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Didactic Engineering (DE) and Professional Didactics (PD): A Proposal for Historical Eesearch in Brazil on Recurring Number Sequences

Francisco Regis Vieira Alves

Didactic Engineering (DE) and Professional Didactics (PD): A Proposal for Historical Eesearch in Brazil on Recurring Number Sequences

Francisco Regis Vieira Alves[1]
Instituto Federal de Educação, Ciências e Tecnologia do Estado do Ceará - IFCE

Abstract**:** The present work presents some examples of data originated from a set of investigations developed in Brazil, at the Federal Institute of Education, Science and Technology of the State of Ceará - IFCE, in the context of teacher education, and the consideration of elements of a historical, mathematical, and evolutionary nature. Thus, from a perspective of theoretical complementarity involving the research elements derived from the notion of the didactic engineering for development and some assumptions of the French aspect of the professional didactics, two sets of data are discussed. The first set involved research carried out between 2017 and 2020. The second set considered involved studies still under development in Brazil, considering the period of 2020 - 2023. The historical landscape of investigation rests on considering the recurring number sequences of Fibonacci, Lucas, Pell, Jacobsthal, Coordonier or Padovan, Perrin, Mersenne, Oresme, Naraynna, and Leonardo. Furthermore, in a broad sense, the study aims to provide a learning scenario for the teacher, affected by a perspective of mathematical knowledge evolution, including the repercussions and applications with current technology.

*Keywords***:** Didactic engineering, Mathematics history, Initial teacher education, Recurring number sequences.

Introduction

The study of recurring number sequences has a limited or little explanatory approach when we consult some mathematics history books adopted in Brazil (Boyer, 1991; Eves, 2004; Burton, 2007; Krantz, 2006; Hodgkin, 2010; Stakov, 2009). In an almost hegemonic way, many authors usually mention, discuss, and stress only some curious aspects and some occasional events related to the Fibonacci sequence and, to a lesser degree or attention, we point out the illustrative case of Lucas number sequence. On the other hand, the authors disregard a dynamic and unstoppable historical, mathematical, and epistemological-evolutionary process of these number sequences, in addition to incurring certain historical inaccuracies or omissions about this mathematical object (Alves, 2017a; 2017b; 2018a; Singh, 1985).

[1] fregis@ifce.edu.br

On the other hand, when we aim at teacher education in Brazil, it is essential to adopt systematic research assumptions, substantially conditioned by the very characteristics of mathematical and scientific knowledge. In this way, we adopt, in a complementary character, the premises of the didactic engineering for development (Perrin-Glorian, 1993; 2012) and the theory of didactic situations (Brousseau, 1997), seeking to develop a historical investigation that echoes on education, propose an extended systematic research in the historical-evolutionary and mathematical context, and describe a conception and structuring of didactic resources for mathematics teachers in Brazil, affected by a historical-mathematical perspective.

Moreover, not to disregard a natural process of historical evolution, on the one hand, and, on the other hand, the necessary acquisition of professional skills, the study also considers some assumptions of the professional didactics (PD), originated in France in the 1990s and applied to our scenario of interest, which, from a methodological point of view, emphasise elements that make it possible to understand the learning resulting from the professional activities of the mathematics teacher through the performance of specialised tasks required in the field of his/her occupation.

Thus, based on some of the assumptions indicated above, we point out the following question guiding our research: How to present and propose an itinerary for the initial formation of mathematics teachers taking as a reference a historical and epistemological context on the notion of recurrent number sequences, to enhance their knowledge on their professional and methodological skills?

The question is related to a more general but ever-present problem concerning the quality component of the initial training of mathematics teachers in Brazil (Alves, 2020b). Therefore, based on this guiding question, we indicate the following objectives to be examined in the course of the work: (i) Conceive, describe and propose a teacher education itinerary, originating in an evolutionary historical and epistemological context about the notion of recurring number sequences; (ii) Aiming at adopting a systematic design of investigation, replication, and reproduction of investigations, assume the concepts of the didactic engineering for development; (iii) Discuss and analyse the teacher’s learning in initial education, focusing on a file that examines the notion of professional competence and the analysis of his/her activity and adult learning, in view of the execution of professional and teaching tasks related to this mathematical object.

As we indicated before, aiming to define a mathematical content that allows for more significant repercussion for the school environment and its multiple implicit conceptual relationships for the teacher's mathematical culture, we chose the notion of recurring number sequences (see table 1), given the scarcity of research in Brazil on the subject, and its deficient approach in specialised textbooks on mathematics history (Alves, Catarino, & Rodrigues, 2020).

Thus, the objectives indicated above (i), (ii), and (iii) take as reference such mathematical content that, in Brazil, is usually discussed in a restricted way, for example, with the limited examination of the Fibonacci sequence, which became popular with the work of the mathematician Leonardo Pisano (1170 - 1250). We note, however, other examples of recurring number sequences, in some cases, recently introduced in the scientific literature (Catarino & Borges, 2020), that encourage the reader to have an expanded understanding of the theme or mathematical subject, as shown in Table 1.

Table 1. *Description of a set of ten recurring number sequences examined*

Name of the recurring sequence	Recurrence relation
Fibonacci sequence	$f_n = f_{n-1} + f_{n-2}$, with initial values $f_0 = 0, f_1 = 1$
Lucas sequence	$L_{n+1} = L_n + L_{n-1}$, with initial values $L_0 = 1, L_1 = 3$
Pell sequence	$P_{n+1} = 2P_n + P_{n-1}$, with initial values $P_0 = 0, P_1 = 1$
Jacobsthal sequence	$J_{n+1} = J_n + 2J_{n-1}$, with initial values $J_0 = 0, J_1 = 1$
Coordonier sequence	$C_{n+1} = C_{n-1} + C_{n-2}$, with initial values $C_0 = 1, C_1 = 0$
Perrin sequence	$Q_{n+1} = Q_{n-1} + Q_{n-2}$, with initial values $Q_0 = 3, Q_1 = 0$
Mersenne sequence	$M_{n+2} = 3M_{n+1} - 2M_n$, with initial values $M_0 = 0, M_1 = 1$
Oresme sequence	$O_{n+2} = O_{n+1} - 1/4 \cdot O_n$, with initial values $O_0 = 0, O_1 = 1/2$
Narayanna sequence	$N_{n+1} = N_n + N_{n-2}$, with initial values $N_0 = 1, N_1 = 1$
Leonardo sequence	$Le_{n+1} = 2Le_n - Le_{n-2}$, with initial values $Le_0 = 1, Le_1 = 1$ (Catarino & Borges, 2020)
Total Set	Ten recurring sequences and initial values

Source: Author's elaboration

In the next section, we will indicate the assumptions of this research that considers as a sample and discussion a set of investigations developed in a context essentially affected by the elements of historical, mathematical, and epistemological order, developed by a research group in mathematics teaching from the Federal Institute of Education, Science, and Technology of the State of Ceará, Brazil. The first set concerns the investigations concluded in Brazil between 2017 - 2020. The second set of investigations, still under development, corresponds to the period 2020 - 2023. From this second set, we will highlight, shortly below, some indicators for future research still under development in Brazil, including some initial indexes of research and of international cooperation with Portuguese mathematicians and educators (Alves, Catarino, & Rodrigues, 2020; Catarino, 2016; Alves & Catarino, 2019). Then, we will indicate the methodological assumptions and the research design adopted aiming at the systematic evolution of the work.

Background

The French strand of the didactics of mathematics (*Didactique des Mathématique*) (Brousseau, 1997) has a multi-theoretical style aimed at the adoption of various theoretical assumptions, with interest in the teaching and learning phenomena in mathematics, that derive from the relations of the classic trinomial student - mathematical knowledge - teacher. In this research context, with international recognition, we register the recurrent use of the theory of didactic situations and the notion of didactic engineering (Brousseau, 1996; Kusniak, 2004). Thus, we assumed the organising principles of our research and its development through a design known as didactic engineering that admits two categories, namely: classical or 1st-generation didactic engineering, didactic engineering for development or 2nd-generation didactic engineering.

To consider an approximate sample of 40 mathematics teachers in initial formation, we chose the didactic engineering for development, which allows us to focus more attentively on the role of the teacher. Admittedly, in the case of the classical didactic engineering, we know that greater interest is devoted to students' learning processes, however, since we showed greater attention to the activity and the teacher's learning, we took the assumptions of the didactic engineering for development (Tempier, 2016).

