The Mathematics Enthusiast

Manuscript 1615

Seeing Pascal's Mystic Hexagon from a Different Angle

Anderson Norton

Seeing Pascal's Mystic Hexagon from a Different Angle

Anderson Norton
Virginia Tech, Blacksburg, VA

ABSTRACT: Hexagons inscribed within circles have the mystic property that their three pairs of opposite sides (when extended) intersect on the same line. This surprisingly simple result seems to support the Platonic notion that elegant mathematical relationships exist in some perfect realm awaiting human discovery. To the contrary, this article introduces a transformational proof that provides for intuitive understanding of the mystic property while supporting the idea that mathematical objects, and relationships between them, arise from coordinations of our own mental actions.

Keywords: epistemology of mathematics, transformational geometry

The Mathematics Enthusiast, **ISSN 1551-3440, vol. 21, no. 1&2**, pp. 35–41

> Most often it is not possible to set [intuition] out logically as in mathematics, because the necessary principles are not ready to hand, and it would be an endless task to undertake. The thing must be seen all at once, at a glance, and not as a result of progressive reasoning, at least up to a point.
>
> Blaise Pascal [Pas2]

Introduction

In the quote above, Pascal might have been reflecting on his early experience with the mystic hexagon. Consider the case of self-intersecting hexagon $ABCDEF$ inscribed in a circle (see Figure 1). Not only do its three pairs of opposite sides form intersections along the same line (the Pascal line); its six interior angles also relate in a mysterious way. Whereas the former property holds under projection so that Pascal's theorem generalizes to hexagons inscribed in any conic section, the latter property is unique to circles and, so, invites a Euclidean proof. Here, I use Euclidean transformations to demonstrate what it might mean to see the thing "all at once."

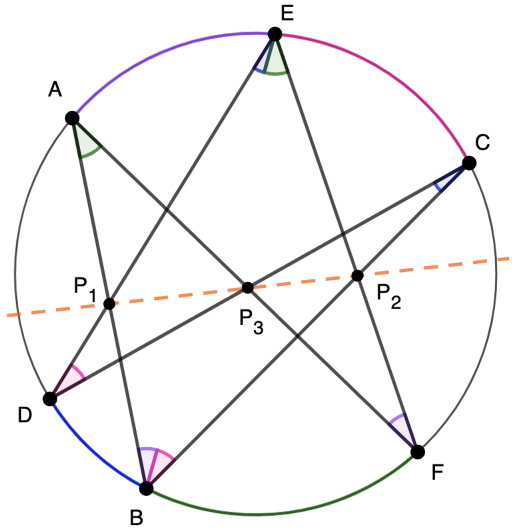

Figure 1: A case of Pascal's mystic hexagon.

Since 1639, when young Pascal first posed his theorem [Bur, Pas1], it has been proven many times, with modern proofs ranging from "simple" [Yze] to "very simple" [SM]. Yzeren's [Yze] simple proof relies on the construction of a new circle (see the dashed circle in Figure 2) passing through two vertices of the hexagon and the point of intersection of one pair of its opposite sides (P_1). From there, the proof really is straightforward, but the motivation for constructing the new circle is less intuitive and more than simple.

Augros [Aug] motivated a proof by engaging readers in a thought experiment to illustrate how Pascal might have formulated his theorem in the first place [Aug]. The conjecture relied on changes in perspective, and its proof relied on projective geometry, to generalize from the special case of a hexagon with two pairs of parallel sides (i.e., as if sides AB and EF were parallel, and sides DE and BC were parallel, in Figure 1).

The allusiveness of an intuitive proof contributes to the mysticism of Pascal's theorem. However, formal proofs require delineation, which stands in contrast to the simultaneities and coincidences that render the hexagon mystical. If "the thing must be seen all at once," how can we unravel it?

In particular, notice that angles BAF and DCB sum to angle DAF (see Figure 1) We can see this via the arc lengths that subtend them: arcs BF, DB, and DF, respectively. The same phenomenon occurs among the three angles on the other side of the Pascal line. Moreover, segments P_1P_3, P_3P_2, and P_1P_2 also have an accumulating effect: they subtend the two pairs of angles and their sums, respectively.

