Stepping Stone Problem on Graphs

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ABSTRACT: This paper formalizes the stepping stone problem introduced Ladoucer and Rebenstock [G] to the setting of simple graphs. This paper considers the set of functions from the vertices of our graph to \( \mathbb{N} \), which assign a fixed number of 1’s to some vertices and assign higher numbers to other vertices by adding up their neighbors’ assignments. The stepping stone solution is defined as an element obtained from the \( \text{argmax} \) of the maxima of these functions, and the maxima as its growth. This work is organized into work on the bounded and unbounded degree graph cases. In the bounded case, sufficient conditions are obtained for superlinear and sublinear stepping stone solution growth. Furthermore this paper demonstrates the existence of a basis of graphs which characterizes superlinear growth. In the unbounded case, properly sublinear and superlinear stepping stone solution growth are obtained.

Keywords: Graph Theory, Combinatorics
Introduction

In a recent episode of the popular YouTube series, *Numberphile* [H], the stepping stone problem on an infinite chessboard ($Z^2$) was popularized. First realized in work done by Ladouceur and Rebenstock [G], the stepping stone problem was originally only considered on $Z^2$. In the notation of Section 1, they consider the graph defined such that $V$ is $Z^2$, or all pairs of integers, and such that

$$E = \{v_1v_2|v_1 \pm \vec{e} = v_2 \text{ where } \vec{e} \in \{(0, 1), (1, 0), (1, 1), \text{ and } (1, -1)\}\}$$

We are using the vector space properties of $Z^2$ in this definition. In the language of Section 1, they show that the stepping stone solution of this graph is linear.

The work of Ladouceur and Rebenstock [G] is informal; one purpose of this paper is to formalize it and place it in a graph theoretic setting. In Section 1, we supply a rigorous definition of the stepping stone solution on a graph. Given $n \in \mathbb{N}$, we assign $n$ vertices of a graph $G$ with the value 1. Then, for $k > 1$, we consider assigning a single vertex of a graph $G$ with the value $k$ on the condition that the sum of its neighbor’s assignments less than $k$ add up to $k$. For instance, a vertex $v$ may have value two iff exactly two neighboring values have the value 1. Since $\mathbb{N} \cup \{\infty\}$ is a discrete set, the sup of an assignment is itself a discrete set, and the arg sup of those sup is realized as a (possibly non unique) maximal function $a_v(n): V \times N \to N$. This maximal function is the stepping stone solution. The goal of this paper is to find properties of graphs which constrain $a_v(n)$’s growth from above and below, placing special interest in linear growth.

Discrete valued functions from graph vertices to $\mathbb{N}$ constitute a large area of research in combinatorial graph theory. The famous four color problem [AH] concerns functions $f: V \to \mathbb{N}$ (Section 1) with smallest maxima and with the constraint that $vw \in E$ (Section 1) implies $f(v) \neq f(w)$. Similarly we consider extremal functions $f: V \to \mathbb{N}$ with a different special property. We rely on this example to motivate how such functions are critical for the classification of graphs that we also aim to perform in this paper. These types of functions may also assist in the converse effort; using graphs to study the structure of $\mathbb{N}$. For instance, the non-existence of arithmetic progressions in sparse sets can be shown by associating to a sparse set with a related bipartite graph (and employing Hall’s theorem [AA]). This is one of the few ways that $\mathbb{N}$ valued functions from graphs advance our understanding of Number Theory. This paper introduces new theorems that could be used in the study of the arithmetic structure of $\mathbb{N}$.

The first result of this paper, Theorem 1, is a proof that the stepping stone solution of a graph is sublinear when the volume of the ball on that graph is polynomially bounded. This is an extension of the approach of Ladouceur and Rebenstock [G]; it relies on the distance of the n-th chip to a unit chip (refer to Section 1 for definition of chip). An interesting consequence of our generalization is that $Z^k$, the graph on $\mathbb{Z}^k$ such that

$$E = \{v_1v_2|v_1 + \vec{e} = v_2, \vec{e} = \sum a_i\vec{e}_i, a_i \in \{0, -1, 1\}, \text{ and } \vec{e}_i \text{ is a unit coordinate vector}\}$$

has sublinear stepping stone solution. Our work proves the same fact holds for the triangular lattice, defined as follows:

$$V = \{\vec{v} \in \mathbb{R}^2|\vec{v} = a\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) + b(1, 0), a, b \in \mathbb{Z}\}$$

$$E = \{vw|v = v’ \pm \vec{e}, \vec{e} \in \{(\frac{1}{2}, \frac{\sqrt{3}}{2}), (1, 0)\}\}$$

The paper then moves to the problem of superlinear growth. In later work done by Howroyd [G], the stepping stone solution on $Z^2$ was determined to be superlinear (and therefore linear). This was shown by finding a nice subgraph of $Z^2$. We give this subgraph in graph form in 3. Since this subgraph will be included in all $\mathbb{Z}^k$ for $k > 2$, our formalization of Howroyd allows us to observe that $\mathbb{Z}^k$ is superlinear for all $k \geq 2$. The graph of Howroyd, however, may not be included in all subgraphs of $\mathbb{Z}^k$ or even in the dual graph of the triangular lattice. We find a family of graphs (we call btg, or beaded triangle graphs), which settles the question of superlinearity in bounded degree graphs. Namely, we show a graph is superlinear iff it includes one of these graphs as a subgraph. The btg family as presented is uncountable; we show any family with the same property must be at least countably infinite and have $\mathbb{N} \cup \{p\}$ as the only infinite...
mutually shared subgraph among the elements of said family (we define what $\mathbb{N} \cup \{p\}$ is as a graph in Section 1). We leave as a conjecture whether all families with this property are uncountable.

Finally, we consider stepping stone solutions for graphs with unbounded degree. We find natural examples of graphs of this kind which are properly sublinear in stepping stone solution ($a(n)$ growing as $\Theta(\sqrt{n})$ in the notation of Section 1) and properly superlinear with arbitrary finite growth rate. We also find a necessary condition on infinite graphs to have finite stepping stone solution $a(n)$, which constrains the problem to graphs with finite clique number. Finding stepping stone solutions of proper superlinear growth on bounded degree graphs remains an interesting problem, with possible connections to the partition structure of $\mathbb{N}$ (look to Bollobás [Bo] for a decent overview of partitions of integers).