On the other hand, since we consider the mathematics teachers' specialised activity, and not neglecting the learning processes arising from their activity at work and teaching, we will extend Piaget's (1947) and Vergnaud's (1990) perspective, admitting specific assumptions of the French strand of the professional didactics (*Didactique Professionnelle*) that enable us to analyse the phenomena of the professional activity and learning of the adults, within the teaching context affected by elements of historical and epistemological order.

Regarding the sample of works considered, in a systematic way, we will take the data of investigations carried out in the period 2017 - 2020, in the context of initial teacher education, as indicated in table 2, given the achievement of some of the objectives declared (see (i), (ii) and (iii)). Soon after, we will present some preliminary indicators of investigations under development in Brazil and still foreseen for the period indicated by 2020 - 2023, which contribute to further referrals and necessary delimitation of future research, given international cooperation with Portuguese researchers. (Alves & Catarino, 2019).

Table 2 helps the reader to understand the first sample of works considered and developed in Brazil in the period 2017 - 2020.

Table 2. *Description of the works and investigations considered in the period 2017 - 2020*

Author	Research title	Description and interest
Dos Santos (2017)	*Uma Engenharia Didática para a noção de sequência extendida de Fibonacci: uma experiência no contexto do IFCE* (A didactic engineering for the notion of extended Fibonacci sequence: an experience in the context of the IFCE)	The dissertation involved the development of research with interest in the generalisation process of the Fibonacci sequence and the proposition of didactic engineering.
Oliveira (2018)	*Engenharia Didática com o tema: relações bidimensionais, tridimensionais e n-dimensionais do modelo de Fibonacci* (Didactic engineering with the theme: two-dimensional, three-dimensional and n-	The dissertation involved the development of research with interest in the n-dimensional process of representation of the Fibonacci sequence and the proposal of didactic engineering for development.

	dimensional relations of the Fibonacci model)	
Guedes (2020)	*Uma Engenharia Didática para o estudo da Sequência Generalizada de Lucas (SGL)* (A didactic engineering for the study of Lucas generalised sequence).	The dissertation involved the development of research with interest in the generalisation process of the Fibonacci sequence and the proposition of didactic engineering for development.
Mota (2019)	*Um estudo sobre as generalizações da sequência de Pell a aplicações com o Maple* (A study on generalisations of the Pell sequence to applications with Maple)	The dissertation involved a systematic analysis of mathematical and epistemological nature on the sequence and its generalisations and use of technology.
Noronha (2020)	*Engenharia Didática sobre os modelos de representação da sequência de Pell* (Didactic engineering on the representation models of the Pell sequence)	The dissertation presents a set of didactic situations addressed to mathematics teachers involving the Pell sequence.
Souza (2020)	*Sequência Generalizada de Jacobsthal: uma engenharia didática sobre seu ensino* (Jacobsthal generalised sequence: a didactic engineering on his teaching)	The dissertation involved the development of research with interest in the generalisation process of the Fibonacci sequence and the proposition of didactic engineering for development.
Total	**6 (six) research projects.**	

Source: Author's elaboration.

Theoretical Framework

The French strand of the didactics of mathematics introduced, decades ago in Brazil, a significant influence on the studies focused on the phenomena arising from teaching and learning in mathematics. Based on an eminently multi-theoretical bias, we found the adoption of frames of reference and a technical and scientific corpus dedicated to the systematic study of scientific principles that shape research (Alves, 2016; 2017a; 2017b; 2018b; Alves & Catarino, 2019), detailed by the teaching-learning binomial in mathematics.

The tradition of an investigation design adopted by the French strand of the didactics of mathematics (DM) has its origins in the 1970s and was consolidated in the 1980s and 1990s (Artigue. 2014; Artigue et al., 2019; Brousseau, 1986). However, a significant return to the fundamentals and their examination was needed, as the French specialists registered obstacles and difficulties of repercussion and the dissemination of a technical and scientific apparatus aiming at the research. In turn, Perrin-Glorian and Bellemain (2016) describe an evolutionary path of employment, in a multi-theoretical character, of the traditional notion of engineering, combined with the use of the situation theory (Brousseau, 1986; 1997). In this sense, Artigue et al. (2019) explain some elements that contributed to its emergence and corresponding historical evolution in the European context of the DM, arguing that:

> The didactics of mathematics emerged in France with the aim of building a genuine field of scientific research and not just a field of application for other scientific fields such as mathematics or psychology. Thus it required both fundamental and applied dimensions, and needed specific theories and methodologies. Drawing lessons from the innovative activism of the New Math period with the disillusions it had generated, French didacticians gave priority to understanding the complex interaction between mathematics learning and teaching in didactic systems. Building solid theoretical foundations for this new field in tight interaction with empirical research was an essential step. Theories were thus and are still conceived of first as tools for the understanding of mathematics teaching and learning practices and processes, and for the identification of didactic phenomena. It is usual to say that French didactics has three main theoretical pillars: the theory of didactical situations due to Brousseau, the theory of conceptual fields due to Vergnaud, and the anthropological theory of the didactic that emerged from the theory of didactic transposition, due to Chevallard. These theories are complex objects that have been developed and consolidated over decades. (Artigue et al., 2019, p. 14)

As a necessary consequence for the constitution, consolidation, and recognition by specialist educators of an autonomous field of investigation, equipped with its own constructs, instruments, and a systematic perspective of approaching problems derived from mathematics teaching and learning, we identified the evolution of the notion of didactic engineering which, according to Artigue (2015, p. 468), had been a consequence of a scenario of cultural interest in France since 1980, where summer schools on didactics of mathematics and its advances were organised every two years. In this sense, Artigue (2015, p. 468) indicates the following scenario of genesis on the didactics of mathematics, according to the French perspective, as a new scientific field of research:

> Accessible documents about this summer school show the shared belief that didactic research should give a more central role to the construction and study of

> achievements in the classroom. French researchers expressed concern about the observed tendency to favour methodologies borrowed from established fields, such as psychology (clinical interviews, questionnaires, pre-test/post-test comparisons...) to ensure the scientific legitimacy of research in mathematics teaching. They pointed out that the didactics of mathematics is a true scientific field whose methodologies must be aligned with its specific objective: the study of the intentional dissemination of mathematical knowledge through didactic systems and the interaction associated between the teaching and learning processes." (Artigue, 2015, p. 468)

As mentioned earlier, we registered a multi-theoretical character in didactics of mathematics, so we chose to use of the theory of didactic situations (Brousseau, 1997) and didactic engineering (Artigue, 2015) as a theoretical framework, investigative design, and organisational and structuring principle research, and to adopt assumptions that guarantee the necessary scientific rigour. Thus, with the support of some important scientific works on the notion of the second-generation engineering (Barquero & Bosch, 2015; Tempier, 2016; Tempier & Chambris, 2017), we point out the necessary adaptations for our case (see figure 1).

In Figure 1, we point out a flowchart developed in research carried out by Barquero and Bosch (2015), in a context of teacher education. From figure 1, we assume the following investigation phases: (i) Preliminary analyses; (ii) Design and analysis *a priori*; (iii) Implementation, experimentation and observation of data; (iv) Post-analysis and development. In figure 1 we see a conceptual research itinerary proposed by Barquero and Bosch (2015) in the context of the initial education of science and mathematics teachers in Spain.

In turn, above, in figure 2, we observe an itinerary followed by Tempier (2013) in his doctoral thesis, which involved the proposal of a pedagogical device and resource for mathematics teachers in the context of French teaching. On the left side of figure 2, we see a concern with a didactic transposition process related to the "numbering" content. Throughout the flowchart indicated by Tempier (2013), we identified interest in the dissemination of research results, supported by the assumptions of didactic engineering for development and its diffusion in ordinary and French teaching. Similar to the works of Barquero and Bosh (2015) and Tempier (2013), we considered the assumptions of a study design, capable of guaranteeing our systematic investigation and preserving the necessary scientific rigour.