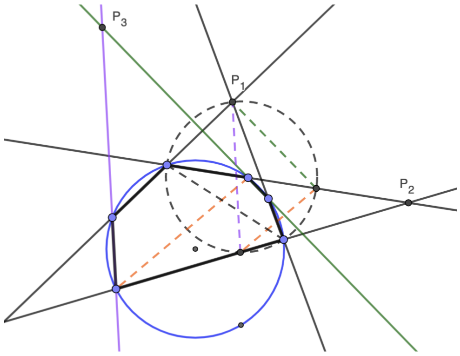

Figure 2: Yzeren's proof of Pasal's theorem.

This happens all at once, on both sides of the line. To prove this relationship and this particular case of Pascal's theorem, I adopt an approach motivated by Felix Klein's Erlangen program [Kle].

Originally used to classify geometries with corresponding groups of transformations (e.g., the group of isometries and dilations for Euclidean geometry, extended to include projections for projective geometry), the Erlangen program extends to the study of algebraic invariants, as seen in the work of Emmy Noether [RK]. This framework has also impacted mathematics education, especially from a Piagetian perspective [Pia], in which mathematical objects arise through a coordination of our own mental actions, or transformations [Nor]. For example, multiplication might be understood as a transformation of units not captured through the linear process of repeated addition [Bou, Bur], "transformational reasoning" undergirds students' understanding of mathematical systems [Sim], and "transformational proofs" support students' meaning-making for axiomatic proofs [HS]. A proof that relies on geometric transformations, then, might provide insights into the inner workings of the mystic hexagon.

1 A Pair of Two-Chords Theorems

As an example of transformational reasoning, consider the pair of chords shown on left side of Figure 3. What might we say about the angles and segments it forms? As it turns out, we can say quite a lot! If we translate point P to center O, the angles remain invariant, but the pairs of opposite arcs, which subtend the pairs of vertical angles at P, average out to reveal the measures of those angles. In other words, each pair of vertical angles is measured by the average of the two opposite arc lengths.

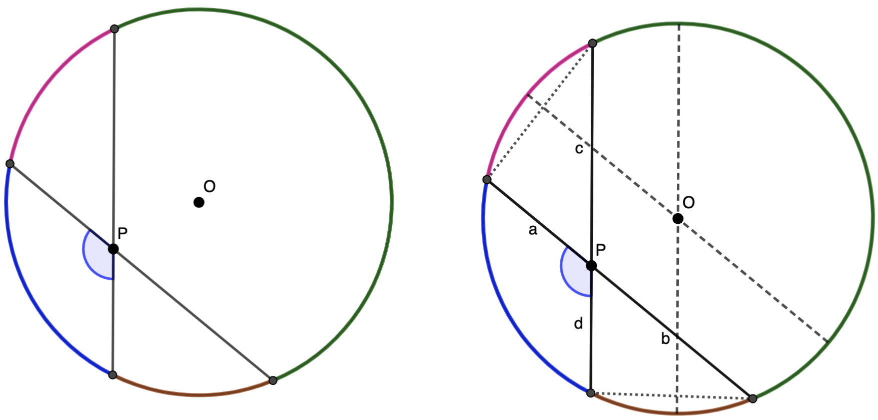

Figure 3: Two two-chords theorems.

In the case of the angle marked in blue, the angle measure is the average of the blue and green arc lengths. As P moves to O, what is added to the blue arc is taken from the green arc, by the symmetry of the circle, until the two arcs have the same length and measure the central angle. This result is known as the two chords theorem, and the inscribed angles theorem arises as a special case, as one of the arc lengths goes to zero.

As shown on the right side of Figure 3, we can use what we know about the angles to prove another two-chords theorem: the intersecting chords theorem. This theorem appears as Proposition 35 in Book III of Euclid's *Elements*, where it is proved without appeal to inscribed angles or the first two-chords theorem [Joy]. However, as demonstrated elsewhere [Bog], we can readily prove $ab = cd$ by referencing the two pairs of inscribed angles within two similar triangles.

2 A Pair of Subtended Angles

Here, we turn to another pair of angles, subtended by segments RS and ST, with lengths g and h, respectively (see Figure 4). Whereas arcs provide for stable measures for inscribed angles, segments are tricky things because their role as angle measures depends upon their particular position within the circle. For instance, if we move P along the circumference of the circle, the lengths of RS and ST no longer have the same significance in measuring the openness of the angles.

