The Mathematics Enthusiast

Manuscript 1586

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A Review of Jordan Ellenberg's *How Not to Be Wrong: The Power of Mathematical Thinking*

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How Not to Be Wrong is a wonderful and entertaining romp through a field of intriguing and thought-provoking anecdotes and social conundrums in which mathematical thinking and mathematics are revealed in all their messiness and power. While reading Jordan Ellenberg's weave of stories for the first time, I found myself retelling them to as many individuals in my social circles as I could. Though the intent of the book is quite serious, [its goal is to show that mathematical thinking is accessible and useful to those who have not mastered complex mathematics, that mathematical thinking can lead to profound insights about the world around us, and perhaps, to reify mathematics as a foundation for truth] I found its tone light and engaging, and I really enjoyed retelling the stories as they sparked thoughtful conversations amongst friends and family. Talking about slime mold decision making helped stave off the feelings of dread and the awareness of crumbling social foundations around us. Ellenberg makes the case for the power of mathematical thinking to open a variety of phenomena to meaningful contemplation. He chooses to highlight topics from the esoteric to ones of concern to the mainstream middle to middle-upper class folks like me. It is the kind of thinking that many of us who also teach mathematics would like our students to engage in.

Ellenberg hooks his readers with the powerful promise that he will help us learn how not to be wrong by demonstrating mathematical thinking in action. In my first, more casual, reading of the book, that is what kept me going from chapter to chapter and anecdote to anecdote. He makes a great case for mathematical thinking as a powerful tool that helps see what would otherwise remain hidden. He takes us deeper into a series of phenomena than we may otherwise be inclined to go, by taking us into the mathematician's head,

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The Mathematics Enthusiast, **ISSN 1551-3440**, vol. 20, nos. 1, 2 & 3, pp. 21-30.

seeing through their eyes and showing that you do not need a PhD in mathematics to get some of that dopamine hit.

Ellenberg's book is a useful contribution to the cannon of books on mathematical thinking and I think it is a standout in that field; but, I'd like to focus this review on how it can challenge, inspire, and provide concrete lessons to those of us who are teaching mathematics. My pitch is that reading this book can help improve how we think about what we are doing when teaching mathematics in a college or high school setting. I can't make claims beyond those settings as my teaching experience has been limited to mainstream and alternative high school (13 years) and community college (20 years) settings in Ontario.

One of the goals of math education is to figure out how to give access to 'the power of mathematical thinking' to as many of our fellow citizens as possible, though I'm not so sure that as a community of teachers we know what mathematical thinking is. Numeracy has provided one avenue of exploration and research in math education where 'giving access to mathematics for the many' has been a core principle. Though numeracy is one of those frustrating constructs that has a myriad of context dependent conceptualizations that make it ill-defined at best, researchers and practitioners have worked hard to bring more individuals into mathematics by making connections to the world our students live in. Though I don't plan to use the term, much of my perspective is informed by the literature in and around the teaching and learning of mathematics loosely fitting under the umbrella of numeracy [also known as quantitative literacy], in particular the work of Steen, 1992, and 2001; Goos, 2012; and Dehaene, 2011.

Through that lens, in my second, less linear, and more contemplative re-reading, I looked for answers to a different, but related challenge: how not to be wrong as a teacher of mathematics? Educators (often and vocally) obsess about how our students keep getting things wrong, and (rarely and privately) obsess about 'being wrong' ourselves as teachers in the classroom. For many years I have been struggling with how to bring the wonders of mathematics to the health science students who are put in front of me by the registrar's office, and to help them be less wrong. The realization that helping my students not be wrong meant that I needed to work on 'not being wrong' as an educator, and then making necessary changes in how and what I teach has been a non-linear, gradual, and lonely process.

An explicit distinction between number and quantity in an essay by Alfred Manaster (in Steen, 2001) has been particularly helpful. He proposes a clear distinction between numeracy/quantitative literacy and mathematics:

> In quantitative literacy, numbers describe features of concrete situations that enhance our understanding. In mathematics, numbers are themselves the objects of study and lead to the discovery and exploration of even more abstract objects. (p. 69)

My first reaction was to think: "I've been so wrong.", which was not a bad thing as I knew that already. Getting students to think *about* vs *with* numbers in explicitly distinct activities provides a powerful lens for examining and improving what I/we do in our classrooms.

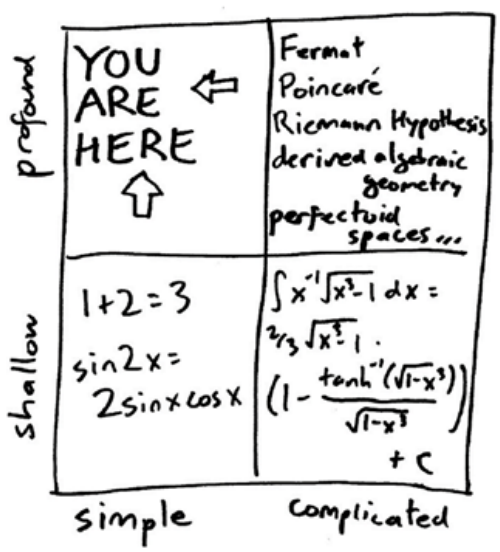
For this review, I have also tapped into research about thinking and decision making, (Kahneman, 2011; Gigerenzer, 2008), about systemic aspects of education (Arum and Roksa, 2011; Davis, 1992; Prosser and Trigwell, 1999), and mastery in mathematics education (Collins et al, 2019; Oakley, 2014; Schoenfeld, 1991).