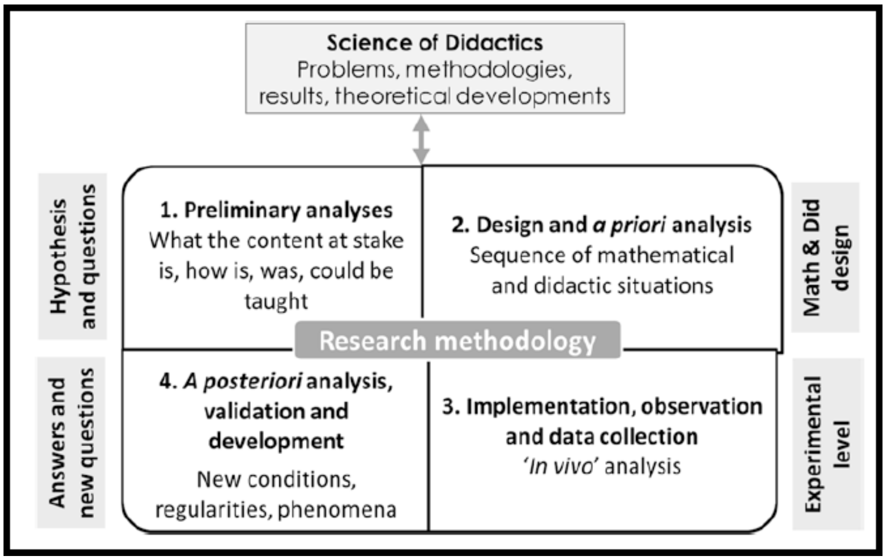


Figure 1: Scheme proposed by Barquero and Bosch (2015) for research with science and mathematics teachers involving the use of didactic engineering for development.

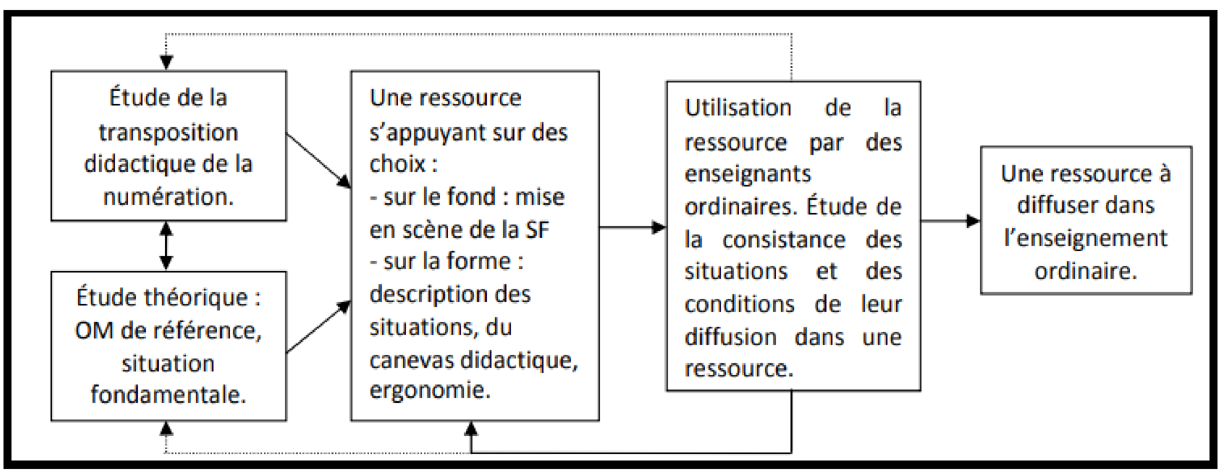


Figure 2: In his doctoral thesis, Tempier (2013) develops a study with the support of didactic engineering for development and interest in proposing a resource on the notion of numbering.

On the other hand, since we expressed interest in examining a set of professional skills of the mathematics teacher by incorporating elements of a historical nature and the interest in the corresponding expansion of their mathematical culture and their professional teaching culture

(Alves, 2018a; 2018b; 2018c; 2018c), we will also embrace some assumptions of the French aspect of the professional didactics (Alves, 2019a; 2019b, 2019c), given the understanding of the phenomena derived from learning, which involve the mathematics teacher's mobilisation of knowledge in the situation, whose nature goes beyond the classic epistemic field that defines mathematical knowledge. Thus, in the subsequent section, we will indicate the assumptions made from another strand of studies originating in France.

Professional Didactics

The French aspect of the professional didactics is constituted in a zone of confluence and interface of studies of the French aspect of work psychology, ergonomics and theory of the conceptualisation of action. The French aspect of the professional didactics has some interface with the didactics of mathematics (Alves, 2020a), both originating in France, insofar as it indicates certain elements forming from the old European notion of engineering for development and the activity developed at work.

In this sense, Pastré, Mayen, and Vergnaud (2006) explain the scenario of interest in professional didactics below and emphasise some assumptions of the notion of engineering for development that differs from the interest in learning of characteristic scientific knowledge in the context of schools. The authors comment and indicate the following field of interest.

> It is a field of practices that consists of building the formation devices corresponding to the needs identified for a known audience, in their group or work environment. School education tends to decontextualise learning. Engineering for development will, on the contrary, insist on the social context in which adult learners should be educated. Such adults are, initially, workers. When they decide to get a qualification, it is usually convenient for their work, and not from disciplinary excerpts that they usually find meaningless." (Pastré, Mayen, and Vergnaud, 2006, p. 147)

In another work, we identified a clear influence of the French aspect of the didactics of mathematics, in view of the delimitation of problems and the scenario of interest in research in professional didactics. In this sense, Pastré (1999) clarifies that:

> The mathematicians - we think of Brousseau in a special way - constructed an extremely suggestive theoretical framework around the notion of situation. However, it is necessary to add that, in the case of professional situations, the relationship between situation and problem acquires a particular bias. In the case of the didactics of the disciplines, we can say that there is a homology between situation and the problem that it considers. But, it is the problem that occurs first and the situation has the function of exemplifying the problem posed and gives it meaning. Otherwise, in professional didactics, a situation has a trajectory beyond it allows to address. We saw that the conceptual structure of a situation is an organisation, therefore, a reduction of the situation as a whole. (Pastré, 1999, p. 28)

The interest of the assumptions of the notion of "engineering," used from a research design point of view in the qualification scenario, differs from the notion of the didactic engineering - whether 1st or 2nd generation - that we mentioned in the previous section, since it considers an epistemically demarcated and conditioned terrain by mathematical and scientific knowledge of reference (*le savoir*). On the other hand, the field of interest and examination of praxis considered by the professional didactics takes into consideration an essentially pragmatic knowledge, of a circumstantial nature, acquired before a set of professional tasks and activity at work (Alves, 2019a; 2019b; 2019c).

In the case of the mathematics teacher, however, the repercussion of the professional didactics lies on the consideration of the knowledge needed for the good performance of activities and routines that are not located only inside the classroom (Alves, 2020). In fact, Pastré (2004; 2011) highlights a qualitative change in perspective and understanding of labour relations at work, insofar as it starts to consider a qualitative bias of strategic application of procedures aimed at solving and examining complex, erratic problems and not trivial at work, as noted below:

> And, consequently, a crisis in the prescription of work: when we are dealing with a dynamic environment, the application of procedures is no longer sufficient as a guide for the activity. Operators must be able to diagnose the situation at any time, which becomes a central element of competence. Other ways of working put operators in problem-solving situations. The interesting thing is that the problems to be solved have multiple dimensions and an acceptable compromise must be found between those dimensions.

> We are in what we could call the strategic intelligence of the situation, where it is no longer a question of finding out where the error or dysfunction is, but a question of building a solution that considers the different dimensions of the problem. (Pastre, 2004, p. 4)

One aspect that stands out from the elements indicated above by Pastré (2004) concerns the polysemic character or notion of what we call professional competence. In this sense, the professional competence of the mathematics teacher will be revealed through the overcoming and facing of erratic, non-trivial, resilient and intrinsically derived problems of the teaching activity. Therefore, the professional didactics allows us to objectify the work and professional development (of the mathematics teacher) from a dialectical point of view (Alves, 2019a; 2019b; 2019c; 2020), as it does not allow us to dissociate knowledge from action and (professional) situation. In this way, the current of the conceptualisation of action, originating from Piaget's thoughts (1974), made it possible to expand a perspective of analysis, since "knowledge is fundamentally an adaptation, how humans can adjust to their environment" (Pastré, 2011, p. 86).

We also observed that cognition, for Vergnaud (2007), for example, is transformed into conceptualisation in action, with a robust Piagetian heritage. Thus, professional didactics considers a relatively prosaic idea: for the analysis of competencies, we must analyse the effectiveness of actions and their forms of organisation, through the identification of organisational ontological entities (cognitive schemes and operative invariants) of the subject's action at work. Consequently, "analysing competencies becomes an analysis of the organisation of action" (Pastré, 2002, p. 11), given the identification of elements that are responsible for their invariance and regularity.

