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Arithmetic: Craft or Art? A Recursive (Re)view of Paul Lockhart's A Mathematician's Lament and Arithmetic

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A Recursive (Re)view of Paul Lockhart's *A Mathematician's Lament* and *Arithmetic*

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Context and Introduction

Paul Lockhart is a former computer programmer, elementary teacher, research mathematician, university professor, and somewhat more recently a K-12 mathematics teacher and author. Lockhart's first written composition appeared two decades ago in 2002. All of 25 pages, his essay *A Mathematician's Lament* began like this:

> A musician wakes from a terrible nightmare. In this dream he finds himself in a society where music education has been made mandatory … Educators, school systems, and the state are put in charge of this vital project. Studies are commissioned, committees are formed, and decisions are made—all without the advice or participation of a single working musician or composer." (p. 1)

The nightmare Lockhart shares is what he sees happening in mathematics classrooms where linguistic fluency reigns supreme and the mission of primary and secondary schools is "to train the students to use this language—to jiggle symbols around according to a fixed set of rules" (p. 1). While not officially published, the paper makes its way to those in mathematics and mathematics education. Mathematician Keith Devlin reads Lockhart's lament and concludes that it was "one of the best critiques of current K-12 mathematics education" (Devlin, 2008, para. 1). He then posts the paper online in his monthly column for the Mathematical Association of America (MAA), *Devlin's Angle*. Lockhart extends the essay one year later and publishes it as book. The book which bears the same title includes the subtitle *How School Cheats Us Out of Our Most Fascinating and Imaginative Art* and features a foreword by Devlin. As in the original essay, Lockhart (2009) argues for mathematics to be an adventure, an experience, hard creative work, and ultimately, that mathematics should be understood as art.

> [T]he fact is that there is nothing as dreamy and poetic, nothing as radical, subversive, and psychedelic, as mathematics. It is every bit as mind-blowing as

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> cosmology or physics (mathematicians *conceived* black holes long before astronomers found any), and allows more freedom of expression than poetry, art, or music (which depend heavily on properties of the physical universe). Mathematics is the purest of the arts, as well as the most misunderstood. (p. 23)

Drawing on mathematician Johann Gauss, Lockhart (2009) asserts that regardless of representation, symbolic or otherwise, at its core, mathematics is about ideas—beautiful ideas. He contends all too often K-12 students are denied opportunity to interact with mathematics in ways that allow them to question, explore, and get to know what "mathematics looks like and feels like" (p. 26). Instead of genuine questions which naturally lead to real problems, where knowledge and skills result from the creative process, students are presented with "a sterile set of "facts" to be memorized and procedures to be followed" (Lockhart, 2009, p. 27) and applied mindlessly and redundantly.

> Gone is the thrill, the joy, even the pain and frustration of the creative act. There is not even a *problem* anymore. The question has been asked and answered at the same time— there is nothing left for the student to do. (Lockhart, 2009, p. 28)

As in the conclusion of the essay, Lockhart (2002) argues that such indoctrination begins in the elementary years where mathematics curriculums, both planned and enacted, require children "sitting still, filling out worksheets, and following directions" (p. 24), where expectations emphasize mastery of:

> a complex set of algorithms for manipulating Hindi symbols, unrelated to any real desire or curiosity on their part, and regarded only a few centuries ago as too difficult for the average adult. Multiplication tables are stressed, as are parents, teachers, and the kids themselves. (p. 25)

"But don't we need third graders to be able to do arithmetic?" (Simplicio in conversation with Slaviati, as quoted in Lockhart, 2002, p. 12). This question, many more years older than two decades, persists today and continues to be one often asked by teachers and parents. Thus, in the spirit of keeping the conversation going, I take the question up again. This time, in the search for potentially new responses inside Lockhart's most recent book.

Exploring *Arithmetic*

Arithmetic is book three for Lockhart, following the publication of *Measurement* in 2012. Similar to his essay and two previous works, the language and themes in this one is also easily accessible and straightforward. Had the author's work been designed as a larger hardcovered text and the line drawings as supersized coloured images, it could have been a (and perhaps the only) coffee table book featuring arithmetic! Each of the fifteen chapter titles clearly identifies a distinct topic (e.g., Language, Rome, Machines, Multiplication) which also includes curious problems throughout (e.g., "Suppose we wanted there to be two zero entities, each having the property that when added to any number leaves it alone. Does anything go wrong?" on page 197). As such, the book can be read from beginning to end or picked up, put down, and read in parts while investigating the various prompts which Lockhart presents along the way.