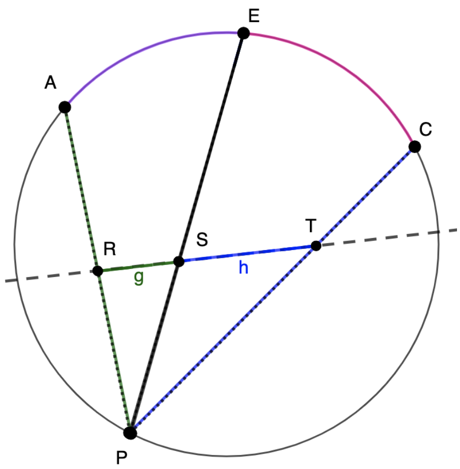

Figure 4: Segments RS and ST subtending angles APE and EPC.

3 A Pair of Transformations

Figure 5 shows what happens when we transform P in two different ways: once along the circumference of the circle to point P_1, while maintaining angle EPA; and once along the circumference to point P_2, while maintaining angle CPE. Notice that these particular transformations place point T in line with points E and P_1, and likewise place R in line with E and P_2. In the process, segment RS is transformed to segment R_1S_1, and segment ST is transformed to segment S_2T_2, with their positions changed but lengths preserved. As we will see, we can further transform each of those segments back on to segment RT. They will lie on opposite ends of RT and have different lengths than before, but those lengths will still sum to the length of RT.

Take segment RS, which subtends angle EPA. When we move P to P_1 and maintain angle EPA (now EP_1A), we transform triangle SPR into congruent triangle $S_1P_1R_1$. Side R_1S_1 has the same length as side RS but is rotated through angle PEP_1 (as measured by half the length of arc PP_1). Thus, we have triangle $Q_1R_1S_1$ similar to triangle TES. Likewise, triangle $Q_2T_2S_2$ is similar to triangle RES.

Theorem 3.1. *When triangle $Q_1P_1R_1$ is dilated from point P_1 so that Q_1 coincides with T, and when triangle Q_2P_2T is dilated from point P_2 so that Q_2 coincides with R, R_1 and T_2 will coincide at a common intersection, I, along the segment RT.*

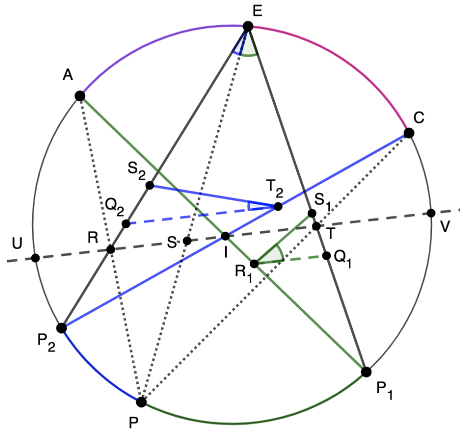

Figure 5: Transforming segments RS and ST.

Proof. To prove the claim, we will label segment lengths as indicated in Table 1.

Table 1: Labels for segment lengths

Label	**Length of Segment**
a	ER
b	RP_2
c	ET
d	TP_1
g	$RS = R_1S_1$
h	$ST = S_2T_2$
g'	R_1Q_1
h'	S_2T_2
j	Q_2S_2
k	Q_1S_1
m	ES
n	SP
u	UR
v	TV

With respect to these values, as labeled in Figure 6, the triangle similarities noted above give us the equalities $mk = mj = gh$. Note that these equalities also imply $j = k$.

Furthermore, the intersecting chords theorem provides us with the following three relations:

$$mn = (u + g)(h + v)$$

$$ab = u(g + h + v)$$

$$cd = (u + g + h)v$$

Now we can see how lengths g and h are transformed into new lengths, g'' and h'', along the line through RT. First g is transformed to g' by a factor of a/m. Then the dilation transforms g' by a factor of $b/(n - j)$. h undergoes a corresponding transformation to h' and then h'', as indicted in the equations below.

$$g'' = g(c/m)(d/(n - k)) = g(cd)/(mn - mk) = g(cd)/(mn - gh)$$

$$h'' = h(a/m)(b/(n - j)) = h(ab)/(mn - mj) = h(ab)/(mn - gh)$$

Finally, we show that the sum $g + h$ is maintained in the transformation of g and h. Here, we rely on the relations provided by the intersecting chords theorem.