Vergnaud (2007) confirms some arguments made explicit by Pastré (2002). Below, Vergnaud (2007) reveals the real content and nature of the notion of professional competence. His description is irremediably linked and conditioned by the notion of adaptive plastic capacity of the individual (worker or teacher), in the face of a set of (professional) situations that are characteristic and fundamental for the effective exercise of a given *métier* and which irrevocably confer a pragmatic component related to the notion of "professional competence in a situation."

> The concept of competence is not for me a scientific concept; it is a pragmatic and practical concept that is useful, that I use constantly, because I am part of a human community in which we need to communicate, including with terms that are not systematically learned. Even so, I will talk about concepts learned, like schemes, operational invariants, concepts in action, theorems in action, because, for teacher trainers, these are essential concepts. If I want to be operational, I need to examine skills in situations and, in particular, classes of situations. In fact, we are not competent for a singular situation, but, in general, for situations that belong to a certain class, that have specific characteristics. We need to analyse the activity in relation to the characteristics of the situations. (Vergnaud, 2007, p. 1)

To conclude this section, we note that we do not intend to provide an exhaustive description of all the fundamental concepts for the professional didactics and, in turn, consider some elements of more significant repercussion for the activity and the formation of the mathematics teacher. In this sense, the professional didactics allows us to target the necessary skills and strategic decisions for professional development and the progressive constitution of identity and the required professional competence in the profession (Alves, 2020).

In this sense, in our works (Alves, 2020b), we have pointed out some fundamental binomials and, based on these relationships, we can establish elements that constitute the notion of professional competence. In fact, we considered the binomial “teacher-students” and, detailed in the classroom, we aimed at the activities developed by the teacher, seeking learning in a landscape of investigation in the context of mathematics history. In turn, the mathematics teacher’s activity cannot be limited to the classroom; thus, we consider the binomial “teacher-teachers.” Regarding the set of relationships considered in this binomial, we can observe that the repercussion of a mathematical culture that values elements of a historical nature requires an understanding on the part of the mathematics teachers, and an understanding of the messages, the exchange of experiences, a set of professional gestures typical of teaching activity and the reproduction of pragmatic knowledge on the topic. For example, a desired pragmatic conception involves the understanding that an in-depth mastery of elements of a historical nature highlights an essential component for the notion of professional competence.

Furthermore, the acquisition of knowledge of a historical-mathematical nature, such as, for example, on the notion of recurring number sequences from its genesis and the current evolution is an element to be recognised for its relevance, among the mathematics teachers themselves. Finally, a third binomial considered is indicated by "teacher-educational institution." For the latter case, we can have as objective whether historical knowledge is valued and, in fact, acquires a representative space in Brazilian school curricula, not only as secondary and accessory elements.

The relationships indicated above contribute to the identification of a mathematical culture. Also, preserving and promoting a professional culture that we have identified is needed for the constitution of the notion of professional competence and, in this case, the assumptions of the professional didactics add other elements disregarded by the didactics of mathematics, whose field of discussion is shown to be eminently conditioned by mathematical and scientific knowledge. However, we understand that for the exercise of their profession, teachers must master not only mathematical knowledge, but also a wide repertoire of pragmatic knowledge and circumstances derived from the their activity that cannot be a goal only inside the classroom.

Some results on investigations done in the period 2017 - 2020.

In this section, we will report some results of research carried out in the period 2017 - 2020. The set of investigations done can be found in table 2. In a traditional way, we state that mathematics history books usually privilege, in a limited way, the discussion of episodic elements of the Fibonacci sequence. Thus, we highlight the research by Dos Santos (2017) and Oliveira (2018). Both works were dedicated to the study of the evolutionary mathematical model and the generalisation of the Fibonacci sequence, taking as reference a historical time frame, from the year 1202 to the present day, with an understanding of the current research. Dos Santos (2017) and de Oliveira (2018) developed a study involving about 30 basic education teachers in the city of Fortaleza, at the Federal Institute of Science and Technology Education of the State of Ceará, Brazil, taking as a reference the context of initial teacher education.

To exemplify, Dos Santos (2017) provides a learning scenario that stimulated the understanding of the historical and evolutionary process about the Fibonacci sequence. For example, in the excerpt of a student participating in the research, we can record his understanding of the process of extending the Fibonacci sequence to the field of integer indexes f (-1), f (-2), by stating that:

> "[...] the surprise was that until then, for the natural numbers, I knew it, now, when it came to the third question, and it was dealing with f (0), f (-1) and f (-2), for me it could not happen, because as far as I knew, the Fibonacci sequence was only for natural numbers [...]" (Dos Santos, 2017, p. 103)

In another case, in work developed by Souza (2020), we recorded examples of didactic situations created to promote an investigative scenario of a historical and evolutionary nature about the Jacobsthal generalised sequence, inspired by the works of the German mathematician Ernst Erich Jacobsthal (1882 - 1965). We emphasise the considerations below of a mathematics teacher participating in the study, when he saw the introduction of a variable 'x' in the recurrence formula, as we see below:

> "The initial sequence and the polynomial sequence behave in the same way, including the initial terms, and differ with the addition of the x value. Thus, for the polynomial sequence, the following terms generated are polynomials. [...] the difference from the first to the second is the variable that we don't know is this x here, so it would be as if it were the result of a polynomial [...]. What we can see is that from the second we will already have a polynomial in the variable x." (Souza, 2020, p. 146).

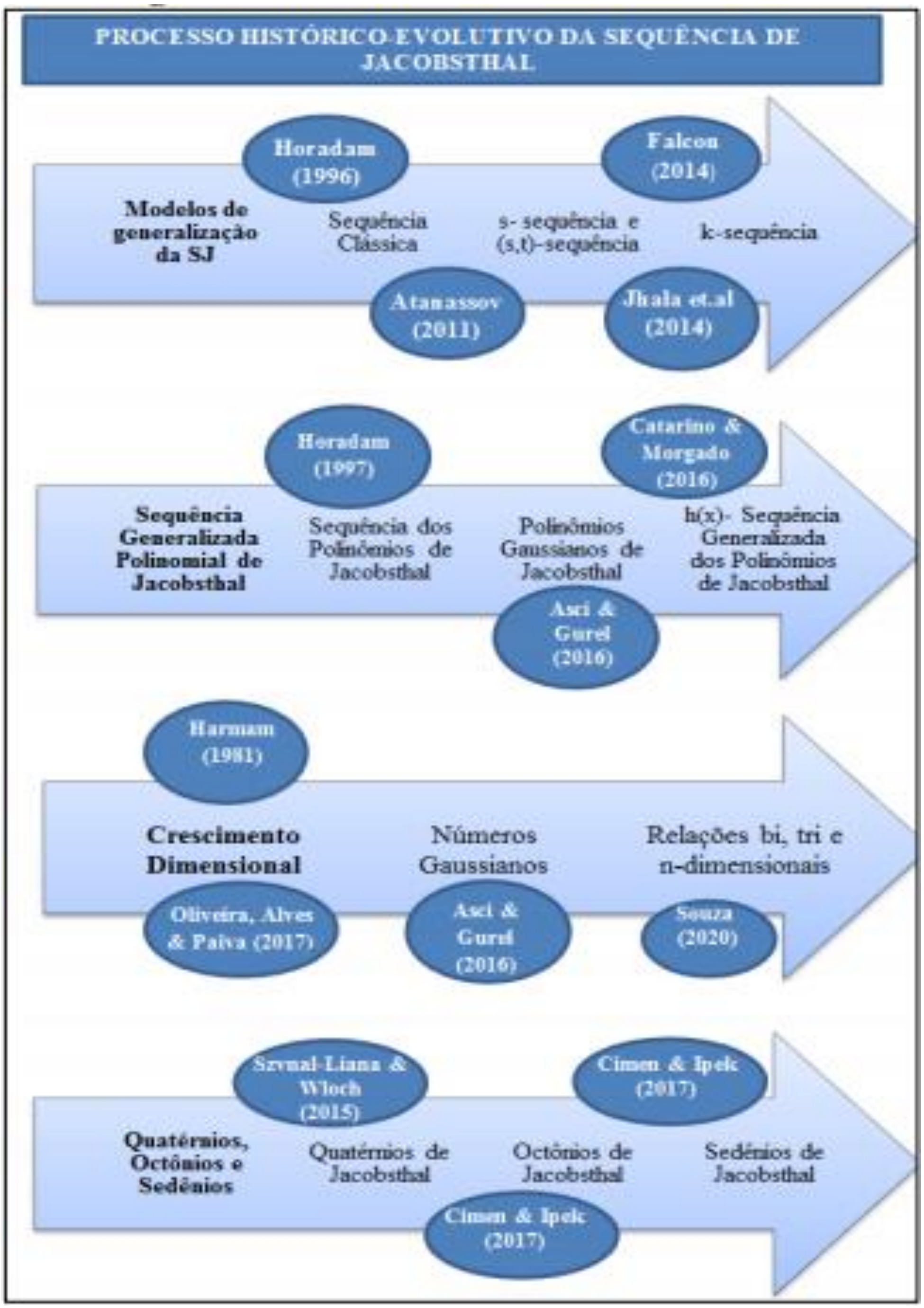


Figure 3. Sousa (2020) developed a model to understand the historical mathematical and epistemological evolution on Jacobsthal sequence to the present day, in the stage of *a priori* analysis.

