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ABSTRACT: After the Hindu-Arabic decimal positional system was introduced in Europe, throughout many centuries textbooks on elementary arithmetic, intended for beginners, had a more or less fixed organization of content, usually starting with chapters on numeration. These chapters, as a rule, contained one or more examples of large integers the purpose of which was simply to be named (read out loud), sometimes also *vice versa*. This tradition apparently began with the two first texts that significantly contributed to the spread of the decimal system in Europe — the Latin translations of al-Khwarizmi's treatise on decimal arithmetic, and Leonardo's *Liber Abaci*, containing examples of reading a 16-digit and a 15-digit number respectively. Throughout the centuries, the order of magnitude of these introductory numbers increased, in general up to some 30 digits, but in some cases to over 60 digits. In this paper we examine the development and extent of this characteristic of introductory arithmetic textbooks from the period 13th–19th century, and the conditions which lead to this, now extinct, practice.

Keywords: history of arithmetic, history of mathematics education, decimal positional system

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Figure 1: Avoiding large numbers in Ancient Rome: Reverse side of a coin of emperor Hadrian minted in 119 AD, commemorating his remission of citizens' debts towards the *fiscus* (Emperor's treasury) in total value of 900 000 000 *sestertii.* The legend: RELIQVA VETERA HS NOVIES MILL[ies] ABOLITA, roughy translates as 'remaining old [debts amounting to] nine thousand *sestertia* remitted'. A *sestertia* (denoted here by HS) is 100 000 *sestertii* — making the total amount 'named' $9{\cdot}1000{\cdot}100000 = 900000000$ *sestertii.*

Introduction

Many centuries before googol got its name, humans have been fascinated with large numbers. The most prominent of such early interest in naming large integers stems from the ancient Indian Vedic period (ca. 500-1000 BC). Several texts from that period survive, demonstrating their fascination with large numbers, for example the *Lalitavistara Sutra* describes the naming of numbers up to 10^{421} [Hoo, Plo, Bel]. The ancient Greek mathematicians had little regard for reckoning, but one of the most famous of them, Archimedes of Syracuse, in the 3rd century BC wrote a text called *Psammites* (*The Sand-Reckoner*), aiming to describe how the number of sand particles in the whole universe could be named, and he devised a system for naming numbers up to $8 \cdot 10^{63}$ [Arc]. A bit later, in 4th–6th century AD, the Chinese also had systems based on powers of 10 for naming numbers up to 10^{4096}, but not with fixed nomenclature [YS]. Even if ancient Indians and Archymedes described systems of naming large integers, all of their examples are 'round' numbers, that is, numbers which in our modern decimal system would end with several zero digits. While after 1000 the *Lalitavistara Sutra* contiunes naming numbers by factor 100, Archimedes used the factor 10000 (myriad) as the basic 'unit'. For example, let us look at the number $10^{12} = 1000000000000$. In modern naming we would think in steps 10^3, so it would be 1/000/000/000/000, a thousand of thousands of thousands of thousands, called a trillion in short scale (or a billion in long scale).[1] In the *Lalitavistara Sutra* system it would be 10/00/00/00/00/000, ten hundreds of hundreds of hundreds of hundreds of thousands, or ten *niyuta* [Plo, Bel]. In Archimedes' system it would be 1/0000/0000/0000, a myriad of myriads of myriads, or a myriad of units of second order [Arc]. In the Chinese *Sun Zi Suanjing* (4th/5th century AD), a system of naming numbers in steps of 10^8 is described, and number 10^{12} would be named *wan yi*, where *yi* was the name for 10^8, while *wan* was a common name for 10^4 [YS].

For centuries, European post-antiquity mathematics was of a low level, and one reason was the dominance of the Roman number system. Roman numerals, besides being inapt for advanced calculations, had no system for designating numbers larger than 500 millions [Brü, Tur, Wuß] (see also Fig. 1). Even the most influential arithmetical text in the Early and High Middle Ages, Boethius' *De Institutione Arithmetica* (early 6th century), besides being written in the classical Greek 'theoretical' arithmetic style (that is, containing essentially number theory, and not reckoning), does not mention numbers larger than a few hundreds [Mas]. A little before Boethius, Victorius of Aquitania, in his *Calculus*, which does deal with reckoning, and contains tables of addition, subtraction, multiplication, and squares, rarely reaches numbers larger than a couple of tens of thousands [Fri].