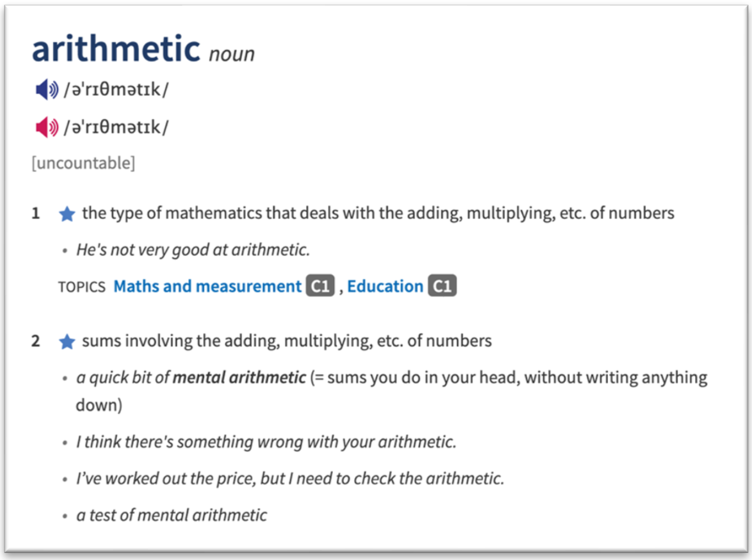
The casual first-person narrative style of Lockhart's writing accompanied with the examples he uses to conceptually walk the reader through the ideas, makes *Arithmetic* suitable for a general audience of ages 12 years and older. Content-wise, parents and teachers could also share excerpts with even younger children as the concepts featured in the book appear in mathematics curriculum documents as early as grade one (e.g., British Columbia Ministry of Education, 2016). More so, I see *Arithmetic* as a companion book of sorts, which I discuss later, one that offers interesting and illustrative problems useful for introducing radical (and beautiful) ideas about arithmetic to teachers and teachers-to-be. The ideas naturally lend themselves to important complementary (and beautiful) pedagogical ideas about what it means to teach and learn elementary mathematics generally and arithmetic specifically.

Ready to examine Lockhart's book closer, other questions emerge: How is arithmetic commonly defined? In what ways are such meanings the same as or different from those communicated within the discourse of educational K-12 mathematics education documents? What meaning does Lockhart hold for arithmetic? How does his definition compare with those typically associated with elementary school arithmetic?

What is Arithmetic?

Consider the following definitions and examples from the *Oxford Advanced Learner's Dictionary*:

Figure 1. Digital image of a definition for arithmetic.



arithmetic *noun*

/ə'rɪθmətɪk/

/ə'rɪθmətɪk/

[uncountable]

1 the type of mathematics that deals with the adding, multiplying, etc. of numbers

- *He's not very good at arithmetic.*

TOPICS Maths and measurement C1, Education C1

2 sums involving the adding, multiplying, etc. of numbers

- *a quick bit of* ***mental arithmetic*** (= sums you do in your head, without writing anything down)
- *I think there's something wrong with your arithmetic.*
- *I've worked out the price, but I need to check the arithmetic.*
- *a test of mental arithmetic*

Next look through K-12 mathematics education standards, curriculums, and association documents from the past 20 years about arithmetic. Key content focuses on number systems and operations as well as algorithms for elementary arithmetic. Skills to be developed include understanding how to represent as well as relate numbers and operations to solve problems, number sense, estimating, applying and explaining methods, computational fluency and efficiency. While conceptually-based with mention of fostering student appreciation, positive attitudes, and confidence in doing mathematics, for the most part, the documents reflect elaborations of the dictionary definition for arithmetic. Consequently, as Lockhart (2002, 2009) argues, being content and competency-driven, such standards, curriculums, and positions for mathematics education do not allow for addressing the aesthetic aspects of mathematics.

Now compare the dictionary definitions and descriptions with etymological meanings for arithmetic.

Figure 2. Digital image of root meanings for arithmetic from the *Online Etymology Dictionary*.

arithmetic (n.)

"art of computation, the most elementary branch of mathematics," mid-13c., *arsmetike*, from Old French *arsmetique* (12c.), from Latin *arithmetica*, from Greek *arithmetike (tekhnē)* "(the) counting (art)," fem. of *arithmetikos* "of or for reckoning, arithmetical," from *arithmos* "number, counting, amount," from PIE **erei-dhmo-*, suffixed variant form of root ***re-** "to reason, count."