1 Notation

By a graph $G$ we mean an ordered pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the vertex set and the edge set are unordered pairs of elements of $\mathcal{V}$ defined to exclude self connections. We may write this as $\mathcal{E} \subseteq \{v_1v_2|v_1, v_2 \in \mathcal{V}, v_1 \neq v_2\}$ where $v_1v_2$ is the same edge as $v_2v_1$.

The degree of a vertex $v \in \mathcal{V}$ is the cardinality $|\{v' \in \mathcal{V}|vv' \in \mathcal{E}\}|$. In this paper, we make an important distinction: a bounded degree graph $G$ is a graph such that $sup_{v \in \mathcal{V}} deg(v) < \infty$; any other graph is called unbounded degree.

For a graph $G$, we define the distance function $d: \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{N} \cup \{\infty\}$ as

$$d(u,v) = \left(\inf\{n \in \mathbb{N}|\{u_1v_1, v_1v_2, \ldots, v_{n-1}v_n=v\} \in \mathcal{E}\}, u \neq v\right)$$

We define the sphere centered at $v \in \mathcal{V}$ of radius $d$ as

$$B_v(d) = \{v'|d(v,v') \leq d\}$$

Now, we proceed to define a stepping stone solution, $a_v(n): \mathcal{V} \times \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$. For $n \in \mathbb{N}$ we consider the set of functions

$$F_n = \{f: \mathcal{V} \rightarrow \mathbb{N}||f^{-1}(1)| = n \text{ and } \forall k > 1, |f^{-1}(k)| \leq 1\}$$

and $\forall v \in \mathcal{V}$ we have

$$\sum_{v' \in B_v(1), f(v') < f(v)} f(v') = f(v)$$

So long as $|G| > n$, this set is non-empty, as we can allow $f(v) = 1$ for $n$ vertices and 0 elsewhere. We can define for $f \in F_n$ the operator

$$\mathbb{F}: F_n \rightarrow \mathbb{R} \cup \{\infty\} \text{ such that } \mathbb{F}(f) = sup\{f(\mathcal{V})\}$$

We would like to choose an arbitrary element of $\arg\max F$. $\arg\max\{\sup\{f(\mathcal{V})\}: f \in F_n\}$ to denote $a_v(n)$.

For $f \in F_n$, $f(\mathcal{V})$ is an ordered discrete space; consequently, the sup of $f$ will be an element of the same ordered discrete space. Taking another sup (over functions), the discreteness of the ordered space guarantees that the sup is actually a maximum. Therefore, the arg max is well defined. As mentioned before, we choose an arbitrary element of this set to denote $a_v(n)$. The fact that the choice may not be canonical does not matter in the context of this work.

We will call for $n \in \mathbb{N}$, an element of $\{v \in \mathcal{V}|a_v(n) = 1\}$ a unit chip. For $j > 1$, we will call $a_v(n)^{-1}(j)$ the $j$-th chip when it exists. We note that for $j \leq a_v(n)$ our definition of $F_n$ forces that $j$-th chip to exist.

We also denote $a,n = sup\{a_v(n)|v \in \mathcal{V}\}$. A stepping stone solution will be called superlinear if $a(n)$ is $O(n)$, sublinear if it is $\Theta(n)$, and linear if it is $\Omega(n)$. It is properly superlinear if it is $O(n)$ but not $\Omega(n)$ and properly sublinear if it is $\Omega(n)$ but not $O(n)$.

In Section 2 we generalize methods of Gerbic and Howroyd [G] to linearly bound the stepping stone solutions of graphs. We prove the following:
Theorem 1. Let $G$ be a graph such that there is some polynomial $p(x) \in \mathbb{R}[x]$ satisfying
\[ |B_v(d)| \leq p(d) \tag{*} \]
for all $v \in G$. Then the stepping stone solution is sublinear. The bounding linear slope is
\[ \min \left\{ \left( \frac{2^{2g}(2g+1)^k}{2^g - (2g+1)^g} \right) | g \in \mathbb{Z}, 2^g > p(g) \right\} . \]

A lemma, obtained by Howroyd [G], obtains a lower bound for a large class of graphs. We provide it and its proof below due to brevity.

Lemma 1. Consider first a graph $G_i$ st $V = \{ v^i_1, ..., v^i_7 \}$ and
\[ E = \{ v_1 v_2, v_2 v_3, v_3 v_7, v_2 v_4, v_4 v_5, v_5 v_6, v_6 v_7, v_3 v_4, v_3 v_5, v_3 v_6 \} . \]

Then let $G'$ be the graph obtained from $\{ G_i \}_{i=1}^{\infty}$ by identifying $v^i_7$ and $v^{i+1}_1$ for $i \geq 1$ (pictured in Figure 1). If a graph contains $G'$ as a subgraph then its stepping stone solution is superlinear bounded below by $5n-4$.

\[ \text{Figure 1: Graph of Howroyd [G]} \]

Proof. Let us attempt to construct an element of $F_n$. We let $f(v^1_1) = 1$, and then for $i < n$, we let $f(v^i_1) = 1$. We then let $f(v^i_3) = 1$. We then let $f(v^i_2) = 2$, $f(v^i_1) = 3$, $f(v^i_6) = 5$ and so on (as in Figure 1). This yields an element $f \in F_n$ such that $\sup \{ f(V) \} = 5n - 4$. This is because for each unit chip placed on some $v^i_j$, we can let $f$ increase by 5 along its neighbors. So, the slope of $f$ is bounded below by 5; we subtract the 4 so that when we have $a(1) \geq 1 \geq 5n - 4$ and furthermore for all $n a(n) \geq 5n - 4$. 