With the introduction of the decimal positional system in India during the 1st millennium AD in India, and its acceptance in the Arabic Caliphate, a new, more practical number system was established. Through contact with the Arabs, the decimal numerals were introduced in Europe during the Middle Ages

[1] Short and long scale are two systems for naming integer powers of 10. In short scale, a billion is a thousand millions, that is 10^9, while in long scale it is a million of millions, that is 10^{12}.

[Wuß, Brü]. Textbooks on arithmetic began to adapt to this new number system. By the 18th century it became common that such elementary textbooks, intended for beginners in arithmetic, begin with chapters on numeration in the decimal positional system. These chapters regularly included examples of large integers, the purpose of which was to be named (read out loud), sometimes also *vice versa*. For example, [Sau, p. 375] found that Prussian basic arithmetic textbooks of the late 18th and early 19th century often contain examples of "pointless" numbers and calculations. He quotes from a 1784 textbook by K. F. Splittegarb: "The weight of the whole globe, according to Schessler's calculation, is said to be 4.591.181.380.813.891.800.000.000 Leipzig pounds". Similarly, it was noted that in a popular 1778 textbook from the Habsburg Empire, [Stei], not only the reading of a very large (29-digit) integer is explained and demonstrated, but on the margins of the studied copy of the textbook there were several examples of very large integers handwritten by its owner (obviously intended to be read out loud) [BS1]. It was also noted that teaching of reading and writing very large numbers in primary education must have been a very common practice at that time, since an instructional book for teachers published in Vienna in 1778 explicitly asked the teacher not to "torment the students with this" [Pac]. In this article we wish to examine the conditions that have lead to this large-number obsession in arithmetic textbooks intended for primary education, its extent, as well as its development from earlier, medieval, times.

Mathematics education in the 8th and 9th century was very limited both with respect to content and to availability, even if in theory it remained a part of the Boethian *quadrivium* (arithmetic, geometry, astronomy, music). With Charlemagne's educational reforms, 'computus' became a part of eclesiastic school curricula. However, until the 12th century all teaching of mathematics was based on Boethius' *De Institutione Arithmetica* and various surviving Latin texts on surveying. The Roman number system continued to be the only number system in usage for several centuries, even if some scholars from 10th century onwards learned about the Hindu-Arabic numerals, mostly through contact with the Arabs on the Iberian peninsula [KS, Brü, Wuß].

Two of the most influential texts that significantly contributed to the spread of arithmetic with decimal numbers in Europe were al-Khwarizmi's text "Book on Addition and Subtraction in Indian Arithmetic" and Leonardo's *Liber Abaci*. Leonardo (Fibonacci) was significantly influenced by Arabic sources, while al-Khwarizmi (and other medieval Arabic scholars) are known to be influenced both by Greek and Indian mathematics of earlier, ancient times [Brü, Oak, Jao, Wuß].

While the Arabic original (written ca. 825 AD) of al-Khwarizmi's treatise on the Indian number system remains lost, there exist several Latin translations dating from 12th century onwards. Later medieval editions have often been copies or adaptations of the earlier Latin translations, and not direct translations [Arn, CH]. No original Latin translation is known, but four surviving 12th–13th century Latin texts, at least partially translations of the original, exist: *Dixit Algorizmi*, *Liber Ysagogarum Alchorismi*, *Liber Alchorismi*, and *Liber Pulueris* [All, Amb, Fol]. The *Dixit Algorizmi* exists in the form of two manuscripts, most probably both from the 13th century [CH, Fol]. Both of them begin with the description of the usage of decimal digits, extensively describe the correspondences between numbers written in the Indian and the Roman system, and then, before going on to arithmetic, give the example of how to read the 16-digit number 1180703051492863.[2]

Leonardo's book is not a classical arithmetic textbook. It is a much more comprehensive mathematical book, but with a huge influence, particularly on the spread of the decimal number system. In the first chapter numeration in the positional decimal system is introduced, and before continuing with reckoning, the reading of large numbers is described by example, in fact two examples: One is a 13-digit number (1007543289081), and the other is a 15-digit number (678935784105296) [Leo].