The form *arsmetrik* was based on folk-etymology derivation from Medieval Latin *ars metrica*; the spelling was corrected early 16c. in English (though *arsmetry* is attested from 1590s) and French. The native formation in Old English was *tælcræft*, literally "tell-craft."

While in stark contrast to the dictionary and described mathematics education documents, these root meanings for arithmetic appear strikingly similar to Lockhart's interpretation. Lockhart (2017) explains in his note to the reader that arithmetic, like a *tælcræf*/tell-craft, is a kind of folk art; that "[a]rithmetic is the skillful arrangement of numerical information for ease of communication and comparison. It is fun and enjoyable activity of the mind and a relaxing and amusing pastime—a kind of 'symbol knitting,' if you will" (p. vii). Lockhart (2017) continues to characterize arithmetic as craft throughout the book—e.g., what once was "requiring years of apprenticeship and experience" (p. 149), is now a "creative and entertaining craft of counting well" (p. 4) which we can all get good at if we want to; and that arithmetic is "not really a big deal; it's just about… how much you care about doing something well—just as with knitting" (p. 30). Symbol knitting is the focus Lockhart takes, developing the metaphor and arithmetic as craft by presenting different skills and techniques ranging from ancient to more current to purely mathematical contexts and situations for the reader.

Re(-)viewing Lockhart's Works

Immediately after reading Lockhart's note to the reader for a second time, his lament starts rattling around in my mind again. First as background noise and then as an emergent framework which I can use to recursively re-view *A Mathematician's Lament* as I review *Arithmetic*. Here I wonder whether Lockhart will continue to champion the idea that mathematics is art and if so, how he might do this given the notion of arithmetic as folk art or craft. In both the original essay and subsequent book, Lockhart (2002, 2009) contends mathematics is art. More specifically:

- Mathematics is inherently interesting and radically relevant because it is a meaningful human experience.
- Mathematics *"is hard creative work"* (2002, p. 11)
- Doing mathematics is about discovering patterns and crafting beautiful and meaningful explanations.

As I ponder them, I remember the point Lockhart makes in his lament that mathematicians be included in the project of mathematics education. Using the three assertions as my frame for reading *Arithmetic*, I make Lockhart's point the target or goal of this (re)view; that is, to gather ideas core to mathematics that could in complementary ways, be substantiated with radical ideas well-known in mathematics education. Said differently, I am eager to discover ways to connect the concepts and examples in Lockhart's book to pedagogical theories in mathematics education. And, how in doing so might not only expose tacit assumptions about elementary school arithmetic but also enrich by deepening value and meaning for it, thereby enabling arithmetic to be enacted differently by teachers, children, and parents. What follows is thus a re-view of Lockhart's three assertions from *A Mathematician's Lament* and a review of *Arithmetic*. By replacing "mathematics" with "arithmetic" for each of the statements, I address key ideas which Lockhart raises in *Arithmetic*. I highlight how they relate to the described queries and the ways they occasion mathematical-pedagogical possibilities for reconceptualizing children's meaningful engagement in arithmetic and particularly, classroom mathematics in the elementary years.

Arithmetic is Inherently Interesting and Radically Relevant Because it is a Meaningful Human Experience

One important difference between *A Mathematician's Lament* and *Arithmetic*, is that despite Lockhart (2017) conceptualizing arithmetic as he similarly does in his essay with mathematics, "as a fun and enjoyable activity of the mind and a relaxing and amusing pastime" (p. vii), he details the role arithmetic plays in human history in the book. From Chapter One onward, Lockhart traces imagined (i.e., Hand, Banana, and Tree tribes) emergences and historical (i.e., Babylonia, Egypt, Rome, China, Japan, Europe, and India) accounts regarding the evolution of number systems. Presenting several practical instances of arithmetic as craft or the "skillful arrangement of numerical information," (2017, p. 20) and its development driven by humans' desire for communication and commerce systems to be increasingly accurate, efficient, and convenient, Lockhart focuses on how human cultures honed their abilities and technologies to group, count, and compare things. In doing so, he draws the reader's attention to arithmetical expressions, operations, and properties, using both image and text to explicate the ways these objects function mathematically, whether physically (e.g., Roman tabula and electric adding machine), written, and/or mentally (e.g., five-barred gate tallies, Hindu-Arabic right to left algorithms). Interspersing examples with questions extend the ideas which Lockhart presents, inviting the reader to think with and through the objects. Doing so allows the reader to experience what each particular arithmetic "looks and feels like" (Lockhart, 2002, p. 4).

