The Mathematics Enthusiast

Volume 19
Number 2 *Number 2*

Article 11

6-2022

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Recommended Citation
Silveira Monteiro, Lúcia Cristina (2022) "Semiosis to Communicate Mathematics: Complementarity in the Circularity of Interpretations in Mathematics for the Development of Creativity," *The Mathematics Enthusiast*: Vol. 19 : No. 2 , Article 11.
DOI: https://doi.org/10.54870/1551-3440.1564
Available at: https://scholarworks.umt.edu/tme/vol19/iss2/11

Semiosis to Communicate Mathematics: Complementarity in the Circularity of Interpretations in Mathematics for the Development of Creativity

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Abstract: Mathematics is a dynamic field of knowledge of human creation and invention that is in continuous expansion. However, the mathematics is presented as a ready and finished field of knowledge in school systems, and there is no concern with the development of cognitive processes other than memorization and symbolic manipulation, therefore, there is no concern with stimulating the development of cognitive processes such as creativity, present in the development of mathematics. On the other hand, the concepts of scientifical mathematics as infinite and infinitely small that are related to the understanding of the dynamics present in phenomena, do not have didactic treatment to be presented to students of basic education. In this work, I will propose an approach of mathematics by mean of the fundamental concepts from the first school levels, approaching mathematics as a semiotic activity based on the possibilities of interpretations of Zeno's aporias, as metaphors of the infinite, to develop creativity in the didactics of mathematics. This work will present description of didactic phenomena during teacher education.

Keywords: Metaphors of the infinite; Zeno's paradox; Semiotics in Mathematics; Creativity.

Introduction

Mathematics today is presented as a set of rules and ready-made formulas. The method for communicating mathematics that is predominant in this modality is exposures of contents due to their operative properties and after this exposure, the realization of some examples of operations are carried out applying the properties, then it is believed that the students are prepared to reproduce these procedures, that is, they must solve exercises similar to the examples presented. The main objective is to encourage symbolic manipulation to achieve an expected response, which is unique. The purpose of these studies is propaedeutic and classificatory evaluations. These studies will serve to be used in other operative structures and that are yet to come in other levels of education, for those who come to reach them. In this way of presenting mathematics, there is no concern with the development of other cognitive processes besides the memorization and skill of symbolic manipulation. Therefore, problems that can stimulate the development of processes such as creativity are not explored.

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For Klein (2009) the didactics of mathematics presents the problem of the gap between scientific mathematics at higher levels and mathematics presented in basic education. Klein (2009) also observes that at all levels of education an approach predominates in which mathematics is communicated using an exclusively deductive method, and highlights that: “The mathematics it develops like a tree, it does not grows only upwards and from the thinnest of its roots, but it extends its branches and leaves, at the same time that its roots penetrate deeper into the soil”. (Klein, 2009, p.20).

When identifying problems in the didactics of mathematics and to contribute to a reflection on these issues, I started studies to communicate a proposal. Then I present a synthesis of the academic production that precedes the present proposal.

In 2000, year of the graduate completion in mathematics at the Federal University of Alagoas I started studying the fundamentals of mathematics for the preparation of the Course Conclusion Work, reading – The Foundations of mathematics (Beth, 1966). This reading led me to understand that there were different ways to looking, and to represent a concept of mathematics. This understanding was central to the conclusion of the work.

After, to continue my studies I introduced the readings – Fundamental Concepts of Mathematics (Caraça, 2000), and – What is Mathematics? An elementary approach to ideas and methods (Courant and Robbins, 1996-2000). These readings endorsed the importance of an approach to mathematics with reference to the concepts and methods of mathematics.

Then, after being select for the master’s degree in Cognitive Psychology at the Federal University of Pernambuco, between 2001-2003, I introduced the reading – Where Mathematics comes from: how the embodied mind brings mathematics into being (Lakoff and Nuñez, 2000), which presented a study on the Basic Metaphors of the Infinite in Mathematics, presenting how human cognitive processes are used in the creation and understanding of mathematical ideas. This reading led me to realize that the intuitive processes present in mathematics. From infinity in Natural numbers, in metaphors such us, 1, 2, 3, …, n, n+1,…, and, this reading led me to understand the importance, for the didactics of mathematics, of the different types of reasoning present in mathematics.

The combination of the work of Lakoff and Nüñes (2000) and Caraça (2000), both with an approach to Zeno's paradox (5th century BC)[2], in which the first highlighted paradoxes as a metaphor for the infinite, describing the intuitive process , represented as successions of halves, half of half, half of half of half ... by the sequence, metaphor of infinity, given by the sum $S_n = \sum_1^n \frac{1}{2^n}$, whose limit is given by series associated with that sequence, described by $\lim_{n\to\infty} \sum_1^n \frac{1}{2^n}$, that is, with *n* going to infinity. Caraça (2000) calls this expression as infinitesimal, and describes it as being something new that appeared in mathematics. This author also highlights the importance of debates and conflicts in the construction of mathematics involving Zeno's paradox and therefore, the concept of infinity.

In that context of my research, another question also motivated my readings, the search for the understanding of periodic tithes that do not have a generating fraction for Euclidean divisions, such as the 0.999…, which led me to the rescue of studies of the sum of infinite numerical series (Munem & Foulis, 1978).

At the confluence of these readings and the specific investigation of periodic tithes without generatrix, I realize that the periodic tithes 0.999…, could be interpreted as the sum, $S_n = \frac{9}{10^1} + \frac{9}{10^2} + \frac{9}{10^3} \ldots + \frac{9}{10^n} = 1$, and Zeno's problem, interpreted as the sum, $S_n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \ldots + \frac{1}{2^n} = 1$, that is, these problems had the same essence when they were interpreted as convergent geometric series. The interpretation that occurred to me was that Zeno's paradoxes are not explained in the current curriculum but, are present when viewed from the perspective of the concepts of current scientific mathematics.

Faced with these questions, a broadening of possibilities occurred, which led me to the conclusion of the dissertation, as I realized that it would be possible to make the reciprocal path, that is, expose the paradox and make interpretations and representations of this paradox accessible to students at different levels.

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[2] "The oldest problem that has been recorded involving the concept of infinity, more specifically the infinitely small, approximately in 450 BC This problem refers to a situation in which the movements of a corridor in history known as Achilles are compared to the of a turtle, both moving at constant speeds. By assuming the infinite divisibility of space, it is also assumed that if the turtle goes a little ahead when competing with the runner, it may never reach it, because when the runner reaches that point again where the turtle should have been in the second moment, it would have advanced a little more, and this situation would repeat itself infinitely". (Monteiro, 2013a, 2013, b, p.2)

After completing the master's thesis entitled **- The development of the metaphors of the infinite in Mathematics Education** (Monteiro 2003a), the research carried out led to an article, extract from the dissertation, entitled – **The concept of the infinite and the perception of movement** (Monteiro, 2003b). This dissertation and this article become important texts to guide investigations during the training of mathematics teachers in the Mathematics Degree and in the Pedagogy Degree at the Federal University of Alagoas, and continuous training for mathematics teachers. Some syntheses derived from these teacher training were presented at regional and national events for the dissemination of scientific production, such as: **The relationship between the dynamics in the study processes and the transformation of geometric tools**: A constructive approach to the concept of perimeter (Monteiro, 2006); and **Iterative processes applied to polyhedra and the development of creativity** (Monteiro, 2007).