From Theorem 1 and Lemma 2, we obtain the following Corollary advertised in Section

Corollary 1. $\mathbb{Z}^k$ and the triangular lattice have linear stepping stone solutions. Any subgraph of either has sublinear stepping stone solutions. The dual graph of the triangular lattice has sublinear stepping stone solution.

Proof. $\mathbb{Z}^k$ with the lattice structure described has $B_0(d)$ equal to a $k$ dimensional hypercube with side-lengths $2d + 1$. Therefore, the ball’s cardinality is $(2k+1)^d$. This gives sublinearity of $\mathbb{Z}^k$ from Theorem 1. Note that the first three linear bounds given by the equation in Theorem 1 are $\{714, 6248, 19572\}$. On the triangular graph, $B_0(d)$ is a growing hexagon; its volume is the $d$’th hexagonal number, or $d(2d-1)$. Similarly, the ball on the dual graph is a growing triangle; its volume is the $d$’th triangular number of $d(d+1) \setminus 2$. Therefore, these graphs are sublinear. The graph of Lemma 2 is a subgraph of $\mathbb{Z}^2$; just let
the v_i be (0, i) and the remaining vertices lie on the lattice from inspection. Since Z^2 \subset Z^k for k \geq 2, this gives superlinearity of Z^k. Together, this obtains linearity of Z^k in stepping stone solution. Similarly, one can check Lemma 2 for the triangular graph. It is linear.

Our next major theorem is unique to this work and thoroughly addresses the superlinearity in stepping stone solution growth in bounded degree graphs. But, before we introduce these theorems, we need to introduce some further notation.

A collection of graphs, denoted \mathcal{A}, will be called a linearity basis of degree b \in \mathbb{N} if, for any bounded degree graph G of degree b, G has a superlinear stepping stone solution. We now omit reference to degree) is \textit{minimal} if for G_1, G_2 \in \mathcal{A} any shared infinite subgraph of G_1 and G_2 is properly sublinear.

A beaded triangle graph of size N \in \mathbb{N} \cup \{\infty\} and of bound b \in \mathbb{N} is a graph whose vertex set is V = \{p_1\}_{i=1}^{N} \cup \{p_0\} where C is a countable index set and whose edge set is E = \{p_ip_{i+1}\}_{i=1}^{N-1} \cup \mathcal{E}' where \mathcal{E}' \subset \{p_ip_{i+1}\}_{i\in\mathbb{N}, \alpha \in \mathcal{C}} and where for all \alpha \in \mathcal{A} we have deg(p_i) = 3 for \infty > i > 1, deg(p_1) = deg(p_N) = 1, deg(p_0) \in \{1, ..., b\}, and the induced subgraph on \mathbb{B}_{p_0}(1) cannot include a path of length longer than 2. This graph will be denoted btg^{N}, where the bound b will be apparent from context.

With these concepts defined, we have the following theorem:

**Theorem 2.** Throughout the statement of this theorem, we will implicitly be using degree b for our bound on G's vertex degree and in our bound for the btg's. Let

\[ \mathcal{A} = \{G|G = \bigcup_{i\in\mathbb{N}}btg^{N}, sup\{k_i|i \in \mathbb{N}\} = \infty, k_i \in \mathbb{N} \cup \{\infty\}\} \]

Then, \mathcal{A} is a linearity basis. Any minimal linearity basis is infinite. Finally, consider for every G in a linearity basis all its subgraphs. The only subgraphs that are infinite and shared among all G in a minimal linearity basis are \mathbb{N} \cup \{p\}, or any graphs such that V = \mathbb{N} \cup \{p\} and E = \{n(n+1)|n \in \mathbb{N}\} \cup \{pp\} for i \in \mathbb{N}.

This proof is given in 3 as it is fairly lengthy. This makes some headway in completely addressing the question of super-linearity in bounded degree graphs G; proper superlinearity is still interesting. As a last look into bounded degree graph stepping stone solution constraint, there is the following general theorem:

**Theorem 3.** Let G be an tree graph. Then G is sublinear.

The proof of Theorem 3 will demonstrate that the stepping stone solution of any graph can be thought of as a special tree with some of the indices identified. This paper collects evidence that the subject of proper superlinearity of stepping stone solutions in bounded degree graphs is related to the partition structure of integers [Bo] by inspecting this tree.

Finally there is case of unbounded degree graphs; in that case the following theorems are proven in Section 5.

**Theorem 4.** Consider the graph G, consisting of countably many vertices V where V = V_1 \cup V_2 is disjointed, and we have deg(v_2) = 1 for v_2 \in V_2, sup_{v_1 \in V_1}deg(v_1) = \infty and vv' is not in E for v, v' \in V_1. Then, the stepping stone solution on this graph is \Theta(\sqrt{n}).

**Theorem 5.** Let G be an unbounded degree graph, and let us fix n \in \mathbb{N} and obtain v \in V st deg(v) \geq n. Let \mathcal{P}_v = \{v_i\}_{i=1}^{\infty} be some countable, now ordered subset, of the neighbors of v. Then, we may define

\[ L(n) = sup_{v \in V, deg(v) \geq n} sup_{v_1 \in \mathcal{P}_v}(\{v_1, v_{i+1}, v_{i+1}, ...\} \in E \text{ and } |v_1, v_{i+1}, v_{i+1}, ...| = l) \]

Then we know that

\[ L(n) \leq C_G \]

iff a(2) < \infty
Theorem 6. Let \( f : \mathbb{N} \rightarrow \mathbb{N} \) such that \( f(N) \geq N \) and \( f(0) = 1 \). Then, for \( k \in \mathbb{N} \) let \( G_k \) be a graph such that \( \{v_1\}_{i=0}^{k-1} \subset V \) and such that \( \{v_2^j\}_{i=0}^{f(k+1)-1} \subset V \) for \( j \in \{0, 1\} \). Then, let \( v_2^1 v_2^+ \subset E \) for \( i \in \{0, \ldots, f(k+1) - 2 - f(k)\} \) and \( \{v_2^1, v_1^g v_2^1, v_1^g v_2^1, \ldots, v_1^{(f(k+1)-1-f(k))} v_2^1, v_2^1\} \subset E \). Then, the graph \( \cup_{k=3}^\infty G_k \) has stepping stone solution \( \Omega(f(n-1)) \) and \( a(2) < \infty \).

