventions in the *milieu* legitimized by the Dialectics of Media and Milieu (Chevallard, 2007) and the Dialectics of Ostensive and Non-Ostensive (Bosch, Chevallard, 1999), which support the materiality and the subsequent manipulation of tangible objects that are characteristic of the MM device.

In *a posteriori* analyses, we identified that the transition between PP and MM devices degenerated techniques and caused alterations in the ostensive dimension of vector representation, triggering a process of praxeological recombinations related to the practical block. Another consequence that we observed during the implementation of the study developed in an HLE is the evidence of the economic dimension, as described by Gascón (2011), in relation to the techniques that are intrinsic to each device. In other words, the practices which emerge from the work with MM are costlier. However, the materialization of the vector representations built by the students generated tendencies which led them to pay attention to the characteristics of vectors in terms of the differentiation between sense and direction. Moreover, it allowed them to compare and verify equality between vectors. The strength of the didactic contract (Chevallard et al, 2011) was revealed through the dependence on Paper/Pencil (PP), even when the didactical device did not include PP in the task solving process, for instance MM. This was also an aspect that attracted our attention, so we will discuss it in detail in the thesis.

6 Conclusion

The connections between institutional recommendations, obstacles and practices developed in the Didactic Systems (DS) describe contrasts between techniques that are naturalized and the ones that are fabricated in the solution process of a task, for example in the MM environment. Thus, there is evidence that working with different media and the circulation between different devices enable us to consider the adoption of a vector teaching/learning process according to a flexible approach (Dreyfus, 1991). This consideration is valid because works based on several didactical investments and different devices generate complements and praxeological recombinations, in addition to allowing the subjects of the DS to develop some *kinesics* – understood here as students' circulation around different devices – through different settings, which can constitute a first step towards breaking with the seeming completeness and uniqueness of solutions in the Paper/Pencil (PP) environment, naturalized in the conventional scope of mathematics teaching and learning.

Thus, supported by the ATD in line with the TGs and the HLE, we understand that this study highlights innovative aspects which insert the extensive viability of the repertoire of alternative techniques for vector tasks into the Mathematics Education literature corpus, especially in the scope of the Didactics of Mathematics.

Furthermore, the comparison between *a priori* and *a posteriori* analyses allowed us to observe the reach of the techniques used to solve the task. Further analyses happened through harnessing a certain number of variables that were linked to the nature of each didactical device, exemplified in this study as

MM and PP, and managed by Task Type Generators (TG), configured by us as the essence of the Hybrid Learning Environment (HLE).

Therefore, we understand that the conception, observation and analysis of experimental schemes based on didactical implementations in the classroom involving Task Type Generators (TG) and a Hybrid Learning Environment (HLE) generated movements that resulted in the creation of techniques, which can bring subjects [students; teachers] closer to mathematical objects *O*. It can interconnect personal and institutional relations between the players in the process and the knowledge in question.

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