The continuity of this proposal for didactics of mathematics arises with the elaboration of the doctoral thesis entitled **- Senses and meanings for a semiotic approach in mathematical education**: an analysis of the discussions on the interpretations of Zeno's paradox – by Universidade Anhanguera de São Paulo – established from 2012 to 2015, under the guidance of the professor and epistemologist of mathematics, Michael Friedrich Otte. To elaborate the thesis, the following themes were introduced: semiotics (Peirce, 2010), epistemology of mathematics in a Peircean perspective (Otte, 2006), complementarity in didactics of mathematics (Otte, 1993), complementarity between Geometry and Arithmetic (Otte, 1990).

A new synthesis of these academic productions cited above was presented in 2019, during a round table at the XIII National Meeting of Mathematical Education, in Cuiabá, which culminates in the article entitled – **Complementarity in the circularity of representations: a semiotic approach to creative in mathematics** (Monteiro, 2019).

This article rescues this path to propose the continuity of investigations by a semiotic approach to the development of creativity in mathematics, proposing the exploration of the mathematics curriculum through the complementarity between concepts, formulating problems through a combination of Zeno's ideas as metaphors of the infinite, elaborating interpretations and highlighting the investigation of abductive reasoning using the formulation of problems through a combination of metaphors of the infinite and metal mental experiments.

For this development, in the first session I present a synthesis in different interpretative perspectives for Zeno's paradox in different authors; in the following session a focus on mathematics understood as the science that is born in the complementarity of contrary thoughts, with quotes from some researchers, such as Caraça (2000), Courant and Robbins (1996, 2000) (2000), Hegel (2013), Kant (1997) and Otte (1993). Next, I present the complementary approach present in the didactics of mathematics proposed by Otte (1993, 2006), followed by a clipping on Percian thought, Peircian triads and the notion of interpretation as a continuum in the production of signs. Also, with reference to the work of Peirce, an introduction to the sense of abduction in that author's work, based on the hermeneutical studies of Souza (2014).

Still with reference to the theoretical focus that underlies the present proposal, I present some concepts about creativity and a relationship between this concept and the resolution and formulation of problems with emphasis on the metaphors of the concept of infinity present in Mathematics, supported by the work of Lakoff and Nüñez (2000). The following are reports based on experiences that took place during mathematics teachers training activities and summaries of publications at scientific events.

Currently, for the proposed development, I conduct research under the hypothesis formulated as follows: If Zeno's paradox that in the construction of mathematical knowledge is considered an aporia (Koiré, 2011), that is, a nucleus of situations and interpretations that create a logical-rhetorical tension and prevent the meaning of a text from being unique, - generates motivation for creations in mathematics, so when they are reinterpreted and adapted to basic levels of education, they can contribute to the accomplish of new interpretations and creations for these levels of education.

Semiotic processes in different interpretative perspectives for Zeno's paradox

The conflicts (Becker, 1965) with reference to the problems proposed by Zeno that narrate the dialectic between the Pythagorean school and the school of Parmenides, from which Zeno came, this one, which was opposed to the idea of a line composed of infinite parts, or monads, culminated in the presentation of the well-known paradoxes. Zeno's ideas narrated problems that arise in the form of mathematical paradoxes by exposing a contradiction in the physical understanding of the mathematical idea about the infinite divisibility of space.

For Otte (1990), the Achilles problem is the classic example that demonstrates the complementarity between arithmetic and geometry, that is, the sense of complementarity between discrete and continuous. "In physics, movements are understood as continuous functions in three-dimensional of time-space […] The function is both qualitative and quantitative, conceptual and constructive. It is knowledge (the general idea) and tool (the calculation formula)" (Otte, 1990, p. 55. Author's translation).

For Otte (1990) "the solution to Zeno's paradox is complementary", and says that a certain solution to a problem will never be forced on us ", but, we need to seek the solution according to a specific type of view of the problem, and, for him, "an absolute vision or intuition does not exist". (Otte, 1990, pp. 58-59)

In the dialogue - What the Turtle said to Achilles, Carroll (1905), when doing an analysis from the perspective of pure logic, states that even the most perfect axioms system is not enough to determine the truth of a logic system, for thoughts with arguments at infinity, whatever the number of axioms, or premises, will always be insufficient.

Ryle (1993), in his work entitled, Dilemmas, in a philosophical perspective, highlighted necessary differences to be considered when recursive reasoning is applied in different situations, such as on a computer, subordinated to a program that leads to a pattern of repetition, in a race like Achilles and the Turtle, or in the subdivision of a cake with its known volume. Ryle (1990) argues that there are ways of thinking that are not antagonistic solutions to the same problem. However, the first issue that occurs to us is that we need to adopt one of them and this can commit us to reject the other. Ryle (1993) clarifies his thinking by saying that when we encounter an apparent antagonism between different ways of describing something, it is often not about antagonistic things, nor about antagonistic descriptions of the same things, but they are two different and complementary ways, to give different types of information about the same thing.

Koyré (2001) interprets the paradoxes as follows:

> a) The dichotomy – Let us take a variable X between the limits O and A. The dichotomy's argument consists in emphasizing that the variable must cover in a certain order all the values between O and A. b) Achilles – Two variables are linked by the relationship Y = AX. Each value of X corresponds to a value of Y and only one, and vice versa. However, Y grows faster than X, until, finally, Y = X + A. c) The arrow – Translating into mathematical language, the arrow argument simply means this: all values of a variable are constant. d) The stadium – This argument only shows that a single and reciprocal relationship can be established between all points of two or more segments – regardless of their respective magnitude. This is expressed by the formula Y = AX. (Koyré, 2011, p.15).

Zeno's paradox in a cognitive perspective

Lakoff & Nünez (2000) conceives Zeno's paradox as a metaphor for describing the infinite in mathematics and they explain that in cognitive terms, these metaphors are descriptions of a continuous and indefinite process. These authors highlight the importance of this metaphor for the conceptualization of the infinite which are used to describe continuous processes that present an iteration, such as, for example, Zeno's paradox. For Lakoff & Nüñez (2000) iteration is a step by step, in which each step is discrete and minimal, and this idea is often applied in continuous processes.