![Figure 2: Graph from Theorem 6](image)

The importance of this graph is that it is a simple instance of a properly superlinear graph with possibly finite \( A(n) \). The author was not able to prove \( a(n) < \infty \), but he obtains evidence around this conjecture. Figure 2 gives a portion of the graph from Theorem 6. With all the theorems stated and required notation established, we may move to proving Theorem 1

2 Proof of Theorem 1

Proof. Let \( a_*(\cdot) : V \times N \rightarrow N \cup \{\infty\} \) be a stepping stone solution of \( G \). Let \( n \in \mathbb{N} \). For \( k \in \mathbb{N} \) such that \( k \leq a(n) \) let \( v \in V \) be such that \( a_*(n) = k \). Then, there are two vertices \( v_1, v_2 \in B_v(1) \setminus \{v\} \) such that \( a_v(v_1, v_2) \leq a_v(n) \) and \( a_v(v_1, v_2, n) > 0, a_v(n) > 0 \). Otherwise, we cannot have \( \sum_{w \in B_v(n)} \{v, a_v(n) < a_v(n) a_w(n) = a_v(n) \} \) as required in the definition of \( a_*(\cdot) \). WLOG we may assume \( 0 \leq a_v(n) \leq a_v(n) \).

Let \( d \in \mathbb{N} \) be such that \( 2^d \leq a_v(n) < 2^{d+1} \). Then, by successively using the above statement, we note \( \exists w \in V \) such that \( a_v(n) = 1 \) and \( v \in B_v(d) \). The same proof implies that for all \( k \leq a_v(n) \exists w \in V \) such that \( a_v(n) = 1 \) and \( v_k \in B_v(d) \) where \( v_k \) is the \( k \)-th chip (as defined in Section 1). This implies at least \( a_v(n) \) vertices are included in \( B_v(d) \) for some \( n \) unit chip. Since there are \( n \) unit chips, this implies

\[
2^d \leq a_v(n) \leq |\cup_{a_v(n)=1} B_v(d)| \leq n \max \{|B_v(d)| \mid a_v(n) = 1\} \leq np(d)
\]

where we’ve replaced the volume of the ball using Equation *. Upon rearranging the first and last terms

\[
\frac{2^d}{p(d)} \leq n
\]

For all \( v \in V \) our above proof demonstrates that there is a single finite \( d \), only dependent on \( n \), such that \( a_v(n) \leq 2^{d+1} \). This is because \( \frac{2^d}{p(d)} \) eventually is monotonically increasing in \( d \), and would exceed \( n \) if that were not the case. Since \( a_v(n) \) is now uniformly bounded by some \( 2^{d+1} \), we have \( a(n) < \infty \). Since \( a(n) < \infty \), there is some \( v_* \in V \) so that \( a_{v_*}(n) = a(n) \). Let \( g \in \mathbb{N} \) be such that \( 2^g \leq a_{v_*}(n) \). From repeating the halving argument again, we know there is \( w \in B_{v_*}(g) \) such that \( a_{w}(n) \leq \frac{a(n)}{2^g} \). Now, put \( A = \{v \in V \mid a_v(n) \in \{1, \ldots, \frac{a(n)}{2^g}\}\} \) we have

\[v^* \in \cup_{v \in A} B_v(g)\]

As before, the same proof implies that for all \( k \leq a_v(n) \) we have the \( k \)-th chip lying in \( \cup_{v \in A} B_v(g) \). Accordingly, \( a(n) \) vertices must lie in the union of balls of radius \( g \) around a unit chip or a \( k - th \) chip for \( k \in \{2, \ldots, \frac{a(n)}{2^g}\} \). So

\[
a(n) \leq |\cup_{v \in A} B_v(g)| \leq |A| \cdot \max (|B_v(g)| \mid v \in A) \leq \left(n + \frac{a(n)}{2^g}\right) p(g)
\]

Where we used the fact that \( |A| = \left(n + \frac{a(n)}{2^g}\right) \) as there are \( n \) unit chips and \( \frac{a(n)}{2^g} \) other non unit chips in \( A \) for \( k \in \{2, \ldots, \frac{a(n)}{2^g}\} \). Thus, when \( 2^g > p(g) \) we have
\[ a(n) \left( 1 - \frac{p(g)}{2^g} \right) \leq n \left( \frac{p(g)}{2^g} \right) \implies a(n) \leq \left( \frac{2^g p(g)}{2^g - p(g)} \right)^n \]

As a result,

\[ a(n) \leq \min \left\{ \left( \frac{2^g p(g)}{2^g - p(g)} \right) | g \in \mathbb{Z}, 2^g > p(g) \right\} n \]

Now that we have this fact established, we note there are immediately some nice corollaries. Namely, we obtain sublinearity for any Ricci flat graph [S]. The notion of curvature in graphs is beyond the scope of this paper, but we will briefly motivate this observation. Just as in the case of manifolds, there is a discrete notion of graph curvature that constrains the growth of volumes of balls. For a complete picture of this, the author refers one to work done by Paeng [S], where the bounds on Ricci flat graph volumes are obtained.

### 3 Proof of Theorem 2

For the context of this proof, a gap of a graph \( G \) is some triplet of vertices \( \{v_1, v_2, v_3\} \subseteq V \) such that if \( v \in V \) has the property \( vv_2 \in \mathcal{E} \) then \( v \in \{v_1, v_3\} \). For \( v, w \in V \), a gapless path is some finite set of vertices \( \{v_i\}_{i=0}^n \subseteq V \) such that \( v_0 = v, v_n = w, \) \( \{v_i, v_{i+1}\}_{i=0}^{n-1} \subseteq \mathcal{E} \), and \( \{v_i, v_{i+1}, v_{i+2}\} \) is not a gap for \( 0 \leq i \leq n - 2 \). We will also need the notion of a gapless connected set of \( v \in V \), \( \mathcal{C}_v \) defined as follows

\[ \mathcal{C}_v = \{ w \in V | \exists \text{ a gapless path between } v \text{ and } w \} \]

Figure 3 is a picture of some \( \mathcal{C}_v \) to demonstrate all the definitions (as well as the result of lemma 2). We note that what \( v \) we use to define \( \mathcal{C}_v \) is not important; if \( w \in \mathcal{C}_v \implies \mathcal{C}_w = \mathcal{C}_v \).