In the discussion scenario about the Jacobsthal sequence (see figure. 3), Souza (2020) followed the investigation of a group of ten teachers around the properties of the recurrence introduced by

Horadam (1997). In this sense, we can compare the classic recurrence relation shown in table 1 with the relation proposed by Horadam (1997) of the form $J_{n+1}(x) = J_n(x) + 2x \cdot J_{n-1}(x)$, $J_0(x) = 0, J_1(x) = 1$, for whole indices $n > 1$. Souza (2020) also presented an investigation itinerary that allowed the participating teachers to understand that, similarly to the case of the Fibonacci sequence, we can also define the polynomial Jacosbsthal sequence for negative integer indices, a sense for the notation $\{J_{-n}(x)\}_{n=0}^{+\infty}$.

In figure 3, we can see that Sousa (2020) described a mathematical and evolutionary itinerary on the Jacobsthal sequence. In Brazil, there are few books on mathematics history that make it possible to provide elements involving the study of the Jacobsthal sequence. In figure 4, for example, we visualise a scenario of a teacher's learning and discovering mathematical properties, when he found matrix relations related to the Jacobhstal sequence and other properties related to basic school concepts.

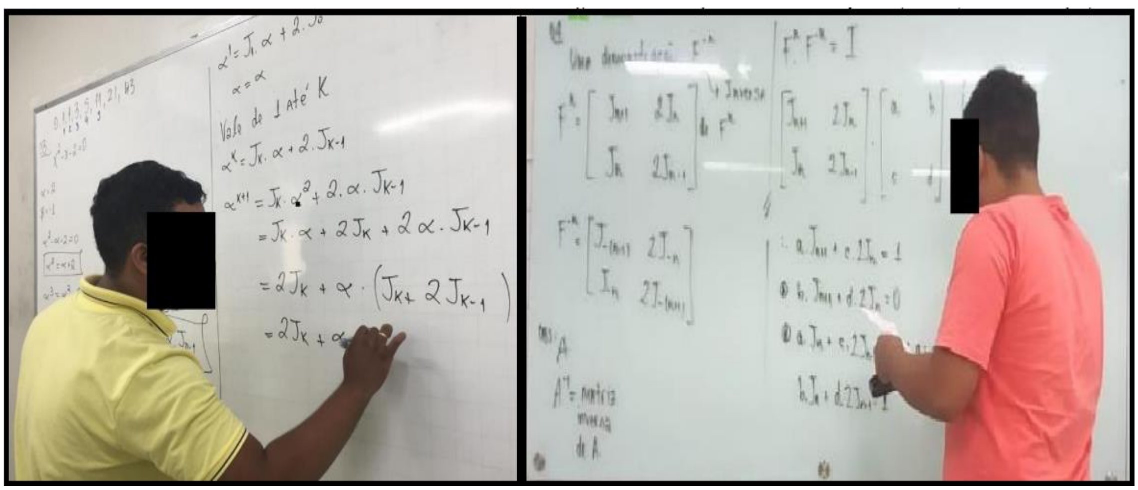
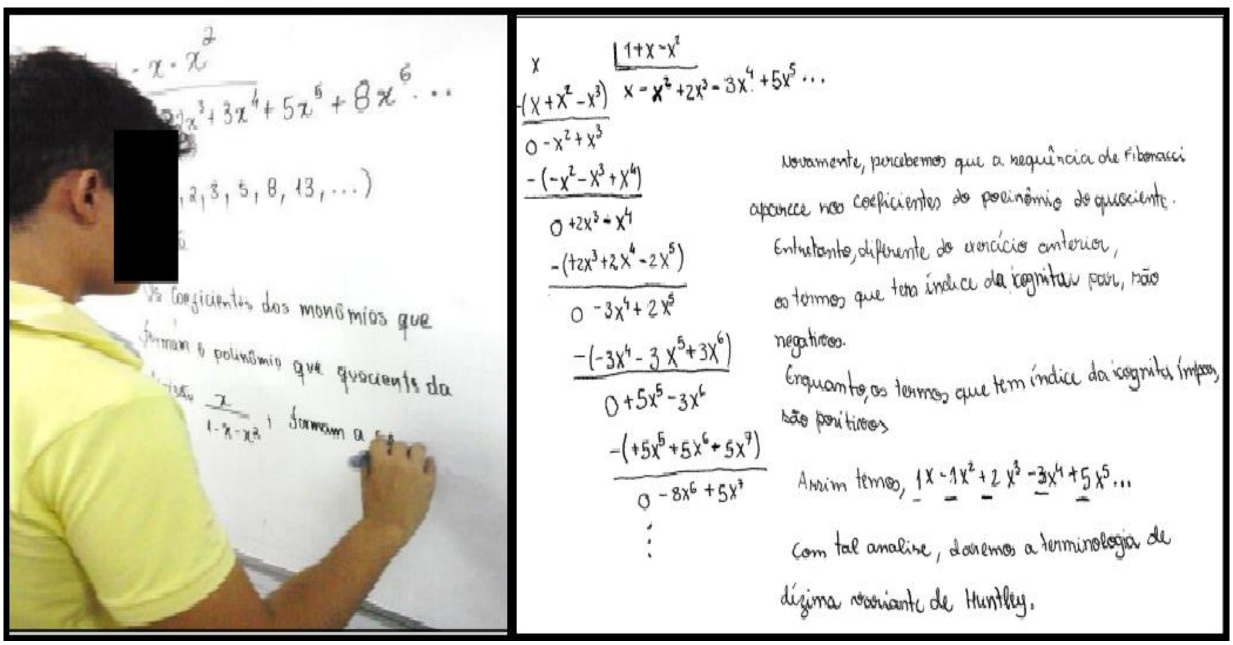


Figure 4. A study developed in Brazil (Souza, 2020) involved teachers in a process of historical research on the Jacobsthal generalised sequence and relations with Fibonacci.

In turn, Oliveira's dissertation (2018) can be characterised as a second part or continuity and extension of Dos Santos's dissertation (2017), since it considers an expanded epistemological and mathematical field of the Fibonacci sequence. Below, a teacher participating (in initial education) in the study by Oliveira (2018) expressed understanding about the generalisation process of the Fibonacci sequence, especially with the use of polynomial representation.

> "[…] the formula of the first one is very basic, elementary. In the second one, he perfected what already existed, added the variables, now you can work with polynomials from these formulas. In the last line, he was already working with polynomials, but at Fibonacci, he will work in complexes. Always extending […]" (Oliveira, 2018, p. 185)

In figure 5, Oliveira (2018) discussed the activity developed by a professor participating in his study, when he detailed specific properties of the Fibonacci generalised sequence, from polynomial representations and the insertion of a real variable and a complex variable to the recurrence model/formula. On the left side of figure 5, we can see the professor carrying out a property discovery activity.



Picture 5. A study developed in Brazil (Oliveira, 2018) involved teachers in the process of historical research on the Fibonacci generalised sequence.

The elements mentioned above and further information can be consulted directly in the papers and scientific articles on the subject (Dos Santos & Alves, 2017; Oliveira & Alves, 2019).

Some results expected over the period 2020 - 2023.

In the previous sections, we indicated some results obtained through a set of investigations carried out in the period 2017 - 2020. Now, in the set of studies and investigations under development in Brazil and planned for the period 2020 - 2023, we will highlight the following themes and objects of study under development at the Federal Institute of Education, Science and Technology:

- Didactic engineering with the theme: the generalised sequence of Padovan or Coordonier;

- An investigation of a historical, mathematical, and evolutionary nature on the Narayanna generalised sequence and Narayanna numbers, whose traces of Indian mathematics are valued and contributed by the Indian mathematician Narayanna Pandita (1290 - 1370);

- Didactic engineering on the notion of recurrent n-dimensional sequences and the sequence hybridisation process, involving correlations with hybrid numbers;

- N-dimensional figures: engineering for development and implications for the activity of the mathematics teacher (Barros & Alves*, 2019;* Barros; Alves; Vieira & Catarino, 2020)*;*

- Implications of international research and scientific cooperation involving Portuguese researchers on the progress of scientific work on recurrent sequences and the publication in international journals (Alves, 2020a; Campos et al., 2014; Catarino, 2015; 2016; 2019; Catarino; Campos & Vasco, 2019).