Monteiro (2003) states that one of the main indications for approaches in mathematics didactics "is the relationship between the concept of infinity in its various levels of abstraction, guided by a fundamental abstraction, the perception of processes, that is, the perception of movement in the phenomena" […], and attentive to the importance to

> […] stimulate the perception of movement intrinsic to continuous processes using mathematical language associated with observable situations; building the conditions for future mathematical approaches and corresponding mathematical models can be added to the initial descriptions of the phenomena. (Monteiro, 2003, p. 22)

Monteiro (2003a, 2003b, 2013a, 2013b) supported by Lakoff and Nüñez (2000) interprets Zeno's problem as a metaphor for the infinite concept. Monteiro also agree with the interpretation of Zeno's problem which is described by the infinitesimal $\frac{1}{2^n}$ (Caraça, 2000), that is, as something that has its limit going to zero. This interpretation allows us to analyse the Infinite divisibility of a knows unit, always in its halves. In this way Monteiro (2003) interprets Zeno's problem as the sum, $S_n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} ... + \frac{1}{2^n} = 1$, **with n going to infinity,** because in addition to being perceived as a known unit, the expression $\frac{1}{2^n}$ when it is described as the function $f(n) = \frac{1}{2^n}$, can be interpreted as a geometric series with ratio $0 < r < 1$. and therefore, a series with convergent sum, given by $S = \frac{a}{1-r}$. Thus, if ***a*** is the first term of the geometric series and ***r***, the ratio, interpreting the Achilles movement that walks in halves, the first term is this series is ½, and the ratio is ½. Therefore, the sum of this series converges to the unit.

Monteiro (2013, 2015), in addition to conceiving Zeno's problem as a series sum, this researcher combines this idea with other perspectives, such as Ryle's (1990) statements and elaborates the problems mentioned a follow.

> Problem 1: let us take a rectangular surface and divide it in two. We will have a new surface with half the area of the first. Let us take the new surface and again divide it in half and repeat the process indefinitely. What is the sum of these areas divided in half? Is it possible to continue dividing the resulting areas in two, infinitely, in any space considered? To construct the answer, one must think of some space in which the problem is possible and another, in which the proposed procedure has limitations.
> Problem 2: How to represent the set of polygons whose halves do not exist? Is this question a paradox? Is this representation in terms of theory possible? Is a descriptive interpretation of this problem possible? Can the idea of the infinite sum $S =$ ½ - ½ + ½…, help to analyze this problem?
> Problem 3: If the subdivisions of problems 1 and 2 were placed in a dynamic geometry software, what would be the concepts and tools used? Is it possible to do experiments that test the limitation of dynamic geometry software? What procedure could describe the problems when using dynamic geometry software? Would be procedures be identical to describe problem 1 and problem 2? Would the two problems face the same possibilities and impossibilities?
> Problem 4: Is it possible to introduce a discourse on the continuous and non-continuous aspects of a given space, starting from the exploration of the previous problems? In the case of software representations, how to describe the continuum represented by the non-continuum? Describe the continuous object represented by a software and comment on the tools and languages of that software to represent that continuum.
> What importance can we attach to logical and intuitive aspects, to think about the problems mentioned in different environments, that is, spaces to represent them?
> Problem 5: In what situations can the above problems have answers, valid in one perspective and not valid in another?
> Given the above, is it possible to consider a better interpretation and representation to start approaches to ideas like those of Zeno? (Monteiro, 2015, pp. 69-70).

To analyze and answer these questions, Monteiro (2015) proposes the didactics of complementarity in the circularity of mathematical interpretations, considering the meanings of the mathematics concepts in different spaces in which a phenomenon can be observed.

Complementary between the interpretations of mathematical concepts

For Courant & Robins (2000) mathematics is a knowledge that is born from the reciprocal influence of antithetical forces, such as: logic and intuition; analysis and construction; generality and individuality, and, for these authors, these forces are the basic elements of mathematics. These authors clarify that "seeking a synthesis of these antithetical forces constitutes life, utility, and the supreme value of mathematical science" (Courant & Robins, 2000, p.10).

Hegel (2013) highlights that the contradiction is precisely the elevation of reason over the limitations of the intellect and the solution of them. The concept pushes itself forward through the negative of the concept, and this is the true dialectical element (Hegel, 2013). For this philosopher of sciences, the infinite exists, and it is at the same time the negation of "the other", the finite, and the finite is in opposition to the infinite as a real existence in a qualitative relationship. Therefore, continuity and discontinuity, equality and inequality etc., are in a qualitative relationship as a real existence. In this sense, Monteiro (2015) highlights the importance of denying a space in which a phenomenon seems to be non-existent, looking for some space in which it is possible to attribute significant existence, at least at some level of interpretation.

According to Otte (1993) since Descartes to Kant, mathematics has been considered intuitive and constructive and the truth of mathematics was based on intuition and not logic. Or, rather, the "logical demonstration and intuition had appeared as inseparable, [...] the demonstration must proceed through the object's intuition". (Kant, Beth E Piaget, 1966, p.16, apud Otte, 1993, p.304). Kant points out that

> […] Our nature requires that intuition be never something other than sensory, that is, it contains only the way we are impregnated by objects. The ability to think the object of sensory intuition, on the contrary, is the reason. Neither of these properties is more important than the other. Without the

> sensory, no object would be given to us and, without reason, none could be thought of. Ideas without content are empty, intuitions without concepts are blind. (Kant, 1997, p.88).

For Otte (1993) the intuition revealed by the reasoning processes by recurrence, reflects the infinity of the mind[3] and proves to be the basis for science in a necessary complementarity between intuition and concepts. Thus, to contribute to didatics of mathematics Otte (1990) proposes the complementarity between Geometry and Arithmetic.

Monteiro (2015) seeks the complementarity between senses and meanings in mathematics. To build this complementarity this researcher proposes a complementarity between the interpretations, immediate, dynamic, and final. To move between these interpretations, Monteiro advises to reflect about the existence of meaning and meaning at these different levels of interpretation and adds:

> We consider the possibility of building sense and meaning in Mathematics Education, introducing a reflection between the object and the object's negation, passing through the negation of its intuitive space of existence. This reflection may be guided by the complementarity between a dynamic and a static conception, a model of continuous and discrete space, experience and intuition, logic and intuition. More challenging is the complementarity of qualitative and quantitative aspects. (Monteiro, 2015, p. 141).

Otte (2003) highlights that the notion of complementarity has been used in mathematics and also in other fields of science aiming to retain essential aspects of the cognitive and epistemological development of scientific and mathematical concepts. Otte (2003) states that a complementary attitude is a consequence of the impossibility of defining the mathematical reality if we consider it to be independent of the activity of knowledge itself, because, "Mathematical practice, which has progressively freed itself from metaphysical and ontological schemes since Cantor and Hilbert, requires a complementary approach – perhaps more than any other field knowledge in order to be properly understood". (Otte, 2003, p.204).

Otte (2003) summarizes the concept of complementarity as pursuing and explaining a universal or general phenomenon in its particular manifestations and cites the complementarity betweens aritmetic and geometry (Otte, 1990) as a first view of the idea of complementarity in Mathematics.

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[3] For Poincaré (apud Otte, 1993), recursion is the affirmation of a property of the mind itself (Otte, 1993, p.307).