The translations and adaptations of al-Khwarizmi's book on decimal arithmetic, and Leonardo's *Liber Abaci* provided the foundation for the spread of the positional decimal system in Europe, but the two most popular Latin texts, in fact pedagogical instructions, on decimal arithmetic (*algorism*) in late Middle Ages, were the versified [Vil] and the prose work [Sac], both written in the first half of the 13th century. Even if surely influenced by al-Khwarizmi's original, or a translation thereof, they do not contain examples of reading large numbers, but just describe the principle of using the positional decimal system. However, in a 15th century English translation of Sacrobosco's treatise the description of the meaning of positions goes up to "A thousande thousande tymes a thousande", that is ten digits [Arn, CH, CH].

On the other hand, the Byzantine 13th/14th century scholar Maximus Planudes in his *Psephophria Kat Indous e Legomene Megale* (*The So-Called Great Calculation According to the Indians*) [Ger] explains

[2]The number is written incorrectly in one of the two manuscripts [Fol].

the usage of Hindu-Arabic numerals using the example 8136274592, which he reads in the Greek tradion in groups of four digits (myriads), and some smaller examples to explain the usage of zeros. It is probable that Planudes learned about the Hindu-Arabic numerals via the Byzantine trade contacts with the Eastern lands, and — during his stay in Venice — he most probably also had an opportunity to see and read Leonardo's *Liber Abaci.* It is worth mentioning that Planudes was apparently also one of the first mathematicians to use the term 'million', which first appeared in the 13th century [Smi2].

During the Late Middle Ages, the typical pre-university mathematical education became part of the so-called abacus schools (*scuole d'abbaco* in Italy, *Rechenschulen* in Germany). The few students who entered them did so generally at age 10 or 11, and the mathematical education started with learning to write numbers in the Hindu-Arabic decimal position system. This type of education flourished in 14th century Florence, and spread to other parts of Italy, and Europe. For the basic arithmetical education books known as *libri d'abbaco* were used (where *abbaco* should be taken in the meaning 'reckoning', and not 'abacus'; this stems from Leonardo's *Liber Abaci*). These abacus books were not school textbooks in the modern sense, but served more as reference books for teachers. Many were shortened versions and extracts from Leonardo's book, and many were based on the *algorism* "translations" of al-Khwarizmi [KS, Ster].

On the other hand, some arithmetic textbooks of that time (like J. Nemorarius' *De Elementis Arithmetice Artis* in early 13th century) were written in the Boethian style, that is, essentially containing number theory. In fact, the term *arithmetica* in these times is mostly used in the Boethian, that is classical Greek sense of theoretical arithmetic (number theory), while for practical calculation with the Hindu-Arabic numerals the term *algorism* was used [Smi1].

In general, the presentation of arithmetic in abacus books began with the topic of numeration, explaning the Hindu-Arabic numerals, but still some textbooks (for example the 14th century *Liber habaci*) used only Roman numerals [Høy, KS]. A selection of reckoning textbooks from this period is given in table 1.

Year	Book	Largest number
11*xx*	[Bon, pp. 25-135]	–
1202	[Leo]	678935784105296
12*xx*	[Vil]	–
12*xx*	[CH]	1180703051492863
12*xx*	[Sac]	–
1307	[Jac, pp. V.4.5-6][3]	234567898765432
13*xx*	[Ghe]	–
13*xx*	[Ger, p. 2]	8136274592
14*xx*	[Chr]	–

[3] There is a discrepancy of how the author writes and reads the number, he reads it as 345000000000+6789000000+8765000+432. Before that there is a correct example of reading number 987644321.

Table 1: Selected arithmetic textbooks from 12th to early 15th century, and the largest integer given as example of the principle of numeration

1 15th to 17th Century

The 15th century saw the advent of the press, which lead to a wider availability of textbooks. During the 15th, and even more so the 16th, century many arithmetic books were printed to meet the increasing demand in education for business and trade, and besides Latin and Italian, a number of notable elementary arithmetic textbooks were published in German, French, English, Portuguese, ... Among them was the first printed arithmetic textbook, *Arte dell'Abbaco* (known as *Treviso Arithmetic*, [Ano1]). It is a so-called *practica*, not intended for the use in schools, but rather for self-study [KS]. The *Treviso Arithmetic* contains a detailed verbal and tabular description of the principle of writing and reading decimal integers (up to thousands of millions).