We observed that in all investigations examined, the phases of preliminary analysis, *a priori* analysis, design and experimentation and, finally, *a posteriori* analysis are developed.

For example, authors of history books comment, recurrently, on the case of the Fibonacci sequence, i.e., that it is possible to extract the number from another $\frac{1+\sqrt{5}}{2}$. In the case of the Padovan or Coordonier sequence, described by $C_{n+1} = C_{n-1} + C_{n-2}$, $C_0 = 1, C_1 = 0$, we determine the characteristic polynomial equation $x^3 - x - 1 = 0$. As explained by Iliopoulos (2015), such characteristic equation has a single real root (plastic number), with approximate value $\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\cdots}}}=\psi=1.324718...\approx\frac{4}{3}$. Thus, based on some of the properties of the Padovan or Coordonier sequence, we can target at possible interfaces with current technology and promote visualisation and mathematical intuition (Alves, Vieira, & Catarino, 2020). In figure 7, we see some trends of generalisation of the Padovan or Coordonier sequence, which should stimulate an understanding of the non-static character and, above all, the evolutionary bias of the mathematical knowledge.

We point out, however, some elements and guiding assumptions for the work in progress, such as the identification and description of relationships involving 2D/3D and n-dim figural numbers with the notion of 2nd-order recurring sequences and the description of teaching structured didactic sequences involving the use of GeoGebra *software* and a particular interest in visualisation (Barros, Alves, Vieira, & Catarino, 2020; Barros & Alves, 2020).

On the other hand, identification and description of relations involving hybrid numbers (Ozdemir, 2018), described by $h=a+bi+c\varepsilon+dh$, in which $i^2=-1,\varepsilon^2=0,h^2=1$, with the notion of 2nd-order recurring sequences and the description of teaching sequences (Mangueira & Alves, 2020; Mangueira, Alves, & Catarino, 2020). Figure 12 presents the image of a formalisation activity developed by a mathematics teacher involving the notion of hybrid number $h=a+bi+c\varepsilon+dh$ (Ozdemir, 2018) and Mersenne recurring sequence. For further details, we suggest consulting the work of Mangueira & Alves (2020) directly.

To exemplify the case of the research in progress on the Padovan or Coordonier sequence, we present the participating teacher's explanation of the didactic situation (at the time or phase of institutionalisation) developed and which we can see in figure 6 (just below), after developing a series of matrix properties and recurrence relationships. He eventually understood and identified the relations of the matrix representations with their representation via linear systems with the Padovan or Coordonier sequence, when he stated that:

> "[...] this formula will be basically the same as Padovan's, and when solving the system of linear equations, we must enter the generalised initial values to obtain the new Binet formula of these numbers similar to those of Padovan [...] " Source: Research data

In the experimentation stage, we present the participating teacher's explanation of the didactic situation (at the time or phase of institutionalisation), who expressed his enthusiasm when he discovered mathematical properties related to certain recurring sequences. Only the teachers involved in the investigation were provided with some examples of the use of technology when some fractals (3D) related to the Padovan or Coordonier sequence were shown, as exemplified in figure 7. Such a learning scenario presented to teachers is an essential element for their initial education and expansion of their mathematical culture.

Then, we present the participating teacher's illustration of a didactic situation involving the discovery of properties of the Padovan or Coordonier sequence, through the use of matrix representations and mathematical induction.

> [...] I felt I was building my knowledge, even more, because I had never seen this sequence before, not even in mathematics, let alone in the way that is being passed on to us [...]. Some of those issues are not found in the literature; this shows more interest from people in the resolution of the exercises [...]. (Source: Research data)

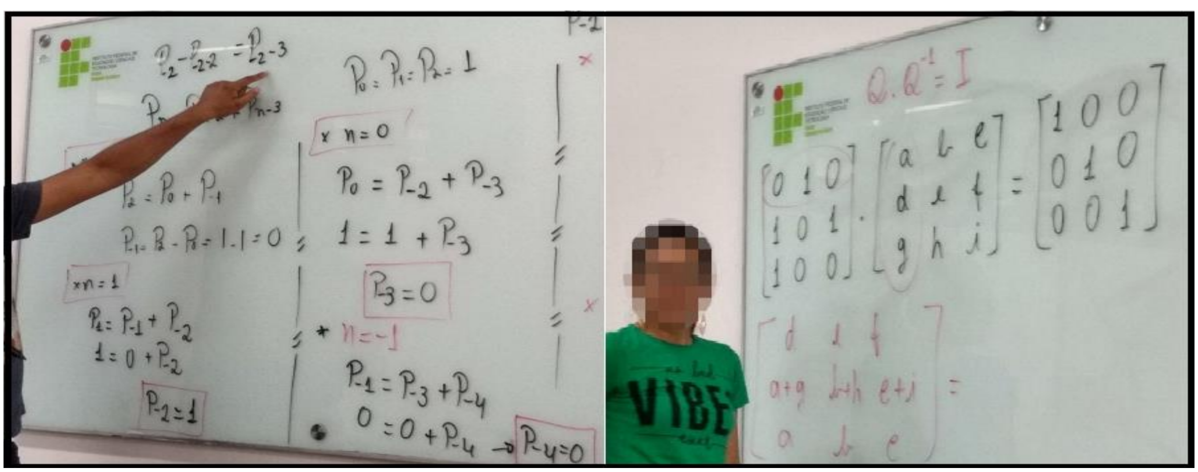


Figure 6. Description (in the experimentation phase) and discovery of properties on the Padovan or Coordonier sequence under development in Brazil. (Source: Vieira, 2020)

In Table 3, some examples of generalisation of two-dimensional recurrence relationships. From the historical point of view, we point out the works of Harman (1981) and Hoggat and Vernner (1979) , pioneers in the introduction of the two-dimensional, three-dimensional and n-dim Fibonacci recurrence relations. As a result of this work, we indicate below the results published in journals related to the works of Mangueira, Vieira, Alves, and Catarino (2020), Vieira, Alves, and Catarino (2020) and Vieira (2020). We point out the three-dimensional, four-dimensional and n-dim representations derived from the sequences (see table 1) that can be developed and become the object of teaching and research for the mathematics teachers involved.

Table 3. *Description of two-dimensional recurring number sequences*

Name of the recurring sequence	Two-dimensional recurrence relation
Two-dimensional Fibonacci Sequence	$f(n+2,m)=f(n+1,m)+f(n,m)$, $f(n,m+2)=f(n,m+1)+f(n,m)$, $f(0,0)=0,\ f(1,0)=1$ $f(0,1)=i,\ f(1,1)=1+i$, with $i^2=-1$. (Harman, 1981) and (Oliveira & Alves, 2019)
Two-dimensional Coordonier or Padovan Sequence	$C(n,m)=C(n-2,m)+C(n-3,m)$, $C(n,m)=C(n,m-2)+C(n,m-3)$ $C(0,0)=1,C(1,0)=0,C(2,0)=1$, $C(0,1)=1+i,C(0,2)=1+i,C(1,2)=1+i$, $C(2,1)=1+i,C(2,2)=1+i,i^2=-1$. (Vieira, 2020)
Two-dimensional Perrin Sequence	$Q(n,m)=Q(n-2,m)+Q(n-3,m)$, $Q(n,m)=Q(n,m-2)+Q(n,m-3)$. $Q(0,0)=3,Q(0,1)=3+2i,Q(1,0)=0,Q(1,1)=2i,$ $Q(0,2)=3+3i,Q(2,1)=2+2i,Q(1,2)=3i,Q(2,2)=2+3i$. (Mangueira, Vieira, Alves, & Catarino, 2020).
Two-Dimensional Leonardo Sequence	$Le(n+1,m)=2Le(n,m)-Le(n-2,m)$, $Le(n,m+1)=2Le(n,m)-Le(n,m-2)$, $Le(0,0)=1,Le(1,0)=1$, $Le(0,1)=1+i,Le(1,1)=1+i$,

	$i^2 = -1$. (Vieira, Alves, & Catarino, 2019).
Total Set	Four two-dimensional recurring sequences

Source: Author's elaboration.