![Figure 3: \( \mathcal{C}_v \) for the vertices within it](image)

To become comfortable with manipulating these ideas, we have this lemma. It will be used to demonstrate the infinitude of minimal linearity bases for Theorem 2.

**Lemma 2.** Let \( G \) be a bounded degree graph, with degree bound \( b \in \mathbb{N} \). Let \( n \in \mathbb{N} \) be greater than \( b \). Then, for \( 2 < k \leq b \), let us denote the \( k \)-th chip as \( v_k \in V \). Then, \( a_v(n) > 1 \) iff \( v \in \cup_{v_k} \mathcal{C}_{v_k} \).

**Proof.** Say \( v \in V \) s.t \( a_v(n) > 1 \) but \( v \not\in \cup_{v_k} \mathcal{C}_{v_k} \). Therefore, \( a_v(n) > b \). Then, the path obtained by the halving argument of Theorem 1 (a path \( \{v_i\}_{i=0}^m \) such that \( v_0 \) is a unit chip, \( v_m = v \), and \( a_{v_i}(n) \leq \frac{a_{v_{i+1}}(n)}{2} \)) can be chosen to include some \( k \)-th chip for \( 2 < k \leq b \). If not, it’s necessarily the case that all neighbors of \( v_1 \) have \( a_v(n) \in \{0, 1\} \) or value greater than \( a_v(n) \). But then, \( \sum_{w \in \mathbb{R}_{v_1}, a_v(n) < a_w(n)} \leq b < a_v(n) \), a contradiction.

Now, since \( v \not\in \cup_{v_k} \mathcal{C}_{v_k} \), said path \( \{v_i\}_{i=0}^{m=n} \) must have a gap (or it would induce a gapless path from \( v \) to some \( v_k \)). Let us identify the gap \( \{v_1, v_2, v_3\} \). We note the path is defined so that \( 1 \leq a_{v_1}(n) <
\(a_{v_2}(n) < a_{v_3}(n)\). Therefore \(\sum_{v \in B_{a_{v_2}(n)}(1), a_w(n) < a_{v_2}(n)} a_w(n) = a_{v_3}(n) < a_{v_2}(n)\), a contradiction. So this path could not have had a gap, and furthermore \(v \in \bigcup_{n \in \mathbb{N}} C_{v_k}\)

To prove the first major part of Theorem 2, we require the following lemma. This lemma relates gapless paths to their btg\(^k\) subgraphs

**Lemma 3.** Let \(\{v_i\}_{i=1}^n\) be a gapless path of a bounded degree graph \(G\). Let the degree bound be \(b\). Then btg\(^{\frac{n}{b+1}}\) is a subgraph of \(G\).

**Proof.** Consider the induced subgraph of \(G\) created by the vertices of the gap-less path and all their neighbors. This gives us set \(\{p_i\}_{i=1}^n\) and \(\{p_i\}_{i \in \mathcal{C}} \subset \mathcal{V}\) where \(p_i\) is just the original path, and \(p_a\) is a choice of neighbor of each \(p_i\). This is already a btg if we can choose \(p_a\) such that \(p_a\) only has paths among its neighbors of length \(\leq 2\). If \(\{p_0, p_0, p_1, \ldots, p_0, p_j\} \subset \mathcal{E}\) for \(j > 2\), then we can find a new gap-less path \(\{q_i\}_{i=1}^{n-(j-1)}\) by taking the same vertices up to \(i\), letting \(q_{i+1} = p_a\), and \(q_{i+2+k} = p_{j+k}\) for all \(k \leq n-i-2\). We use the same \(q_a\)'s as was used before for all \(q_i\) that already had them, leaving only \(q_{i+1}\) without an associated \(q_a\). We give \(q_{i+1}\) the \(a_0 = p_{i+1}\). This will eliminate the path of length greater than 2, and decrease the overall length of the gap-less path by \(b\). If we walk along the original gapless path, and perform this shrinking operation every time we arrive at a too high degree \(p_i\), we can only do this at every \(b'th\) vertex along our walk. Therefore the subgraph we obtain by performing this operation on some gapless path exhaustively is a btg with length no smaller than \(\left\lfloor \frac{n}{b+1} \right\rfloor\).

Now, we supply the first part of Theorem 2 in the form of a lemma. Again, we are under the implicit assumption that the bound on the btg and on the graphs under consideration for the definition of a linearity basis is the same \(b \in \mathbb{N}\).

**Lemma 4.** Let

\[\mathcal{A} = \{G|G = \bigcup_{i \in \mathbb{N}} \text{btg}^{k_i}, \sup\{k_i|i \in \mathbb{N}\} = \infty, k_i \in \mathbb{N} \cup \{\infty\}\}\]

then \(\mathcal{A}\) is a linearity basis.

**Proof.** Let us first prove that \(G' \in \mathcal{A}\) and \(G' \subset G\) for \(G\) a graph implies \(G\) is superlinear. It would be sufficient to show \(G'\) is superlinear. Furthermore, it’s sufficient to show that for any of the graph family btg\(^{k_i}\) has \(a(k) \geq k\) for \(k \leq k_i\). This is because \(\cup_{i \in \mathbb{N}} \text{btg}^{k_i}\) (including btg\(^{\infty}\)) is composed of btg\(^{k_i}\) for increasing values of \(k_i\). Therefore, for any \(k \in \mathbb{N}\), there is some btg\(^{k_i}\) \(\subset G\) such that our \(a(k)\) for \(G'\) is greater than or equal to \(a(k)\) for \(\text{btg}^{k_i}\), which itself is greater than \(k\).