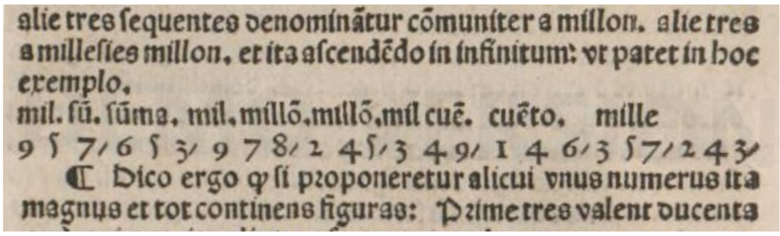

alie tres ſequentes denominātur cōmuniter a millon. alie tres
a milleſies millon, et ita aſcendēdo in infinitum: vt patet in hoc
exemplo.
mil. ſū. ſūma. mil. millō. millō. mil cuē. cuēto. mille
9 5 7/6 5 3/9 7 8/2 4 5/3 4 9/1 4 6/3 5 7/2 4 3/
Dico ergo q ſi proponeretur alicui vnus numerus ita
magnus et tot continens figuras: Prime tres valent ducenta

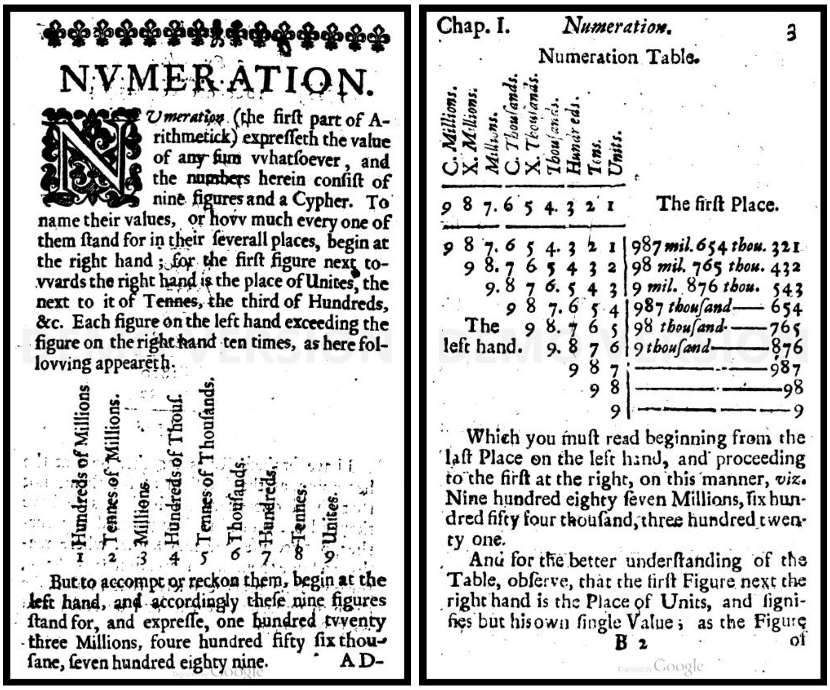
Figure 2: An example of a large number (1495) [Cir]. Source: a digitalized copy from the Internet Archive (copyright notice: public domain mark 1.0).

Year	Book	Largest number
1478	[Ano1, p. 9]	$(9 \cdot 10^9)$[4]
1484	[Bor, p. 6]	$(56789 \cdot 10^{14})$
1484	[Chu, p. 42]	7453248043000700023654321
1489	[Wid]	–
1495	[Cir, p. 4]	957653978245349146357243
1514	[Bös, p. 1]	364957
1522	[Rie, p. 14]	86789325178
1534	[Schö]	–
1537	[Köb]	44559386
1544	[Peu]	–
1546	[Sti, pp. 2-3]	5239089562800
1554	[Pis, p. A5]	9876543210
1556	[GP, p. 8]	23456345678
1557	[Tre, p. 14]	9999999999999
1569	[Ram, p. 1]	1234567890
1575	[Mau]	–
1583	[Cla, p. 6]	42329089562880
1593	[Sue, p. 3]	258147927351
1598	[Uni, p. 4]	$(3643 \cdot 10^{62})$
1618	[Rec, p. 49]	230864089105340
1622	[Blu, p. 3]	34545678594
1623	[Joh, Ch. 1][5]	237856354302567
1649	[Fro, p. 7]	438213506357228
1651	[Jag, p. 48]	123456789
1657	[Ley, p. 6]	736842708645
1682	[Taq, p. 96]	96638908003030460243709
1689	[Pre, p. 20]	20123500890408067574230
1695	[Ayr, p. 7]	84639042724536

[4] Here and in two of the rows below the exponential notation is used for the benefit of the modern reader.
[5] The 1649 edition of the same book contains only the example of how to read 654732, and the description of the general principle of reading.