Monteiro (2015, 2019) highlights the importance of complementarity between geometry and arithmetic in the circularity of interpretations, senses and meanings of the discourses of mathematics and exemplifies with the complementarity in the circularity of interpretations between discrete and continuum aspects in construction of mathematics present in conceptualization of incommensurability since Eudoxo (3th century B.C) to Dedekind (19th century AC).

Interpretation of a sign in Peirce's Semiotics and the continuous circle of interpretations.

Semiotics is the science that studies signs. Peirce (2010, CP303, p.74) describes the sign as anything that leads to something – in this case, the sign is the interpretation of that something. The object of the interpretation transforms this interpretation into a new sign, "and so on, successively ad infinitum". According to Santaella (1985, p.68) to know something, our conscience produces a sign, that is, a thought as a mediation between a subject and a phenomenon, and this is an interpretation.

Santaella (2000) describes the levels of interpretations with reference to Peirce's semiotic theory as following: Immediate interpretation is an abstract level, consisting of a possibility. The character of immediate interpretation is that it is exempt from mediation and analysis; Dynamic interpretation derives its character from the category of action. It can also be said that dynamic interpretation is a "determination of a field of representation outside the sign" (Santaella, 2001, p.98); The final interpretation is on an abstract level, however, more elaborate and of a more formal nature.

We cannot understand any of these levels of interpretation or interpretants of Peirce's semiotic theory (2010) presented previously to the static way, because there is a fine line between them. It is only possible to understand one level by relating it to the other. The interpretations of the signs are themselves members of an infinite series in which each interpretation is a sign of some object for further interpretation and, "every interpretation is a sign, and every sign is an interpretation of an object" […] and, object-sign-interpretation, they are all a sign nature (Santaella, 2001, p.88).

Therefore, a sign is both a thing and a process of establishing a relationship between the object and the interpretation given to the object. We have a flow of meaning, that is, an interpretation that suggests a new interpretation, in an endless flow. The conception of the process of a sign, presented by Peirce (apud Rosa, 2003) is also an interpretation of the continuity of the semiosis of a sign. That description inspired the elaboration of the diagram below, which admits that a complete symbol tends to a continuum, however, only in the infinite.

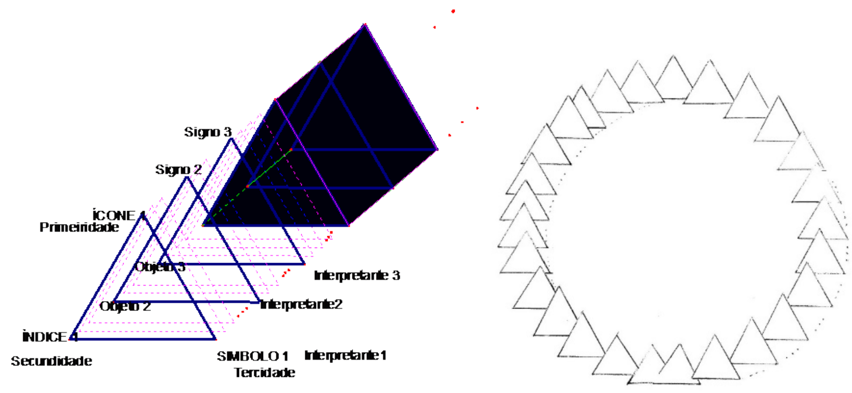

Figure 1: Process of a sign: A continuum of interpretations ad infinitum on the triad: object, sign, interpretant. Source: Monteiro (2015, p. 135)

For Garnica (1992, APUD Souza 2014), hermeneutics can be understood as the theory of interpretation that includes: saying, translating, explaining, and states that the theory of hermeneutics deals like a game between explanation and understanding. For Palmer (1969),

> Understanding is an essentially referential operation; we understand something when we compare it with something we already know. "What we understand groups into systematic units, or circles made up of parts. The circle defines the individual part, and the parts together form the circle [...] through a dialectical interaction between the whole and the parts, each gives meaning to the other; understanding is therefore circular". (Palmer, 1969, p.93-94).

For Heidegger (apud Palmer, 1969), "understanding is the power to capture the possibilities that each has to be, in the vital world in which each of us exists". (Palmer, 1969, p.135). Thus, Monteiro (2015) proposes that this game between explanation and understanding, happens through a relationship between different interpretations, exploring reasoning such as the metaphors of the infinite, stimulating semiosis[4], seeking to understand the concepts in the circularity of interpretations of mathematical thinking.

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[4] By 'semiosis', writes Peirce (2010), "I think of an action, or influence, that involves the cooperation of three subjects such as a sign, its object and its interpretant" (CP 5.484); a sign process.

Creativity in Peirce's semiotic theory: interpreting abductive reasoning

Peirce lived between the 19th and 20th centuries and defined science by contradicting definitions of his time. In the encyclopedias of the time de Peirce we find science concept as: a systematized and organized knowledge structure[5]. On the other hand, Peirce redefines science saying that: Science is a process, "it is the main fruit of the concrete for a real world [...] as something in perpetual and persistent growth". (Santaella, 1992, p.69, apud Souza, 2014).

Peirce classifies mathematics as a Heuristic Science, that is, science of discovery (Souza, 2014), and to elaborate his semiotic theory about science, Peirce (1975, apud Souza, 2014) states that logic is the reasoning that guides human thought. Para Peirce, the main objective of logic for building a method for understanding something, should be to learn the way in which it is possible to conduct any research.

In this pursuit, Peirce understood that he would need to classify reasoning to propose a scientific method. He started by defining scientific method as the method to know something, and in the elaboration of this scientific method, this author states that "from induction, deduction and abduction, it is possible to reach belief, regardless of the science in question". (Peirce, 1975, apud Souza, 2014, p.47). Thus, Peirce proposes that all investigation be conducted from a doubt to certainty and explains that the "stimulus of doubt leads to the effort to reach a state of belief. We call this effort, research, the stimulus of doubt is the only immediate reason for the effort to arrive at the belief". (Peirce, 1975, p.77, apud Souza, 2014, p.77).

To understand the meaning of the term doubt used by Peirce, it is necessary to understand the meaning of the term abduction, which is fundamental in work of this author, and also for the construction of this proposal for didactics of mathematics.

Souza (2014) in his research, attributes interpretative possibilities to the term abduction found in part of Peirce's work, such as: retroduction, that is interpreted as provisional adoption of hypothesis to be checked (Peirce, 2003, p.5 apud Souza, 2014); also interpreted as an original argument described as an argument from which some immediate consequence captured (Peirce, 2003, p.5 apud Souza, 2014); In other

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[5] Abagnano (2000, apud Souza, 2014).

excerpts of Peirce's work[6] the term abduction appears in the sense of a hypothetical adoption of hypothesis; or, the only type of argument that starts a new idea (Peirce, 2003, p.5, apud Souza, 2014); and also as a presumption (Peirce, CP 2.774), presumption of presuming, in the sense of presupposing, predicting, making an advance judgment; in another enlightening moment, the statement is that “the abductive suggestion comes to us like a flash. It is an act of insight, although an extremely fallible insight”. (Souza, 2014, p.61), interpreted by Souza (2014) as “a sudden manifestation or a brilliant idea”.