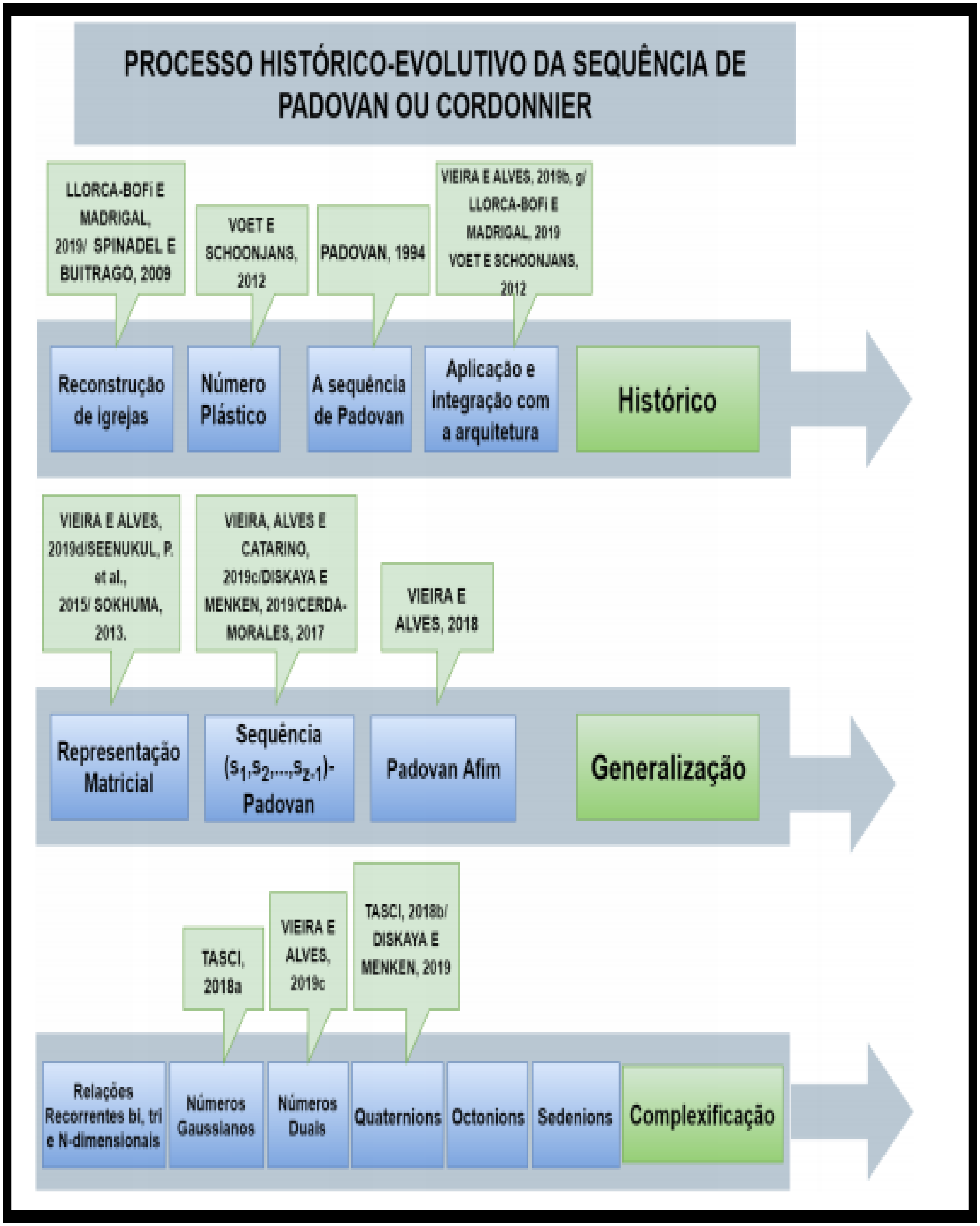


Figure 7. Scheme that indicates the trends of generalisation the Padovan or Coordonier sequence corresponding to research under development in Brazil, in the *a priori* analysis stage. (Vieira, 2020)

Yet, we cannot disregard a scientific cooperation movement involving Brazilian students and researchers from Brazil and Portugal, whose work in research in pure mathematics drives countless developments, and research on the design of structured teaching sequences on subjects usually limited to articles, and others introduced as a result of the research. Starting from this process of increasing scientific cooperation, we pointed out the possibilities of seeing further developments and the generalisation of numerous mathematical properties and their interface with current technology, through the use of *software* such as *GeoGebra*, *CAS Maple*, *Maxima*, and *Google Colaab*. (See figures 8, 9 and 10).

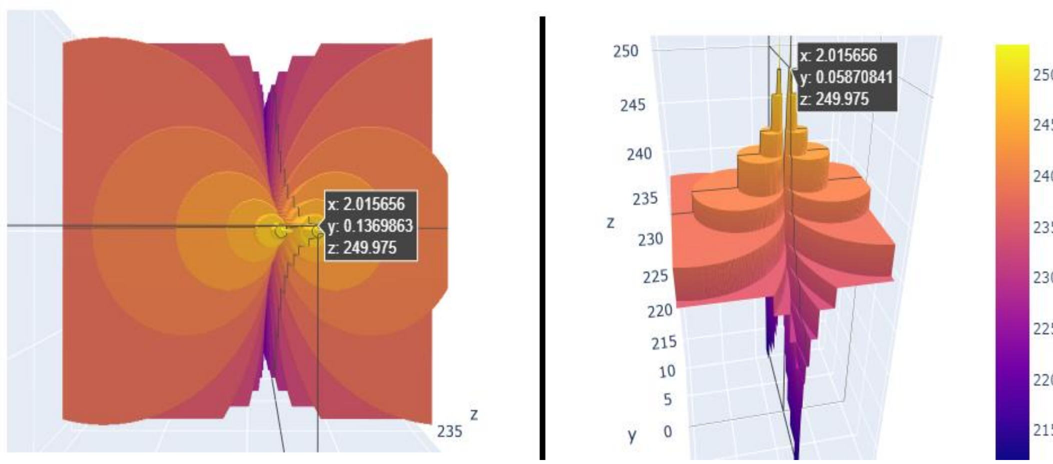


Figure 8. Study and visualisation of the 2D and 3D Mersenne Fractal using Google Colaab discussed in the Brazil x Portugal cooperation work developed by Alves, Vieira, and Catarino (2020).

Recently, we can observe the exploration of multiple representations related to the notion of number sequences. In the works of Alves and Vieira (2020), we identified the exploration of 2D and 3D fractals related to the Leonardo sequence. In works by Alves, Vieira, and Catarino (2020) and Alves, Catarino, Vieira, and Mangueira (2020), we find several examples of fractal representations little known in the literature, related to the set of the ten number sequences that we indicated in table 1. In figure 8, we see an example of a fractal related to the Mersenne sequence, and then, in figure 9, we see an example of 2D/3D fractal related to the Leonardo sequence. Such number sequence was recently introduced into the scientific literature by Catarino and Campos (2020).

In the figure below, using the GeoGebra *software,* we present an example of a 3-dimensional stellar figural number. With the methodological use of the *GeoGebra software*, we register a research in development that allows the creation of a landscape of learning and investigation on the n-dimensional figures and their multiple geometric representations, with computational interface.

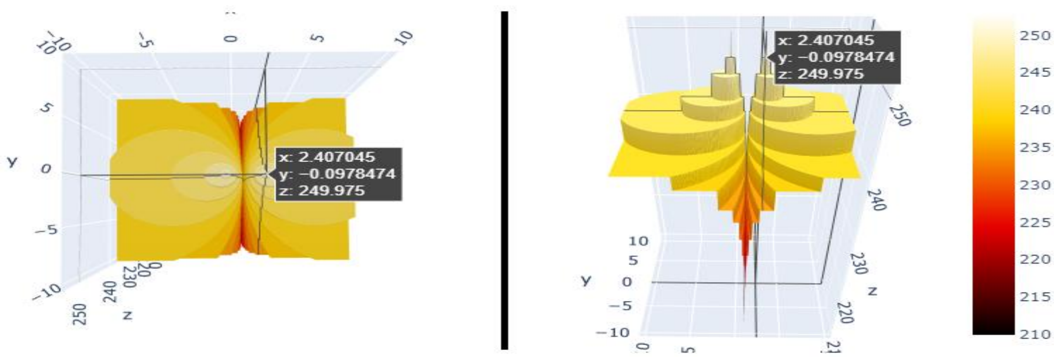


Figure 9. Study and visualisation of the 2D and 3D Mersenne Fractal using Google Colaab discussed in the Brazil x Portugal cooperation work developed by Alves, Vieira, and Catarino (2020).