Table 2: Selected arithmetic textbooks from period 15th to 17th century, and the largest integer given as example of the principle of numeration

In the 16th century the educational system developed further. Probably the best known educational innovation of the 16th century was the *gymnasium*. The first gymnasium was organized by Jean Sturm in Strasbourg 1539. The Protestant Reformation, and the Catholic Counter-reformation, significantly influenced education in Europe, and divided the styles of teaching and learning in Protestant and Catholic countries. In the Protestant system mathematics, mostly just basic arithmetic, was marginally taught at


NVMERATION.

Numeration (the firſt part of Arithmetick) expreſſeth the value of any ſum vvhatſoever, and the numbers herein conſiſt of nine figures and a Cypher. To name their values, or hovv much every one of them ſtand for in their ſeverall places, begin at the right hand; for the firſt figure next tovvards the right hand is the place of Unites, the next to it of Tennes, the third of Hundreds, &c. Each figure on the left hand exceeding the figure on the right hand ten times, as here follovving appeareth.

Hundreds of Millions	Tennes of Millions	Millions	Hundreds of Thouſ.	Tennes of Thouſands	Thouſands	Hundreds	Tennes	Unites
1	2	3	4	5	6	7	8	9

But to accompt or reckon them, begin at the left hand, and accordingly theſe nine figures ſtand for, and expreſſe, one hundred tvventy three Millions, foure hundred fifty ſix thouſane, ſeven hundred eighty nine. AD-

Chap. I. *Numeration.* 3

Numeration Table.

C. Millions.	X. Millions.	Millions.	C. Thouſands.	X. Thouſands.	Thouſands.	Hundreds.	Tens.	Units.	
9	8	7.	6	5	4.	3	2	1	The firſt Place.
9	8	7.	6	5	4.	3	2	1	987 *mil.* 654 *thou.* 321
	9	8.	7	6	5	4	3	2	98 *mil.* 765 *thou.* 432
		9.	8	7	6.	5	4	3	9 *mil.* 876 *thou.* 543
			9	8	7.	6	5	4	987 *thouſand* —— 654
The				9	8.	7	6	5	98 *thouſand* —— 765
left hand.					9.	8	7	6	9 *thouſand* —— 876
						9	8	7	—— 987
							9	8	—— 98
								9	—— 9

Which you muſt read beginning from the laſt Place on the left hand, and proceeding to the firſt at the right, on this manner, *viz.* Nine hundred eighty ſeven Millions, ſix hundred fifty four thouſand, three hundred twenty one.

And for the better underſtanding of the Table, obſerve, that the firſt Figure next the right hand is the Place of Units, and ſignifies but his own ſingle Value; as the Figure of

B 2

Figure 3: Two examples of tabular descriptions of reading large integers (1651 [Jag] and 1702 [Hod]). Courtesy of the British Library, digitised by the Google Books project.

the gymnasia (each with its own school study regulations), but by the mid-16th century arithmetic became a part of nearly all Protestant gymnasium curricula. On the other hand, the Catholic secondary school system became based on Jesuit gymnasia and was regulated by the end of the 16th century, making it more uniform, but mathematics (mostly Euclidean-style geometry) played a minor role here too, even if some (most prominently Christoph Clavius) advocated a more significant role of mathematics. Consequently, in most places mathematics continued to be predominantly part of higher, university, education, with the exception of basic reckoning, which was mostly taught by private tutors in 'primary' education. Basic arithmetic was then repeated and extended in gymnasia, for those who continued with their education. For this reason the arithmetic textbooks for primary education were essentially the same as introductory parts of mathematics for secondary education.

The 17th century mathematics education essentially continued on the same lines, but in the second half of the century the Catholic countries started to promote learning of applied mathematics for sea fare, engineering, military etc., leading to the necessity of specialised teachers. Also, in France the Oratorians organized their colleges, which were more oriented towards sciences and mathematics than their Jesuit counterparts [KS].