In a more profound way, Lockhart reveals how our past, current, and future evolution as biological beings (e.g., perceiving quantities up to four or five, numerical systems based on fingers/toes/persons or geographical location) and cultural collectives recursively (re)produce arithmetic. Arithmetic, like all mathematics, results from us being human and cultural creatures (Barton, 1996; Bishop, 1988; Lakoff & Núñez, 2000; Núñez et al., 2012; Varela et al., 1993). From clapping to drawing/writing, to step-by-step procedures, to tools and machines, Lockhart shows how these developments have directly shaped our beliefs, understandings, and the ways we carry forth arithmetic today. The accounts illuminate how culturally specific and contextually dependent our mathematics practices are, especially when it comes to school arithmetic. While the reader might wish to search for more detailed histories regarding the forms of arithmetic mentioned in the book, pedagogically speaking, what Lockhart offers brings awareness to the danger and missed opportunity when one particular kind of arithmetic is favored at the expense of others.

More than an argument for addressing individual learning styles of children, providing alternative procedures for carrying out computations, or even engaging children in the imaginative realm of pure mathematics, learning such skills and techniques encourage re-cognition of arithmetic as distinct and recurrent biological/psychological/cultural patterns (i.e., algorithms) that contribute to what it means to be human and alive (Davis, 2014, para. 4). More than algorithms as step-by-step procedures for solving computational problems, arithmetic as living algo-rhythms (Glanfield et al., 2020) enables re-cognizing beautifully unique forms created for grouping, counting, and comparing that (have) enable(d) humans to make sense of their immediate environment and the world at large. Whether a Soroban, Egyptian marked-value system, or Quipu, each embody distinct cultural ways of knowing, doing, and being mathematical.

Arithmetic *"is hard creative work"*[2]

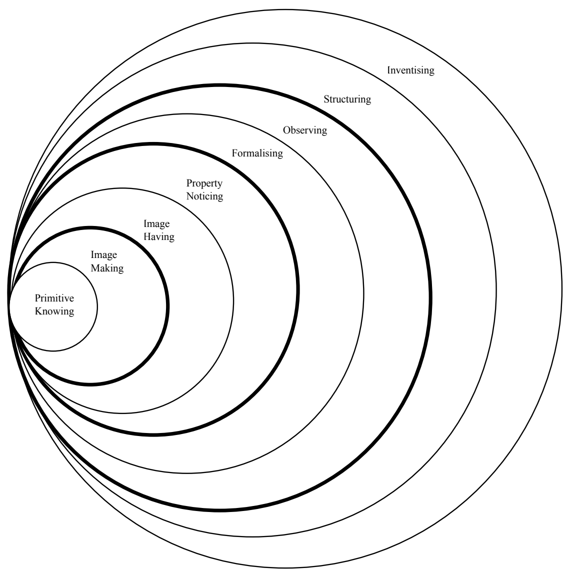
While Lockhart's book showcases arithmetic as culturally created conceptual, linguistic, and technological objects, the questions and examples also easily connect with well-established theories within mathematics education. This point is particularly relevant as it creates the possibility for further explicating the kinds of thinking that Lockhart demonstrates in his book. Specifically naming and theorizing mathematical thinking can facilitate teachers' and parents' know-how of what such thinking "looks and feels like" (Lockhart, 2002, p. 4) which can then better help to engage children in the creative and effortful work which Lockhart (2017) views as necessary for "counting well" (p. 4). Doing so would also go a long way to promote children's curiosity and love for arithmetic, not to mention enable them to experience, differentiate, and articulate their thinking as well as provide a framework in which they could deepen their study of mathematical ideas.

Within mathematics education, 'good' understanding involves all that Lockhart (2002, 2009) alludes to when he speaks of the joy, pain, and frustration that the creative act of thinking mathematically brings (e.g., Boaler, 2019). The Pirie-Kieren model (see Figure 3) and theory for the dynamical growth of mathematical understanding (e.g., Pirie & Kieren, 1994) is an example of how arithmetic thinking can be more explicitly observed, articulated, and facilitated in the ways that Lockhart suggests in his book. In brief, Pirie and Kieren (1994) characterize mathematical understanding to be thinking, knowing, and

[2] Lockhart, 2002, p. 11.

doing mathematics. As inherently dynamic, recursive, and embodied, 'good' understanding entails flexible and fluid integration of informal and formal mathematical knowledge which is facilitated by the process of *Folding Back* (and forth) within eight levelled but non-linear realms of sense-making. Pirie and Kieren (1994) refer to the eight realms as: *Primitive Knowing, Image Making, Image Having, Property Noticing, Formalising, Observing, Structuring,* and *Inventising*.[3]

Figure 3. Model of the Pirie-Kieren dynamical theory for the growth of mathematical understanding.