So, we need to show \(a(k) \geq k\) for \(k \leq k_i\) in the stepping stone solution of btg\(^{k_i}\). We generate \(f \in F_k\) by setting \(f(p_j) = j\) for \(1 \leq j \leq k\), and then for the \(1 < j \leq k\) we take for some \(\alpha_j\) st \(p_jp_{\alpha_j} \subset \mathcal{E}\) and let \(f(p_{\alpha_j}) = 1\). The remaining unit chips may have been placed disconnected from any \(i\)th chip for \(i < k\), and there were enough vertices in \(\mathcal{V}\) to do this as \(k \leq k_i\). One may check that \(\sum_{v \in B_{a_v}(1)} a_v(k) = a_{p_{\alpha_j}}(k) + a_{p_{ \alpha_j}}(k) = i-1+1 = i\), so \(f\) is, in fact, an element of \(F_k\) and \(a(k) \geq k\).

Now, we must show that if no element of \(\mathcal{A}\) is a subgraph of some bounded degree graph, then \(G\) is not superlinear.

Let \(G\) be superlinear. Let \(n \in \mathbb{N}\) be st \(n > b\). Since \(a(n) \geq n\), we have that \(a(n) \geq n > b\). By lemma 2, we know the \(k\)-th chip for \(k \in \{b+1, \ldots, n\}\) lie in \(C_{v_k}\). We also know that there is a path from the \(n-th\) chip to the unit chip of at least length \(\log_b(n)\). This is because if \(a_v(n) = n\), we know \(v\) must have a neighbor \(w\) st \(a_w(n) \geq \frac{n}{2}\). We also know that since \(a_v(n)\) monotonically increases along this path, each vertex must have an off path neighbor. Otherwise \(\sum_{v \in B_{a_v}(1)} a_v(n) < a_{v_{n+1}}(n)\). This implies btg\(^{\log_b(n)}\) \(\subset G\). Since this holds for arbitrary \(n\), we must have an element of \(\mathcal{A}\) as a subgraph of \(G\).

A natural task is to now find a minimally sized linearity basis. The definition of a minimal linearity basis (as defined in Section 1) is so defined because it cannot be reduced in length by considering a shared subgraph of multiple elements of \(\mathcal{A}\). It is in this sense irreducible and minimally large. The reason we look for minimal linearity bases is that while btg graphs are fairly simple, the collection of them have
a natural surjection onto \(2^\mathbb{N}\) (this should be obvious; just append two different types of subgraphs to make a distinct \(btg^\infty\) for any \(2^\mathbb{N}\)). Therefore, \(\mathcal{A}\) is uncountable. It would be desirable to find a finite (or even countable) linearity basis which is simple in nature. We will show in the next part of the proof of Theorem 2 that a finite linearity basis is not possible. The proof will show that the any minimal linearity basis \(\mathcal{A}\) is infinite. Let us state the last part of Theorem 2 as another lemma.

**Lemma 5.** Any minimal linearity basis \(\mathcal{A}\) is infinite, and the only subgraph that is infinite and shared among all \(G\) in a linearity basis is \(\mathbb{N} \cup \{p\}\), or any graph \(st\) \(V = \mathbb{N} \cup \{p\}\) and \(E = \{n(n+1)|n \in \mathbb{N}\} \cup \{p,p\}\) for \(i \in \mathbb{N}\).

**Proof.** Note that \(btg^\infty\) is superlinear, therefore any linearity basis would have an element contained within it. Therefore the only shared infinite subgraphs among any linearity basis are mutual subgraphs of all \(btg^\infty\). Consider the simplest \(btg^\infty\) which connects to each \(p_i\) its own external edge \(q_i\). Any infinite subgraph of this graph consists of a spine and some number of retained external edges. To show that \(\mathbb{N} \cup \{p\}\) is the only possible mutually shared subgraph of a linearity basis, we must find \(btg\) whose subgraphs don’t contain more than one singular external edge. Let \(n \in \mathbb{N}\) and consider the \(btg^\infty\) formed by pairing up \(p_i\) and \(p_{i+n}\) whenever \(kn \leq i \leq (k+1)n\) for some \(k \in \mathbb{N}\). In this case, there are no pairs of unconnected external edges \(n\) apart, so any subgraph cannot have isolated external edges \(n\) distance apart. If we consider mutual subgraphs of this family and the single graph mentioned before, they must consist of a single spine with unconnected external edges, except they cannot have any pairs of unconnected external edges. Otherwise, said subgraph could not be among the family of graphs who don’t have isolated external vertices of some distance. The only graph this leaves is a spine with only a single external vertex, or \(\mathbb{N} \cup \{p\}\)

Any element \(G \subset \mathcal{A}\) of \(\mathcal{A}\) of a minimal linearity basis must itself be bounded degree and infinite. Therefore, by König’s lemma [Bo], one of the connected components of \(G\) contains \(\mathbb{N}\). We can use a copy of \(\mathbb{N}\) given by König’s theorem on the subgraph induced by \(v \in V\) such that \(a_v(n) > 0\). If our copy of \(\mathbb{N}\) eventually had no external vertices, then \(a_v(n)\) would eventually not be positive along it (a contradiction). By choosing a sufficiently far external connection along our \(\mathbb{N}\) and removing a finite starting tail, we can obtain the same graph of type \(\mathbb{N} \cup \{p\}\) inside all elements of the linearity basis \(\mathcal{A}\). This proves all portions of our lemma devoted to mutually shared subgraphs. To complete the lemma, we must now show \(\mathcal{A}\) is uncountable for any minimal linearity basis.

Let’s refer to a specific minimal linearity basis \(\mathcal{A}'\). Say there existed an infinite collection of \(btg^\infty\) st pairwise any mutual subgraph they had was sublinear. We know that each contains an element of \(\mathcal{A}'\), and if they ever shared the same element, it would imply that the contained graph of \(\mathcal{A}'\) is sublinear (and therefore \(\mathcal{A}'\) not minimal). So, each contained element is distinct, meaning the cardinality of any minimal linearity basis would be at least countable.