Mathematics education in the period 15th–17th century continued to begin with reading and writing numbers with the Hindu-Arabic numerals. In contrast to earlier times, it is now much harder to find an introductory arithmetic textbook which does not begin with a chapter on numeration, describe the digits in the decimal system, and the meaning of the positions, and then include one or more examples of large integers and how they should be read (see table 2 and Fig. 2). Rarely, the description of numeration is not at the beginning (for example [Taq]). Occasionally, the books do not contain examples of explicit pronunciation of large numbers; instead, the chapters on numeration contain descriptions of the principle of reading large numerals, with or without tabular presentation (e. g. [Ano1, Bor, Bös, Chu, Pis, Cla]). Also, some authors from the 16th century onward resort to simple examples 123456789(0) or 987654321(0), for example [Ram, Jag, Hod], often with a tabular presentation as shown in Fig. 3. However, it is quite clear that after the advent of print not only the number of published arithmetic textbook increased, but that the authors started to regularly include descriptions and examples of reading large integer numerals. Among the rare exceptions some continue to present arithmetic in the Boethian style (for example [Mau]), and some (for example [Wid, Peu]) simply give a short introduction ([Wid] explains the digits and positions, [Peu] not even that) before continuing with arithmetical operations.

2 18th and 19th Century

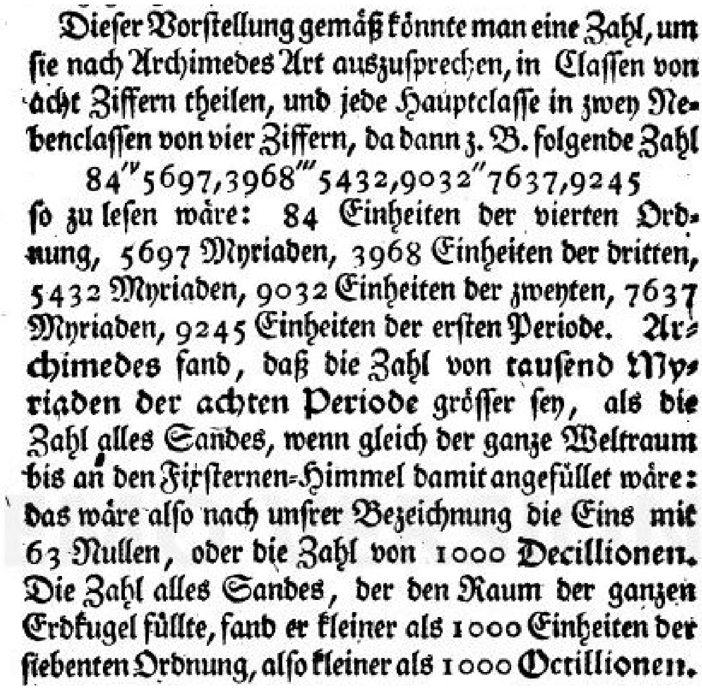
The 18th century marks the high point of the age of Enlightenment. In German lands it also known as the 'pedagogical century', as it brought several important innovations in education. Of interest here is the introduction of *Realschulen* (the first one was founded in Halle in 1747), intended for liberal education independent of university studies, with mathematics significantly represented in their curricula. This in turn changed the gymnasial lectures too, increasing the mathematical content of their curricula, and mathematics was included in their final exam (*Abitur*). In both cases, as in the older forms of education, the mathematics education started with arithmetic, and then proceeded to more advanced topics. Mathematical textbooks started to appear in various local languages, as well as in America [KS, Jon]. During the second half of the 18th century, under the influence of the philosophy of enlightenment, and also because of centralist political tendencies, the governments of most European states began reforming their respective educational systems, making education accessible to larger parts of population, and in most cases arithmetic became just one of the mathematical disciplines covered in secondary mathematical education, besides algebra and geometry, but in most cases it continued to be the exclusive part of mathematics taught in primary education [Lee, Sta, HH].

Year	Book	Largest number
1701	[Jea, p. 15]	3479841234
1702	[Hod, p. 3]	987654321
1702	[Coc, p. 4]	4578235782
1710	[Wol, p. 44]	2125473613578432597
1721	[Spe, p. 5]	6429357461282
1729	[Wes, p. 9]	897643291678546789987654954321659- -863960543953267854986736598
1742	[Wil, p. 5]	86897563974857296473859658
1757	[Reü, p. 5]	79873425318954
1758	[Bol, pp. 2-3]	7638101005900112360
1762	[Cla, p. 35]	620448401733239439360000
1772	[Har, p. 18]	437821506930582103
1773	[Pfl, p. 3][6]	620448401733239439360000
1776	[Pac, p. 68]	54321654321654217
1778	[Stei, p. 11]	53402689510437058007823608024
1780	[Kar, p. 16]	75840937528296364709007283
1781	[Béz, p. 4]	23456789234565456
1784	[Sau, p. 375]	4591181380813891800000000
1791	[Mal, p. 24]	538760053141125987604
1791	[Bar, p. 4]	345678912357986421357
1796	[Ebe, p. 105]	4356782936400008523
1799	[Gru, p. 7]	23456789234565456

[7] Note that the largest number in [Pfl] is the same one as in [Cla]. In both cases it is said that it is the number of ways in which the 24 letters of the (German) alphabet "can change their positions", that is it is the number 24!. The other examples of large numbers in these two books are different, but somewhat smaller (in both cases the second largest number is a 20-digit one).