Given that my second reading was non-linear, it felt more like a mining expedition. As such, I was able to mine a few nuggets from *How Not to Be Wrong*, nuggets which I hope to make the case that Ellenberg's approach can help show us how to build bridges between the concrete situations and abstract mathematical objects for students in mathematics classrooms [that is, between thinking *with* and thinking *about* numbers and other mathematical objects]. My review will consist of a description of these nuggets, and a pitch for how each can help us be less wrong as teachers of mathematics.

Nugget One - Simple and Profound

Ellenberg sets the tone early by dividing the mathematics universe into 4 quadrants (see my attempt at reproducing his sketch in Figure 1). His promise, which he lives up to, is to stay in the simple and profound quadrant.

Figure 1. Ellenberg's 4 quadrants of the mathematical universe.



Ellenberg agrees with many in the math education community: complex calculations, computations and manipulations of sophisticated mathematical objects are one of the pillars of mathematics. Nevertheless, one can do a lot of mathematical work with what most of our fellow citizens have at hand. The difference between professional mathematicians and the rest of us then is that they have many more math ingredients to work with. Like Ellenberg, the authors I have leaned on have shown themselves to excel at communicating complex ideas to those who don't have their depth of knowledge.

Too often our curriculum, discourse, textbooks, and assessments reside in the shallow zone of formalized mathematics. The over-emphasis on procedures of formalized mathematics has been well described by Davis (1992) and is expressed clearly by Schoenfeld (1991):

> Teachers give you rules for solving problems, which you memorize and use. Those rules don't have to make sense, and they may not, but if you do what you're told you will get the right answer, and then everybody will be happy. (p. 323)

My pitch: Use Ellenberg's conceptualization of mathematics, and the goal of simple and profound as a starting point [a thinking tool] for re-imagining the content of secondary and college math courses. I have been examining topics, lessons, exercises, and especially tests in the courses that I teach through this lens and have been surprised (and dismayed) by the amount of my testing that resides in the shallow zone even as I attempt to go a bit deeper in the lessons that I create.

Nugget Two – Teaching and Coaching: Show How Mathematical Thinking Happens

Ellenberg invites us onto a thinking ride during which we are immersed into the thinking world of a variety of mathematicians. [The book is held together in part by his delving into the minds of many figures in the history of mathematics and the mathematical ideas they were involved in creating.] We see into and through Ellenberg's mind…yes, a privileged, slightly quirky, male, academic mathematician's mind…which invites the reader to participation and encourages action.

Ellenberg provides a rough model for how we can bring students into mathematical practice by showing how mathematicians gain insights into the phenomenon at hand and their ability to shift back and forth between *thinking with* and *thinking about* numbers and other mathematical objects.

My pitch: Math teachers are often called 'content experts', but very few of us are experts in the art of communicating the simple and profound, in demonstrating the shift between 'thinking *about* and *with* numbers' nor in the art of creating exercises or experiences to take students there. Teachers themselves rarely spend time slowly solving tricky, ill-defined, open-ended, and/or wicked problems rooted in concrete situations where the mathematics to be used is not clear and which evoke the switching back and forth between concrete and abstract. By challenging ourselves to 'be mathematical' we can better model this for our students.

Nugget Three – Math as a Thinking Prosthesis Needs Grounding

"Math is like an atomic-powered prosthesis that you attach to your common sense, vastly multiplying its reach and strength" (Ellenberg, 2014, p. 12). The work of researchers in numeracy and decision making along with Prosser and Trigwell (1999) would agree that all mathematical thinking's power source is grounded in naming and counting. Counting is a fundamental thinking tool used by humans to make sense of the world and numbers are the most basic abstraction of those counts as quantities (Dehaene, 2011). Measurement, for example, is a count of units of measure, while measuring an abstract concept like uncertainty is grounded in naming and counting the set of possible outcomes. The phenomena that Ellenberg analyses are all grounded in naming and counting, and the insights that we get stem from his ability to show us that in order not to be wrong it is crucial to think about what it is that we are counting, how we are counting, and how we organize the counts. Ellenberg doesn't belabour the point, but he does state it quite emphatically: "Dividing one number by another is mere computation; figuring out *what* you should divide by *what* is mathematics." (p. 85)

My pitch: Bring on the common sense and the meaningful concrete situations that encourage 'figuring out *what* you should divide by *what*' as an explicit challenge. Take an algebraic formula that describes a concrete phenomenon and create challenges that bring the formula to ground level with counting at their core. An example from F. James Rutherford's chapter in Steen (2001) suggests the following challenge: If a worm traveled 2 meters in 3 hours what was its speed? Imagine what that actually means – imagine watching the worm. My extension: How might GPS be used to track the speed of the worm, or any other object moving on earth? Or take a measure like incidence rate of Covid-19 (new cases of disease): Where do the numbers that are being reported come from? Are they accurate? How and what is being counted?