Yu (2006) based on Peirce’s semiotic theory summarizes the types of reasoning stating: abduction creates, induction verifies, and deduction explains. Pimentel and Vale (2013) also with reference to Peirce presents abduction as a creative phase of producing exploratory hypotheses and its success is conditioned to intuition[7] and prior knowledge. Souza (2014) concludes his research about abduction sense in Peirce's work, saying that “abduction is the reasoning that values creativity and opens up the possibility of producing knowledge”. (Souza, 2014, p.88).

Creativity: other contributions

Vygotsky (1990) calls creative activity a combinatorial activity. This researcher states that creativity being the result of the activity of the subject, everyone has it and manifests itself whenever the human imagination combines, changes, and thus creates something. Vygotsky (1990) adds that creative imagination activity is completed by the crystallization of the image in an external form, depends on previous experience and is a vitally necessary function in the face of new situation.

For Otte (2012) the creative process operates in the interaction between variation and repetition and adds that a theory being an interpretation of a phenomenon is also a process of creating an interpretation of the given interpretation, and so on. Otte (1993) simplifies, saying that creativity is seeing an A as a B. Monteiro (2015), adds that the formulation of the hypothesis, the insight, an abduction or the first moment of creation it can be stimulating to see an A as an ~ A, or that is, object and negation of the object, in a

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[6] Collected Papers (Souza 2014).

[7] The term intuition can also be defined as a representation, an explanation, or an interpretation accepted directly by us as something natural, self-evident, or immediate (Fischbein, 2002).

qualitative relationship by the affirmation and denial of the possibility of the space of real existence of the object.

For Vale and Pimentel (2015), creativity emerges with an approach that leads to a new discovery and emphasizes the synthetic nature of creativity and states that “intuition, by itself, is not enough: it is necessary a conscious analysis, an understanding of how things work, even if you don’t know all the rules of the game. Thus, creativity results from a combination of synthetic thinking and analytical thinking”. (Vale and Pimentel, 2015, p.1). Vale (2015) also highlights that creativity is a transversal capacity that depends less on the content to be explored and more on the methodology, experiences, culture and interaction.

For Feldman (1988) creativity is a phenomenon of coincidence between, who creates, where something is created, and what is created. Moraes (2015) says that creativity manifests itself in the areas of greatest individual skills, but motivation is needed. For Dineem (2006), creativity requires associations of information, multidisciplinary knowledge at different levels and not just in-depth knowledge in the area in which you want to create something.

Cropley (2009) warns of the subjective dimension of creativity, as it is related to the influence of the eyes of others, which can be the teacher evaluating students, the art critic, or even an academic community with the limitations of the research methods proper to socio-historical moments. These people can filter what is creativity and what is not.

Sternberg & Lubart (2003) warns that the idea of insight, a mental process that is associated with creativity, is not synonymous with sudden inspiration, because insight happens after intense work and persistence.

Dimensions of creativity, problem solving and problem posing

To judge creative productions in mathematics, it is necessary to admit criteria accepted by the scientific community. In this search there is a confluence around three main components/dimensions of creativity which are: fluency, flexibility and originality (e.g. Guilford, 1967; Leikin, 2009; Silver, 1997). Fluency is the ability to produce different solutions for the same task. Flexibility is the ability to think in different ways to produce a variety of different views of the same issue, that is, to view the problem from different perspectives. Originality is the ability to think in an unusual way, producing new and unique ideas (e.g. Leikan, 2009; Silver, 1997), thinking outside the obvious and having a rare idea.

Another important dimension of creativity is called elaboration (Torrance, 1967; Guilford, 1967). This refers to the facility to add many details to the information already produced, from outline to an organized structure or system. In the elaboration, one or more aspects can be changed, that is, replaced, combined, adapted, enlarged, removed, rearranged, and one can also speculate how this change could have a cascade effect among other aspects of the problem or situation. Thus, it is the combination of these dimensions of creativity that allow us to characterize, through the analysis of the proposed tasks, manifestation of creativity.

According to Liljedahl & Siraman (2006), the manifestation of new insight and / or solutions in mathematics is considered as an indicator of creativity and therefore, tasks that allow various solutions, arouse curiosity and involvement, provides possibilities for fluency of mathematical ideas, flexibility of thought and originality, therefore, must be presented.

For Polya (1945, 2003) creativity is an innate characteristic of the individual, but educators have a responsibility to simulate students' creative mathematical thinking, seeking to offer suitable environments for creative manifestation. Polya (2003) also refers to the fact that problem-solving tasks is impoverished if it is not articulated with the problem proposition. Brown & Walter (2009) exposes the importance of proposing new problems by looking at old problems from different perspectives.

Leikin, (2009), Vale, Barbosa & Pimentel, (2014) argue that creative thinking can be taught and developed in proposing activities that enable multiple solutions and involve the use of different representations and different properties of mathematical concepts.

Methodology: Elaboration of new problems using metaphors of the infinite

In this section, some tasks performed during the initial and continuing training of mathematics teachers will be presented and briefly commented.

Semiosis with puzzle

The tangram is a well-known puzzle with only seven pieces, but, very peculiar for its versatility. With its few pieces you can build thousands of shapes and not just a single figure, characteristic of other puzzles. This puzzle can be interpreted as a careful sub-division of a square surface into quadrilateral and triangular surfaces, as shown in figure 2.

Figure 2: The "flat" puzzle

This material was produced by undergraduate students in Mathematics at the Federal University of Alagoas - UFAL, who built individual plans to compose the idea, "from solid to plan", recorded in figure 3. In this example, the idea that led to the elaboration was the orientation of doubling, tripling, quadrupling etc., the height of the puzzle in figure 2. The process of the transformation the puzzle started with the debate over the representation of the plan. An important idea for to guide this task, was highlight that even when the puzzle is built with a piece of paper, with its height measurement given by fractions of a millimeter, this will be the representation of a flattened solid. This activity driven by the metaphor of infinity, a process thar indicates the finite possibility of reducing the height of the puzzle quietly, always in half, aims to lead to an abstraction, a mental process, going to zero, which can be mediated by numerical values.