Table 3: Selected arithmetic textbooks from 18th century, and the largest integer given as example of the principle of numeration

Even if the topics of most elementary arithmetic textbooks mostly remained the same as before, during the 18th century, mathematical textbooks (see table 3) began to include justifications (according to [Tro], the first such textbook was [Wol]). Consequently, some of arithmetic textbooks started to more clearly state the teaching outcome to be achieved by the examples of reading of large numbers and/or include the inverse problem to write a spoken number (for example, "The 2. task. To orderly write a



Dieser Vorstellung gemäß könnte man eine Zahl, um sie nach Archimedes Art auszusprechen, in Classen von acht Ziffern theilen, und jede Hauptclasse in zwey Nebenclassen von vier Ziffern, da dann z. B. folgende Zahl

$84^{IV}5697,3968'''5432,9032''7637,9245$

so zu lesen wäre: 84 Einheiten der vierten Ordnung, 5697 Myriaden, 3968 Einheiten der dritten, 5432 Myriaden, 9032 Einheiten der zweyten, 7637 Myriaden, 9245 Einheiten der ersten Periode. Archimedes fand, daß die Zahl von tausend Myriaden der achten Periode grösser sey, als die Zahl alles Sandes, wenn gleich der ganze Weltraum bis an den Fixsternen-Himmel damit angefüllet wäre: das wäre also nach unsrer Bezeichnung die Eins mit 63 Nullen, oder die Zahl von 1000 Decillionen. Die Zahl alles Sandes, der den Raum der ganzen Erdkugel füllte, fand er kleiner als 1000 Einheiten der siebenten Ordnung, also kleiner als 1000 Octillionen.

Figure 4: A large integer read 'according to Archymedes' in [Kar, p. 18] (1780). Source: Deutsches Museum, München, Bibliothek, 1903 A 854 (1

spoken large number with digits, that is, to place each said digit in its correct place", and an example of how to write "eighty-three trillions four hundred and nine thousand seven hundred sixty one billions fifty-two thousand seven hundred millions eighty thousand and six" [Cla, pp. 35–36]).[7] The textbook [Cla] is also interesting as the earliest example we have found which includes the inverse problem, that is, not only the problem of reading a given number, but to write a spoken number as well.

During the 18th century an increasing number of textbooks on arithmetic included not only reckoning with integers and vulgar (common) fractions, but also with decimals (for example, [Coc][8] and [Wil]), and even non-decimal number systems (for example, binary) [BS2].

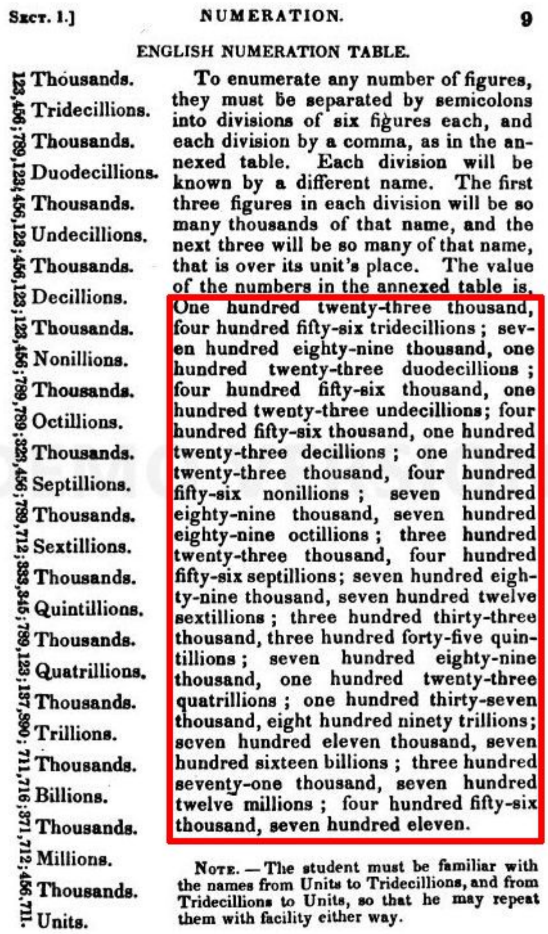
Some textbooks from the 18th century include the Archimedean numeration style in myriad steps ([Reü] only mentions the ancient Greek myriads with 416666 as an example, while [Kar] also gives a much larger integer, 84569739685432903276379245, with the description of how to read it 'according to Archimedes', as shown in Fig. 4). The few 18th century exceptions from this trend, not containing chapters on numeration and consequently also no large integers, some practical (commercial) arithmetic books (for example, [Muc]), and more advanced textbooks and compendia, for example, [Pic, Tan].