Consider a \(btg^\infty\) such that any \(p_n\) always is connected to pairs \(\{p_i, p_{i+1}\}\). These graphs appeared to the author to look like triangles all beaded along some common thread (hence the name \(btg\)). We may find a \(btg\) st each triangle is spaced out uniformly from another by a fixed distance \(n \in \mathbb{N}\). Graphs of this type are drawn explicitly in Figure 4 for understanding.

![Figure 4: Caption](Image 213x202 to 383x279)

This gives us an infinite sequence of \(btg^\infty\). If we look at any pair of these \(btg^\infty\) that have two different periods \(p\) and \(q\), than any infinite mutual subgraph of the two necessarily has a gap at least every \(pq\) distance along a spine. The reason this is the case is that finding an infinite subgraph of some \(btg^\infty\) of Figure 4 consists of finding a simple path along \(p_i's\) which could deviate through a triangle. If we never deviated through a triangle to find a mutual subgraph of two incommensurately periodic \(btg^\infty\), then we necessarily find a gap when we have traversed a length \(pq\) along our simple path. This would be the resultant maximal period between gaps; using the ends of a triangle to form a new spine just introduces
further gaps. From lemma 2, \( a(n) \) must be contained in the gapless connected set of the 2nd and 3rd chip, meaning that we have \( a(n) > 3 \cdot pq \) for any shared subgraph of the \( blg^{\infty} \) corresponding to distinct numbers \( p \) and \( q \). So, there exists an infinite collection of \( blg^{\infty} \) st pairwise among elements any mutual subgraph is sublinear.

\[ \Box \]

4 Proof of Theorem 3, \( a(n) \) sublinear for trees

Let us give the proof of Theorem 3 now. It will be our last look at bounded degree stepping stone solutions, although it applies to non-bounded trees as well. In this proof, a jump refers to any \( v \in V \) st \( a_v(n) \geq 2 \) and \( w \in B(1), v \) and \( a_w(n) < a_v(n) \) implies that \( a_w(n) \in \{0, 1\} \).

Proof. Let \( G \) be a bounded degree graph and let \( a_v(n) \) be a stepping stone solution. Let \( v \) be the \( k \)-th chip for \( k > 1 \). If we cannot select from the neighbors of \( k \) a smaller non-unit chip, then \( v \) is a jump. Otherwise we place each non-unit chip neighbor in a new set. If we call \( C_0 = \{v\} \) and \( C_1 \) this new set, we can introduce an inductive decomposition of the \( k \)-th chip by defining \( C_{i+1} \) to be the set of all non unit chips used to generate the any non-jump in \( C_i \). Since \( k \) is finite and the elements of \( C_i \) are atmost \( k - i \), this process necessarily concludes with some set of only jumps, \( C_f \), where \( f \in N \) such that \( f < k \). We will call this collection \( \{C_i\}_{i=0}^f \) the decomposition of \( k \).

Reversing this process, we note that the following expression holds true

\[
\sum_{v \in C_i} a_v(n) = \sum_{w \in C_{i+1}} a_w(n) + |\{v|v \text{ is a unit chip used to decompose some } v \in C_i\}|
\]

By induction this implies

\[
k = \sum_{w \in C_f} a_w(n) + \sum_{i=0}^{f-1} |\{v|v \text{ is a unit chip used to decompose some } v \in C_i\}|
\]

Each \( w \in C_f \) is a jump, and therefore requires its value worth of unit chips to be generated. This implies that unless the same vertex was used multiple times as either a unit chip or a larger chip in our decomposition there must be at least \( k \) distinct unit chips to see the \( k \)-th chip. So \( a(n) \leq n \) if \( G \) is a tree, as there are no loops to facilitate multiple uses of the same vertex. This proves Theorem 3.

\[ \Box \]

One may refrain from identifying the vertices in the decomposition of the \( k \)-th chip (including the unit chips), and notice that \( \{C_i\}_{i=0}^{final} \) creates a natural tree structure. We obtain this tree from connecting each parent vertex in \( C_i \) by an edge to its children in \( C_{i+1} \) and its generating unit chips. In this manner the stepping stone solution on any graph can be thought of as a couple disconnected tree with vertices identified. We will call this tree the decomposition tree. The number of unit chips used to generate some \( k \)-th chip may be reduced by either reusing elements of \( C_i \) or reusing unit chips. Say this identification was used to generate properly superlinear graphs. Relying on using just unit chips forces us to look towards unbounded degree graphs. In the case of bounded degree graphs of bound \( b \), we may still only use a unit chip \( b \) times. Therefore if we only identified unit chips, \( a(n) \) is still linearly bounded by \( bn \) where \( b \) is the degree bound.

Instead we could consider identifying chips in our decomposition’s proper. This could lead to superlinear solutions within the category of bounded degree graphs. In order to discover these graphs, one would have to inspect how to partition \( k \) into smaller integers, and these into smaller integers (and so on), only using the same integer at most \( b \) times. This is a generalization of a partition [Bo], where we consider multiple layers of partitions and are constrained on what integers were used between layers. Therefore the existence of such graphs would correspond to a nice combinatorial problem. We now proceed to the last part of this paper, which addresses stepping stone solution growth on graphs with unbounded degree.
5 Unbounded Degree Graphs and Intermediate Growth

This section is devoted to studying stepping stone solution growth on unbounded degree graphs. We will introduce instances of unbounded degree graphs with provably $\Omega(N^\alpha)$ stepping stone solution for $\alpha < 1$ and $\alpha > 1$, suggesting that among unbounded degree graphs such instances are much more commonplace. We will also obtain $O(N^\alpha)$ in the $\alpha < 1$ case and conjectural evidence for the same result in the $\alpha > 1$. We will also show that stars (induced subgraphs on vertices with high degree) in any graph with unbounded degree cannot have neighbors with too many connections among themselves. Otherwise the stepping stone solution is infinite at finite values of $n$. The author wants to stress that a key graph not thoroughly explored in this paper are bounded degree graphs with proper superlinear stepping stone solution. This should constitute the subject of future work. As already mentioned in Section 4, these graphs, due to their decomposition trees, would need to rely on partitioning integers into b smaller integers many times. In this section, we use the other method described in Section 4 to obtain proper superlinear growth, and that is making multiple use of unit chips.