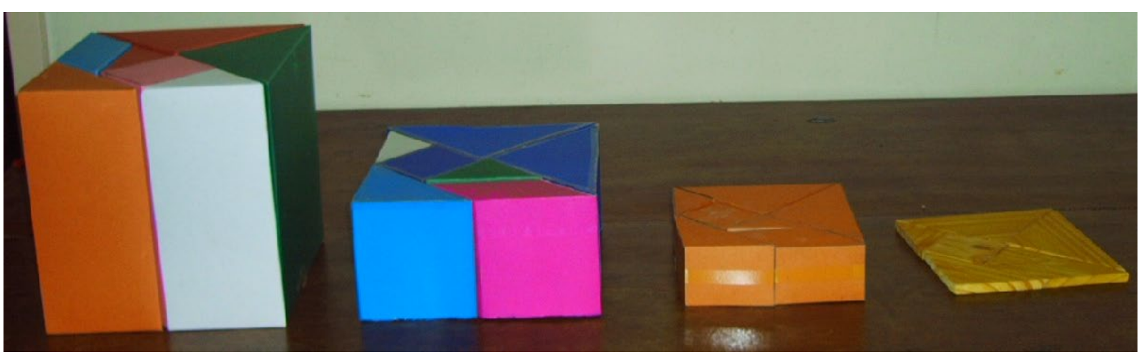


Figure 3: A metaphor of the infinite: from solid to plane. Presented at an event in the category Source (Monteiro, 2006)

Many problems were proposed from of the metaphor of the infinite represented with this material, problems involving numerical and algebraic expressions, equivalences of areas and volumes, fractions, ratios and proportions in areas and volumes, prism sections, geometric progression etc. This material continues to flow in elaborations, among others, it was produced a base planification, figure 4, called tangram in three dimensions.

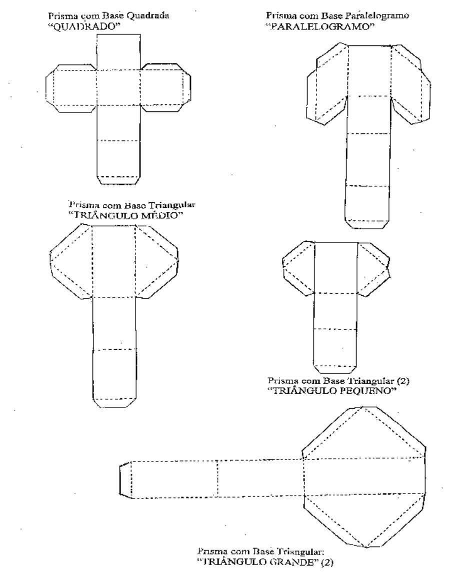


Figure 4: This puzzle has seven (7) pieces. Here, we present five (5) planification because two of these pieces are repeated, that is, the "small triangles" planification and "large triangles" planification.

The use of these plans presented fluency, flexibility, originality and elaboration. An example is exposed in the characters created to tell stories, as in figure 5. The students used printers to elaborate and solve problems involving the proportion between the characters in their stories.

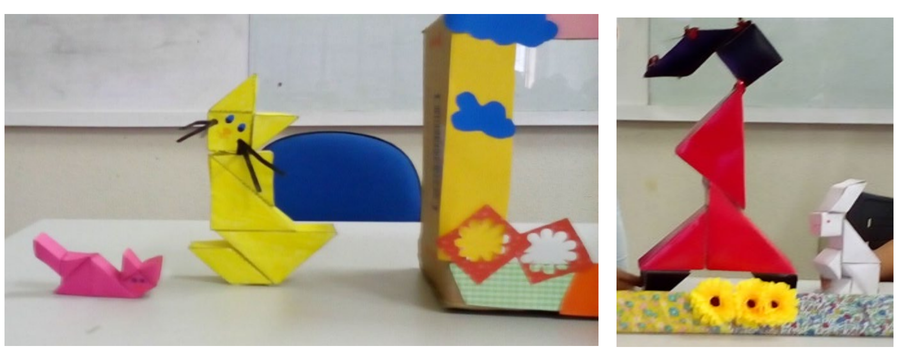


Figure 5: Telling a story with puzzle in three dimensions.
Source: classes in undergraduate courses in Pedagogy at UFAL

It is also important to note that with the seven pieces of this puzzle it is possible to build your own pieces, that is, you can build isosceles right triangle, square and parallelogram. This observation proved to be very fluent. Thus, it is possible to state that the transformations with this teaching material are endless. For example, we will use the metaphor of infinity, puzzle within puzzle, elaborating them in the appropriate dimensions and replacing them in their equivalent parts, and we have a new puzzle, as shown in figure 6.

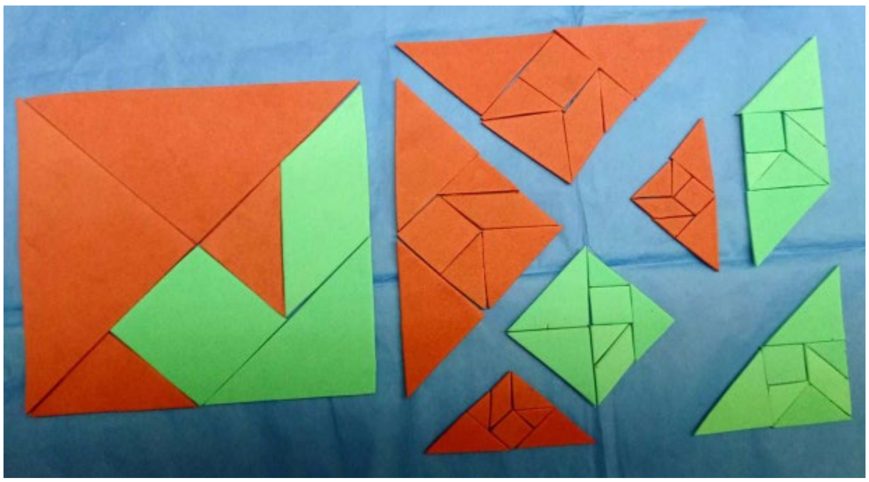


Figure 6: The new puzzle has 49 pieces and many possibilities

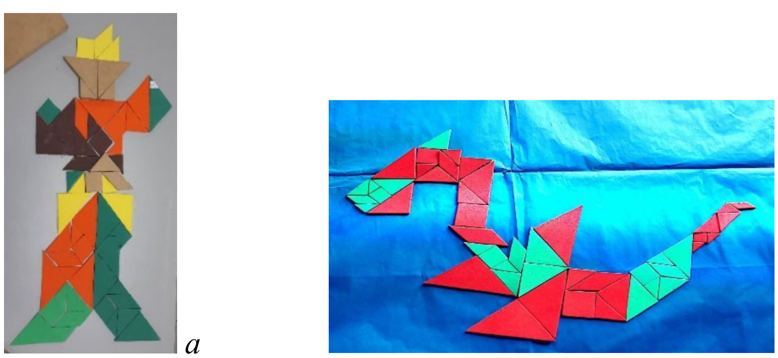
One of the main guidelines for the flow of ideas is the proposing of problem involving the elaboration this material. Along this path, students are guided to the identification of the problem, the verification of possible answers and explanation to expose in the classroom. These students are also oriented to highlight the concepts explored and elaborate suggestions for different levels of approaches, such as, for example, problems that involve a degrowth pattern of the puzzle pieces. This way of exploration by the metaphors of infinite, introducing visualization and construction of materials like this provoke many hypotheses about equality and inequality, that is, path to other elaborations.