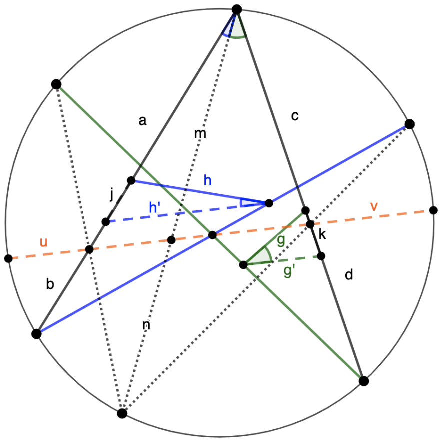

Figure 6: The invariance of g+h.

$$g'' + h'' = [g(cd) + h(ab)]/(mn - gh) =$$

$$[g(u + g + h)v + hu(g + h + v)]/[(u + g)(h + v) - gh] =$$

$$[(g + h)(uh + gv + uv)]/(uh + gv + uv) = g + h$$

Because of this equality, we know that the two transformed segments, g'' and h'' meet at a common endpoint, I, along the line through segment RT. Thus, with reference to Figure 1, hexagon sides AF and CD intersect along the line through P_1 and P_2 — the Pascal line.

□

4 Conclusion and Remark

In this transformational proof of Pascal's theorem, we find the substance of Pascal's lament. Although the transformations themselves are simple and provide for intuition, delineating and marking them for others to see seems an endless and painstaking task. Existing proofs admit more elegance by appealing to general principles from linear algebra or projective geometry but provide few insights into the mystical phenomenon.

Transformational proofs are intuitive because they explicitly refer the reader to mental actions they can perform rather than formal relationships between static figures. Following Felix Klein and Emmy Noether, we can also consider figurative relationships that remain invariant under transformation. In the present case, the sum of lengths of the subtending segments is unchanged by the pair of transformations. When we consider that, as mathematical objects, the figures themselves are results of mental actions (e.g., sweeping out a circle or a line, as allowed within Euclidean geometry), the whole enterprise of mathematics becomes a coordination of our own mental actions.

References

[Aug] Augros, M. (2012). Rediscovering Pascal's Mystic Hexagon. *The College Mathematics Journal, 43*(3), 194–202.

[Bog] Bogomolny, A. (2021). Intersecting chords theorem—a visual proof. https://www.cut-the-knot.org/Curriculum/Geometry/GeoGebra/IntersectingChordsTheorem.shtml

[Bou] Boulet, G. (1998). On the essence of multiplication. *For the Learning of Mathematics, 18*(3), 12–19.

[Bur] Burton, D. M. (2007). The history of mathematics: An introduction. New York: McGraw-Hill.

[Dav] Davydov, V. V. (1991). A psychological analysis of the operation of multiplication (J. Teller, Trans.). In L. P. Steffe (Ed.), *Psychological abilities of primary school children in learning mathematics* (Vol. 6, pp. 9–85). Reston, VA: National Council of Teachers of Mathematics.

[HS] Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *American Mathematical Society, 7*, 234–283.

[Joy] Joyce, D. (2021). Euclid's Elements. https://mathcs.clarku.edu/ djoyce/elements/elements.html

[Kle] Klein, F. (1893). A comparative review of recent researches in geometry. *Bulletin of the American Mathematical Society, 2*(10), 215–249.

[Nor] Norton, A. (2019). An Erlangen Program That Empowers Students' Mathematics. *For the Learning of Mathematics, 39*(3), 22–27.

[Pas1] Pascal, B. (1963). *Essai pour les coniques.* Éditions du Seuil.

[Pas2] Pascal, B. (1966). *Pensées* (A. J. Krailsheimer, Trans.). London: Penguin (Original work published in 1670).

[Pia] Piaget, J. (1970). *Structuralism* (C. Maschler, Trans.). New York: Basic Books (Original work published 1968).

[RK] Rowe, D. E., & Koreuber, M. (2020). *Proving it her way: Emmy Noether, a life in mathematics.* Cham, Switzerland: Springer.

[SM] Stefanović, N., & Milošević, M. (2010). A very simple proof of Pascal's hexagon theorem and some applications. *Proceedings-Mathematical Sciences, 120*(5), 619–629.

[Sim] Simon, M. A. (1996). Beyond inductive and deductive reasoning: The search for a sense of knowing. *Educational Studies in mathematics, 30*(2), 197–210.

[Yze] Yzeren, J. (1993). A simple proof of Pascal's Hexagon Theorem. *American Mathematical Monthly, 100*(10), 930–931.

Department of Mathematics, Virginia Tech, Blacksburg, VA 24061
Email address: norton3@vt.edu