5.1 Properly Nonlinear and Infinite Stepping Stone Growth Among Unbounded Degree Graphs

First, we prove our graph in Theorem 4 has stepping stone solution which grows as $\Theta(\sqrt{n})$.

**Proof.** Let $n \in \mathbb{N}$, and let us find $\{v_i\}_{i=1}^n$ such that $\deg(v_i) \geq i$. Furthermore, let us consider $\{v_i\}_{j=1}^i$ such that $f(v_i^j) = 1$. One can observe that $|f^{-1}(v_i^j)| = \sum_{i=2}^n i = \frac{n(n+1)}{2}$. Furthermore, we can now choose $f \in F_{\mathbb{N}^2}$ to have $f(v_i) = i$ and $f(v) = 0$ where $f$ is otherwise undefined. The supremum over these functions, therefore, is larger than $n$, and $a(\frac{n(n+1)}{2}) \geq n$ so $a(n)$ is $\Omega(\sqrt{n})$.

Note that in any stepping stone solution we must have for $k \geq 2$ the k-th chip placed on some $v_i \in V$ such that $\deg(v_i) \geq 2$. Therefore, from inspection of G, the nonzero neighbors of the $k - th$ chip for some $f \in \mathbb{F}_n$ must be precisely k units, for if there are more it cannot be true that $\forall v \in V$ we have $\sum_{w \in B_v(d), f(w) < f(v)} f(w) = f(v)$. Therefore, any $f \in \mathbb{F}_n$ actually behaves like the $f$ obtained above, and the stepping stone solution is $\Theta(\sqrt{n})$.

Now, let us prove Theorem 5. This theorem has some interesting consequences. It means that infinite graphs, which don’t have a finite clique number, have infinite stepping stone solution $a(k)$ for $k \geq 2$. Therefore, the stepping stone solution is truly interesting in the case of graphs with more sparse connections.

**Proof.** Let $\mathbb{L}(n)$ not be bounded in $n$. Then, let us fix $m \in \mathbb{N}$. There is some $n \in \mathbb{N}$ and $v \in V$ such that $\deg(v) \geq n \geq m$ and there is some path $\{(v_i, v_{i+1}, v_{i+2}, \ldots)\} = m$ in the neighborhood of $v$. Let us consider $f \in \mathbb{F}_2$ st $f(v_i) = 1$ and $f(v_{i+j-1}) = j$ and $f(v) = 0$ otherwise. We can check $f \in \mathbb{F}_2$ as $\sum_{v \in B_v(j) \setminus v_{i+j-1}} a_v(n) = a_v(n) + a_{v_{i+j}}(n) = 1 + j - 1 = j$ (we implicitly relied on induction). Therefore, $a(2) > m$. But m is arbitrary; therefore, the stepping stone solution for two unit chips is $\infty$.

To complete our look at graphs with unbounded degree, we also provide a graph with properly superlinear stepping solution of very general growth rate in our proof of Theorem 6:

**Proof.** First, we show G’s stepping stone solution grows $\Omega(f(n-1))$. Consider $G_n$ in our collections of graphs. Consider $f \in F_n$ such that $f(v_i^j)$ to $f(v_i^{n-1})$ are all 1. Then, we may have $f(v_i^0) = k$ for $k \leq f(n-1)$. This is because if we placed the jth chip at $v_i^{j-1}$ for $j + 1 < f(n-1)$, then we have $\sum_{v \in B_v(j)} a_v(n) = a_{v_{i+j-1}}(n) + a_{v_{i+j}}(n) = j + 1$. This vertex assignment is demonstrated in Figure 5. Therefore, $a(n) \geq f(n-1)$. Now, we must show that any stepping stone solution grows $O(f(n-1))$.

This graph clearly has $a(2) \leq \infty$ from Theorem 5, as it is triangle free. Therefore there are no paths around $\deg(g) \geq n$ vertices for $n \geq 4$ as used in the definition of $\mathbb{L}$.
Let us devote some time to demonstrating useful lemmas for the conjectured finite \( a(n) \) of the graph of Theorem 6, or very similar graphs. First the size of the smallest loop of this graph is 6. We can, however, for \( f(N) << N! \), find graphs with the same \( v_1^{\mathbb{R}^N} \) and \( v_2 \) vertices such that the edges \( v_1^{\mathbb{R}^N} \) rather than being connected periodically are connected in special permutations of the \( N \) vertices to consecutive \( N \) vertices. These permutations would be chosen such that the smallest loop in the graph grows with order \( N \) (we would still have \( \Theta(f(n - 1)) \) from the same proof above). Then from our decomposition tree argument, if the \( k \)-th chip lies in \( G_N \) for \( N \geq k \) we would require \( k \) unit chips. This implies that most chips lie in smaller \( G_i \). With the fact that each \( G_i \) itself is finite, this strongly suggests that those distributions which minimize use of unit chips are constrained in size, and that therefore \( a(n) < \infty \).

6 Conclusion and Remark

In this paper the stepping stone problem of Ladoucer and Rebenstock [G] was rigorously defined in the setting of Graphs. A sufficient condition by which graphs of bounded degree could have sublinear stepping stone solution was obtained, and then all graphs of bounded degree with superlinear stepping stone solution were characterized by showing that they must contain some subgraphs among a set \( A \). Next the question of graphs with unbounded stepping stone solution was addressed with examples of graphs with properly superlinear stepping stone solution. A proof was not obtained to demonstrate that these graphs had an upper bound in stepping stone growth. The author would leave this as a conjecture; that the graph obtained in Theorem 3 is \( \Theta(f(N)) \). He believes that natural directions for future work would involve characterizing graphs with proper, but bounded, superlinear stepping stone. Furthermore that this property could be a useful property for studying connectivity properties in fast growing graphs.

Acknowledgment. The author would like to thank Hailey Leclerc and Santiago Aranguri for their help in composing this work.

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