Another important idea is presented for to contribute to the fluency of the problem's elaborations: A generalizing thinking guides the tasks is the perception of the conservations of the area or volume of the initial puzzle. It is observed that, regardless of the subdivisions or shape given to the versatile puzzle, it will remain with its area being that of the square. Other question that can lead to the elaboration of problems is the understanding of the possibility variation in the perimeter of the puzzle when its shape changes. To explore the perimeter change, we can do it in a playful way. The orientation is to realize that the area converts to the area of the square, but there may be variation in the perimeter. We can elaborate the perimeter metaphor going to infinity, as in figure 7a.

a *b* *c*

Figure 7: Images of some tasks for proposing problems, the perimeter going to infinity, in figure 7a. Iterative process in the square piece of the puzzle, figure 7b, Using Dynamic Geometry software to propose problems, figure 7b and figure 7c.

Below is a verification by a playful activity of the perimeter going to infinity, continuously replacing the parts by equivalence between areas, as in figure 8a - the cowboy, produced by students of the mathematics degree, in the classroom, in the face of the challenge of those who produce form with larger perimeter, presented by Monteiro (2019), and in figure 8b, the dragon, produced by the author of this article, to encourage originality and elaboration. Thus, an increase in the possibilities to express new equality and inequality, using not only geometric equivalences, but also numerical and algebraic expressions.



a b

Figure 8: Variation of perimeter and equivalent areas for visualization. To stimulate hypotheses about what would happen if other iterations were performed on the puzzle, that is, if to put other equivalent puzzles within these smaller ones.

With each elaboration, other possibilities may arise, for example, infinite ways to represent a unit at different levels of representations, such as: ½ + ½ = 1; ¼ + ¼ + ¼ + ¼ = 1; ½ + ¼ + ¼ = 1 etc.; or $\frac{x}{2}+\frac{x}{2}=x$; $\frac{x}{4}+\frac{x}{4}+\frac{x}{2}=x$ etc., are questions that cannot go unnoticed, because they are levels of verification of the presented statement. A multitude of fractions can compose the unit in question, that is, the square area. Such problems allow to approach mathematics by a complementarity between geometry, arithmetic, algebraic operations, and at different levels. This task also makes possible other explorations, such as one for a semiotic approach to mathematics, the concept of unity in mathematics.

Diagrams that can be interpreted as a puzzle

The puzzle presented also can provoke abductions by analogy, for example, diagrams known as mental experiments[8], they can also be interpreted a puzzle. Thus, the mental experiment on double the area of the square, diagram in figure 9a., Can be interpreted as a puzzle, and moreover, by a pattern that is repeated, like figure 9b, that is, always doubling the area of the square, this allows the creation of new problems and stimulate the emergence of hypotheses. The combination of mental experiments with metaphors of infinity represented by a puzzle, presents possibilities for fluency, flexibility, originality, and elaboration.

a) b)

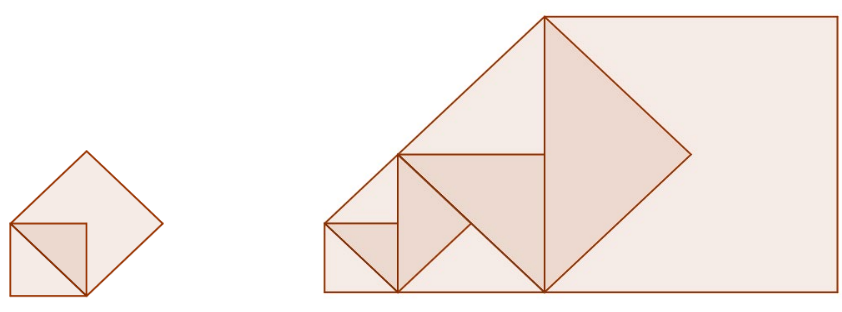


Figure 9: A proof by mental experiment, of how to fold the area of a square, figure 9a. and proposition of replicating the idea in figure 9b, suggests the elaboration of a new puzzle for verification and elaboration hypotheses.

[8] Solved by Menon the slave of Plato as in figure 9a, approached by Socrates (6th century BC)

Iterative processes applied to polyhedra

In this experience, iterative processes were applied to the faces of polyhedron. The motivation for the task was caused by the possibility to build a ball, making a polyhedron tend to the sphere. The finding that every convex polyhedron tends to the sphere, is the generalizing idea that guides the tasks.

The task of making a polyhedron tended to the sphere was driven by the application of iterative processes to the faces of regular and semi-regular polyhedral, that is replacing each face with a composition of smaller faces. The sum of the smaller faces must be equivalent to the area of each face initial. This done, the number of faces is increased, but the diameter of the ball is preserved if it were transformed from the original polyhedron.

Students can choose type of material to be used; the number of iterations applied to the chosen polyhedron faces; the choice of the measurement on the side of the face of the initial polyhedron.

a) b) c)

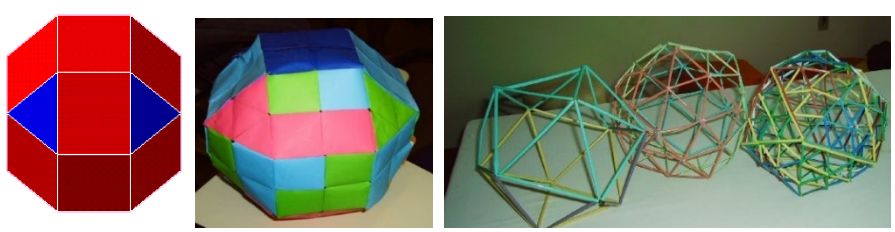


Figure 10: Cuboctahedron tending to the sphere, figures 10a and 10b. Regular icosahedron tending to the sphere, figure 10c (Monteiro, 2007)

The dimensions of creativity, fluency, flexibility, originality, and elaboration can arise, both for teachers and for students who are involved with such tasks. By proposing tasks with the guidelines suggested above, an appropriate environment is created so that hypotheses can be manifested, as many interpretations arise in the teaching and learning process. Thus, it is easy to observe the creation of new objects, that is, semiosis to communicate and do mathematics.

Analysis and perspectives from the Diophantus's proof

Let us observe the Diophantus proof (Garbi, 2006), a mental experiment that represents a moment of discovery in mathematics, ahead of the development of the structures of arithmetic in its time. Diophantus presents, at the same time, abductive, inductive, and deductive reasoning while it presents a complementarity between geometry and arithmetic, intuition, and concept. This passage in the history of mathematics expose self-evident equality that culminates in a generalization, that is, algebra as a generalizer of arithmetic. But, mainly, an experience that indicates extended possibilities.