In works by Alves and Barros (2019) and Barros, Alves, Vieira, Mangueira, and Catarino (2020), we can examine a script for the investigation of figural numbers, their unexpected relations with the notion of recurring number sequences, and implications for the exploration of technology involving the use of some software (*Cas Maple, GeoGebra, Google Coolab*, etc.), in the context of teacher education, with an emphasis on visualisation. In figure 10, we show an example of a representation of a 3D stellar figure number.

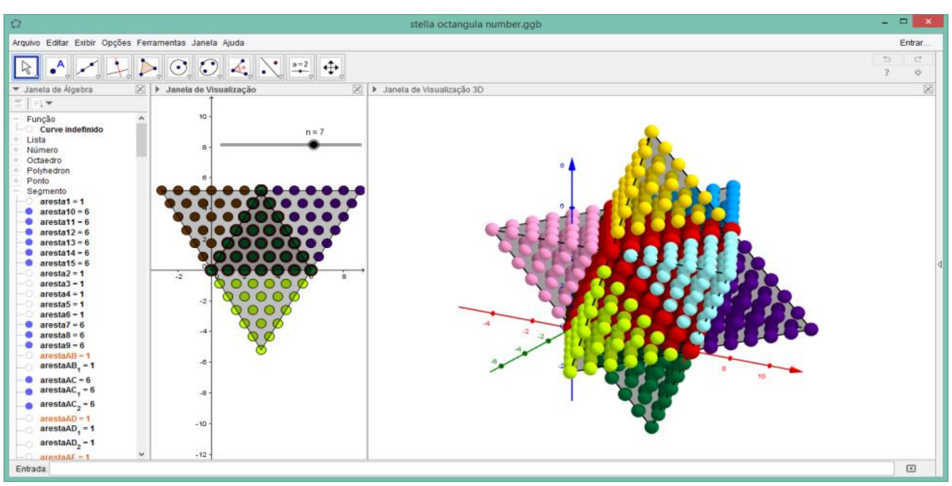


Figure 10. Research in progress in Brazil on 2D, 3D and n-dim figural numbers and their relations with recurring sequences and interfaces with technology.

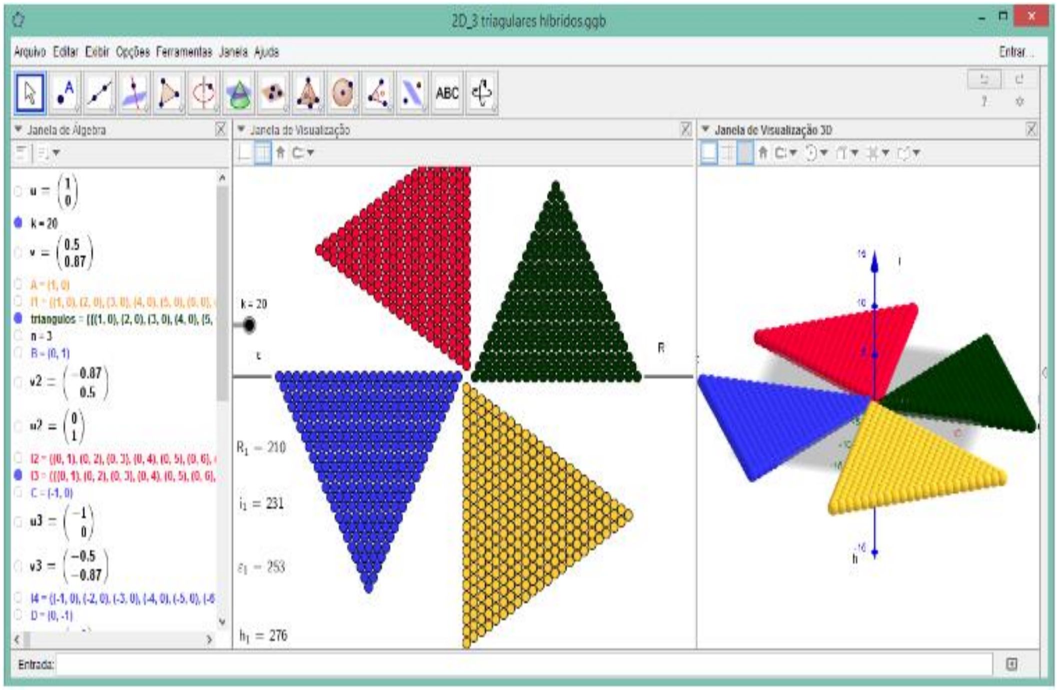


Figure 11. Barros, Alves, Vieira, and Catarino (2020) explore the visualisation and understanding of properties of figural hybrid numbers with the GeoGebra software and 2D/3D representations.

Final considerations

In the previous sections, we presented some results on a set of research supported by the assumptions of the didactic engineering for development and the professional didactics in the context of teacher education and with an emphasis on particular historical and evolutionary elements derived from the notion of recurrent number sequence. From an expressive set of ten recurring number sequences (see table 1), we observe a distinction and the contribution of scientific and national and international scientific research works related to the topic during 2017 - 2020. On the other hand, we add the preliminary results of another set of investigations under development in Brazil and planned for the period 2020 - 2023. The results should contribute to the proposition of resources and an improvement in the field of mathematics teachers education, aiming at the significant appropriation of a mathematical culture on the subject, with repercussions for their professional activity and significant professional evolution.

The research of initial teacher education in Brazil is still quite anachronic (Alves, 2020b), given the presence of specific knowledge needed for professional activity, however, they are either far from the classroom or involve curricular components that require a scientific update. In this sense, mathematics history component lacks a non-static but an evolutionary and dynamic perspective, a current scientific and mathematical knowledge, so that teachers in initial education can understand an unstoppable evolution and current research on the notion of recurring number sequence that also allows multiple relations and interconnections with important elementary mathematical concepts, such as matrices, determinants, numerical sequence, and polynomial functions.

However, we do not intend to exhaust the discussion in this paper. The research continues to evolve in the scenario and context of teacher education in Brazil. We highlight the importance of the professional didactics (Pastré, Mayen, & Vergnaud, 2006) to provide some modelling principles for the understanding of the professional knowledge of the teacher, who, in the course of their professional activity, experience circumstances exercising historical research tasks that contribute to their evolution and professional experience.

On the other hand, the adoption of an investigative design, called didactic engineering for development, very relevant to the French aspect of the didactics of mathematics, associated with the set of premises assumed by the theory of didactic situations, made it possible to conceive and develop an apparatus and a wide itinerary of investigations and structured didactic situations, with well-defined and precise mathematical objects (see table 1) and with interest in historical, mathematical, and epistemological biases. Robert (2005) warns, for example, the essential character of research on (professional) teacher education and the proposition of updated devices.

> There is little significant research in mathematics teaching on the actual professional - initial or continuing - education of high school mathematics teachers. Whether in France or abroad, there are more and more educational proposals

> substantiated in reference to various jobs that have been little evaluated. (Robert, 2005, p. 210).

That said, starting from the three specific objectives stated at the beginning of the present work, we can say that we presented an itinerary for teacher education substantially affected by elements of a historical nature, above all, of an evolutionary nature, which makes it possible, on the part of the participating teachers, a non-static understanding of mathematical knowledge correlated to the set of sequences considered (see table 1). Second, the didactic engineering for development, combined with the assumptions of the professional didactics (Alves, 2020b) that aim at teaching activity (Habboub & Lenoir, 2011) allow the establishment of modelling principles for a systematic understanding of the necessary professional pragmatic knowledge mobilised and acquired during the proposed activities and ongoing investigations in Brazil.

To conclude, we note the growing scientific cooperation involving Portuguese researchers (Alves & Catarino, 2019a). In fact, the mathematical research developed in Europe, in a standard way, is limited to a selected scientific dissemination circuit, on the part of specialised international journals, which allows us to aim at possible developments and repercussions, in view of the context of the - initial or continuing - education of mathematics teachers. Resuming the thought of Tempier (2013) expressed in his thesis, and that can be seen in figure 2, the research resource or itinerary proposed here, with emphasis on historical elements, may have two important, not mutually exclusive, interests, namely: the scientific diffusion in initial or continuing education and the professionalisation of mathematics teachers, and the necessary repercussion for the students' basic learning in the Brazilian school context.

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