Nugget Four – Pull on the Thread (I Dare You) and Tinkering as Mathematical Habits of Mind

Much different than being skeptical or questioning everything, the idea of pulling on a thread came to me from a video describing Richard Feynman's approach to knowledge (2017), in which we are prompted to pull on threads like 'Is a dolphin a fish?' among others. Ellenberg, like Feynman, delights in pulling on the thread of phenomena that he is describing, whether they be in the concrete world of how Americans elect their presidents, in a chapter titled There is No Such Thing as Public Opinion, or in the abstract world of mathematically described phenomena like the Law of Large Numbers. He also knows when not to pull on the thread too, such as during a rare mention of math education – his careful description the 'Math Wars'. Ellenberg also shows us how mathematicians can use pulling on a thread to get to the core of the phenomenon at hand, then instead of leaving us with a pile of disconnected threads, he follows that up with mathematical models that provide powerful means of tying up the loose ends by describing, predicting and making sense of the world around us.

My pitch: Build a community of thread pullers in mathematics education who take a mathematical object, or concrete scenario that is common in a math/stats course and pull on the thread to see what we find. One that I have been struggling with lately is the standard deviation. Instead of following the typical path to simply think about how the formula works, try bringing it down to a count of objects (Nugget 3) – does it make sense? What does the standard deviation actually measure? Why not teach the average distance from the mean $\sum_{i=1}^{n} \frac{|x_i - \mu|}{n}$ (average deviation) alongside or, dare I say, instead of standard deviation? Should we?

Nugget Five – Mathematics is Not Mystical

Even though mathematical thinking is powerful and has helped achieve a dominant position as a species, Ellenberg does not agree with Devlin's (2009, p. 97) suggestion that "natural patterns are a result of hidden mathematical laws". Instead, he compares the search for the inherent structure of mathematics to conservative United States supreme court judge Antonin Scalia's formalist view of the constitution, or baseball's view of an umpire's ruling as the truth irrespective of what actually transpired (no video replays allowed). Ellenberg sees mathematics as a co-created body of knowledge that can demystify phenomena that seem out of our grasp of experience without resorting to formalist trappings. If needed, it is ok to embrace not being sure. As Ellenberg (2014) asserts:

> Math gives us a way of being unsure in a principled way: not just throwing up our hands and saying "huh", but rather making a firm assertion: "I'm not sure, this is why I'm not sure, and this is roughly how not-sure I am." (p. 426)

My pitch: Teach about mathematics. If the job of mathematics is to help demystify the phenomena around us, then why do our students tend to see it as a set of rules almost biblical (or constitutional) in nature? Before answering that question, we must ask ourselves to what extent we as teachers see ourselves as Judge Scalia. Every time we mark a test, we are invoking the mathematics constitution as a core principle in judging the correctness of our students' work. There must be standards of course, but we may be over-emphasizing the formalist aspects of mathematics (the ones that are easily assessed), to the detriment of the others.

In our secondary and college math classes there is rarely a time and place for students to think about mathematics itself, to examine it in all its humanity and messy creativity. Surely, we can make space in our secondary and college math classes to allow students to think about something as simple as the history of the number line, which Ellenberg alludes to in his discussion of a 16th century mathematician's disdain for negative numbers as fake (p. 78). This would require the deletion of a few topics and exercises from our courses such as: ladders leaning on walls, trajectories of cannonballs, and Jane being 5 cm taller than Judy who is 4 cm shorter than Joan.

Nuggets in a Wrap, and Some Left Behind

I was asked recently by a colleague to describe what a college math course would look like in which I had the freedom to teach whatever and however I wanted. I haven't come up with a direct answer to that question yet, but I have a clearer idea of a course that I would love to take. It is a course informed by the ideas in Ellenberg's *How Not to be Wrong* with the added proviso of how not to be wrong as a teacher of mathematics. I imagine it as a professional development experience with math teacher colleagues from a variety of institutional settings. It would be a course that would allow us to develop a clearer perspective on the topics from the courses we teach, inviting us to pull on the threads of the mathematical objects we are asking our students to learn how to use, to reconnect the mathematical objects to the concrete world they came from, and prod us to examine the historical context from which they emerged. Then, as a community of math educators, we can reconstruct the courses as ones that evoke mathematical thinking in both teacher and student.

How Not to Be Wrong was the right book at the right time for me in my journey of thinking about how and what to teach in a mathematics classroom through a numeracy lens. There is a lot more to explore in the book and many paths not taken in this review. In these final words, I do want to put in a pitch for the idea of mastery that runs throughout the book; especially as it lies as a foundation of the last chapter titled *How to Be Right*. The book contains many riches beyond the ones I have highlighted here, and I heartily recommend a careful reading. You will come away more ready for, and less likely to be wrong in whatever mathematical or educational tasks you take on.

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