The self-evident equality is that a rectangular surface has its area represented by the product between the measures of its two perpendicular sides. That is the general idea, or main premise of the reasoning that follows. Here is the experiment to build equality that reached an abduction:

1) This total area can be subdivided into four other rectangular areas and it is conceived that the sum of these four areas is equal to the area taken initially.
2) Equality is verified.
3) To maintain the logical compatibility between the constructed equality, it is necessary that "the product between negative signs is a positive sign". This is the Diophantus hypothesis.
4) This insight from Diophantus, who lived in the 3rd century AD, is a property of operations, but it did not make sense to mathematicians in his day. However, this check is explanatory even today. Using diagrams, a rectangle whose area is the product *ac*, we have figure 11.

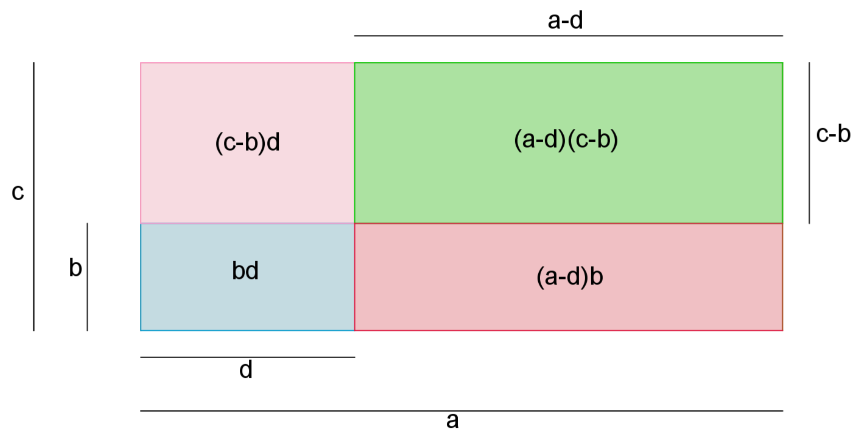


Figure 11: diagram to interpret the Diophantus proof

And representing equality between these areas:

$$ac = (a - d)(c - b) + (a - d)b + (bd) + (c - b)d$$

Manipulating the equation, it turns out that to maintain logical compatibility with the self-evident truth, we have that the product between negative signs is a positive sign, or, (-) x (-) = (+). Thus, isolating the terms that contain the product (-d) x (-b), we have that:

$$ac - ba - dc + db = (a - d)(c - b)$$

The exposure of experiments such as the Diophantus proof are important during the training of teachers because they present initial hypotheses and can to stimulate tasks like those presented above, showing possibilities, whether in the elaboration of problems that may be in the numerical fields, algebraic, and geometric, and highlight the complementarity between intuition, and concept in the construction of mathematics.

When we approach mathematics in this way during the training of mathematics teachers, most of them behave as if they are discovering another mathematics.

Naturally, following the proposal for the approach to mathematics presented in this paper, we also propose to elaborate metaphors of the infinite with Diophantus proof. To propose tasks this way expands possibilities, leads to the creation of signs, new interpretation and, can lead to abductive reasoning. The Diophantus diagram could be reinterpreted as in figure 12.

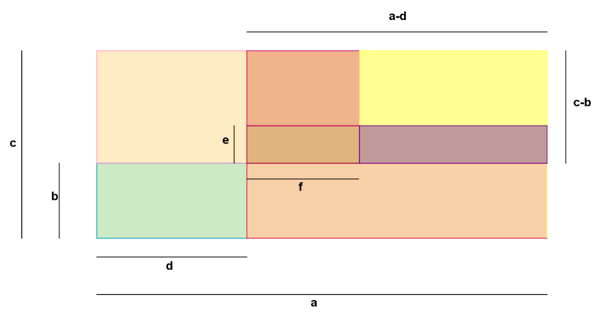


Figure 12: Diagram for proposing problems

Thus, the provocation of abductions, checks and explanations of the mathematical phenomenon possible to occur in the classroom at different levels of teaching and visualization, can be pursued by the scientific method for the elaboration of signs.

Considerations and expectations

Gontijo (2015) considers that creativity is fundamental to explore the social and technological challenges that are emerging today and that the development of creative skills can favor conditions for innovative solutions to problems encountered. This author emphasizes that the creative process does not occur in a systematic and organized manner from beginning to end, and that, among other things, the availability of time and resources for the development of activities must be taken into account. Therefore, time, available resources, and motivation in the individual's relationship with the environment should be variables considered in investigations for the development of creativity (Moraes, 2015).

Other parallel issues that require necessary attention are related to the development of a Didactics of mathematics that allows and encourages autonomous, self-confident, tolerant behaviors with ambiguous, persistent issues, as highlighted by Cropley (2009), as well as resilient, allowing to look to mathematics by different perspectives (Smith & Amnér, 1997). It is also necessary to value and stimulate the production of the interpretation objects. As well as to stimulate to observe them, to compare them and make syntheses (Ward, Smith, and Fink, 1999, apud Morais, 2015). Furthermore, to stimulate the elaboration metaphors, that is, development of languages to communicate what they perceive in some phenomenon (Sternberg & Lubart, 2003).

The approach to mathematics based on the interpretations of fundamental concepts at different levels as presented in this article, can also narrow the gap between higher and basic levels of education, pointed out by Klein (2009).

Lannin, Ellis, Elliot (2011), describe that "mathematical reasoning is an evolutionary process that includes conjecturing, generalizing, investigating why, developing and evaluating arguments" (Lannis, Ellis, Elliot, 2011, p.10).

Peirce (1958) considers mathematics as a science of mediation, and stresses that this should be the greatest concern for every mathematical educator. The fertility and consistency of our intuitions and hypotheses must be emphasized and valued, rather than stressing only importance the formal aspect and symbolic manipulation by appeal to memorization during all educational process, including evaluation, because, for Peirce, Mathematics is essentially diagrammatic thinking and diagrams, and diagrammatic figures are intended to be applied to better understand the state of things, whether experienced, or read, or imagined. (Peirce, 1931-1935, 1958, CP 3.419).

Looking at mathematics as a science that produces signs, semiosis at different levels, makes us aware that different signs cause different cognitive experiences (Peirce, 2010), and, thus, we can effectively reach the development of other cognitive processes besides memorization.

For Otte (2006) the fact that Mathematics on the one hand enables the mediation between the process of intuition and abduction and, on the other hand, enables inductive verification, puts the mathematics adequate to be conceived in semiotic terms. Otte (1993) considers that creativity requires the combination of formal and free thinking. For Monteiro (2015), provoking movement between a formal thought of mathematics and a free thought with meaning and, mediated by metaphors of the infinite, seems to have great potential to produce new interpretants and conceive mathematics as a semiotic activity.

Mueller (1981) interprets some diagrams in Euclid's elements as being mental experiments. In this perspective, Monteiro (2015) indicates that based in the proposal presented in this paper, there seems to be evidence a gap in the interpretations of diagrams with respect to the work of Euclid, that deserve to be investigated by a complementarity in the circularity of interpretations using metaphors of the infinite to elaborate problems, that is, semiotic activity for the development of creativity and, creativity to communicate mathematics.

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