In the same manner that the different arithmetic ways of thinking featured in Lockhart's imagined and historical accounts can be identified, located, and distinguished using the Pirie-Kieren model (e.g., subitizing as *Primitive Knowing*, figuring out that one rock can represent one sheep as *Image Making*, knowing that VIII is equivalent to 8 as *Image Having*), so too can children's arithmetic thinking be observed and distinguished (e.g., Thom & Pirie, 2006). Perhaps even more importantly, the Pirie-Kieren model/theory substantiates Lockhart's argument that there's much more to thinking mathematically than simply knowing (i.e., *Image Having*) your basic arithmetic facts. Pirie and Kieren make this clear in the structure of their model and in their conception of the theory that

[3] For detailed explanations and empirical studies regarding the Pirie-Kieren model and theory, see for example, Pirie and Kieren (1989, 1994), Martin and Towers (2015), Thom and Pirie (2006), and Thom and Glanfield (2018, 2022).

regardless of age or level of mathematics, any and all concepts do not end at *Image Having* (e.g., Thom & Pirie, 2006) but can and should move dynamically and fluidly inside and across all eight realms.

Doing Arithmetic Is About Discovering Patterns and Crafting Beautiful and Meaningful Explanations.

While Lockhart (2017) drops hints of this in earlier chapters (e.g., commutative and associative aspects on pages 185 and 186), it is not until the end of the book in Chapter 15 where he reveals how arithmetic transforms from the craftwork of symbol-knitting to "the fine art … of counting *beautifully*" (p.197). Here the author explains that arithmetic includes counting:

> very patterned, orderly, and symmetrical in some way, the situation may allow for clever and imaginative ways to count, without the tedium of actually counting. Of course, this may require quite a bit of extra mental labor, but this is all in keeping with the mathematician's general philosophy: *to be willing to think really hard in order to find clever ways to get out of doing any actually work*. (Lockhart, 2017, p. 197)

He describes the transformation as resulting from effortful creative work, when the parts of arithmetic fade into the background and the pattern or relationship between the parts comes to the foreground. It is in these moments, that Lockhart claims coherent and breathtakingly beautiful ideas emerge. For example, when solving the 9x facts with one's fingers changes from being an amazing and unexplainable trick to a choreographed dance about the relationality of 10 digits. Such thinking is characteristic of *Property Noticing* within the Pirie-Kieren model/theory (1994), where manipulating or combining aspects of ideas/meanings enables distinguishing relevant features of some context-specific mathematics. For example, when Lockhart prompts the reader to generate numerical expressions for a drawing, such as the triangular and triangular/rectangular designs on page 200, he shifts the reader's attention away from focusing on the individual dots in the design (i.e., *Image Making* and *Image Having*) and toward the arrangements that the group of dots make as the horizontal rows subsequently increase/decrease by one dot (i.e., *Property Noticing)*. With each example Lockhart carefully stops just short of moving into 'for any' or 'for all' thinking, which according to Pirie and Kieren would be indicative of *Formalising* (I wonder if *Algebra* will be Lockhart's next book?). Focusing on specific patterns as opposed to formulaic ones, Lockhart's demonstrations coupled with the Pirie-Kieren model/theory elucidate arithmetic interrelations and distinctions across the realms of Primitive Knowing, Image Making, Image Having, and Property Noticing. In these ways, Lockhart's conceptual modelling connects well with documented instances of mathematics learners doing just this (e.g., Thom, 2012; Thom & Pirie, 2006).

Arithmetic as Craft and Art

To conclude, *Arithmetic* picks up where Lockhart's 2002/2009 lament leaves off. The book serves as both a response to the question "But don't we need third graders to be able to do arithmetic?" (Lockhart, 2002, p. 12) and an elaboration of how the three assertions from *A Mathematician's Lament* (Lockhart 2002, 2009) apply to elementary arithmetic. True to the root meaning of arithmetic, Lockhart elucidates this branch of mathematics as skills and techniques to be honed as well as that which necessitates emotions, feelings, and vision to bring it forth as craft *and* art. Clearly then, not only should third graders be able to reckon and reason arithmetically, but such opportunities which hold potential for their aesthetic engagement and mathematical transformation can also serve as occasions in which children "create … profound simple beauty out of nothing and [be] change[d] in the process" (Lockhart, 2002, p. 5).

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