The 19th century brought many (and frequent) changes in education in general, and also specifically in mathematical education. Mathematics, at least basic arithmetic, became a part of obligatory education in most countries. New pedagogical and didactical theories were developed, so education methods and goals slowly changed during the 19th century, and many countries went through several educational reforms [KS]. Besides arithmetical (and generally mathematical) textbooks, methodological and pedagogical companion books for teachers were now published too in large numbers. In many countries mathematics education slowly moved away from the classical 'state rules — give examples — do exercises' style, and basic arithmetical education moved towards better understanding and away from the purely technical proficiency [Ung, Jon, Spr].

For example, in German speaking countries Johann Heinrich Pestalozzi's ideas on the principle of

[7]The mentioned textbook [Cla] also includes some 'practical' examples with large numbers, for example, "The smallest distance of the Moon to the Earth is taken to be 45580 miles in the Cassini calculus, that is nine hundred and eleven millions and six hundred thousand feet. How is this number to be written? Answer: 911,600, 000, 000 or 911600000000 feet".

[8]This is the earliest example we found that also describes, on p. 5, the reading of decimal fractions. The author calls their integer parts increasing numbers, and the fractional parts decreasing numbers. Books before 19th century rarely mention them, as the full decimal system was not yet common in commercial and daily life [Tro].



SECT. 1.] NUMERATION. 9

ENGLISH NUMERATION TABLE.

123,456;789,123;456,123;456,123;123,456;789,789;323,456;789,712;333,345;789,123;137,890;711,716;371,712;456,711.

Thousands. Tridecillions. Thousands. Duodecillions. Thousands. Undecillions. Thousands. Decillions. Thousands. Nonillions. Thousands. Octillions. Thousands. Septillions. Thousands. Sextillions. Thousands. Quintillions. Thousands. Quatrillions. Thousands. Trillions. Thousands. Billions. Thousands. Millions. Thousands. Units.

To enumerate any number of figures, they must be separated by semicolons into divisions of six figures each, and each division by a comma, as in the annexed table. Each division will be known by a different name. The first three figures in each division will be so many thousands of that name, and the next three will be so many of that name, that is over its unit's place. The value of the numbers in the annexed table is, One hundred twenty-three thousand, four hundred fifty-six tridecillions; seven hundred eighty-nine thousand, one hundred twenty-three duodecillions; four hundred fifty-six thousand, one hundred twenty-three undecillions; four hundred fifty-six thousand, one hundred twenty-three decillions; one hundred twenty-three thousand, four hundred fifty-six nonillions; seven hundred eighty-nine thousand, seven hundred eighty-nine octillions; three hundred twenty-three thousand, four hundred fifty-six septillions; seven hundred eighty-nine thousand, seven hundred twelve sextillions; three hundred thirty-three thousand, three hundred forty-five quintillions; seven hundred eighty-nine thousand, one hundred twenty-three quatrillions; one hundred thirty-seven thousand, eight hundred ninety trillions; seven hundred eleven thousand, seven hundred sixteen billions; three hundred seventy-one thousand, seven hundred twelve millions; four hundred fifty-six thousand, seven hundred eleven.

NOTE. — The student must be familiar with the names from Units to Tridecillions, and from Tridecillions to Units, so that he may repeat them with facility either way.

Figure 5: A mid 19th century example of an 84-digit number (vertical), and how it is read (framed text) in [Gre]. Adapted from the digitalized copy from Harvard Library (CC BY 4.0 license)

Anschauung (concrete observation, perception) significantly influenced school curricula, including the arithmetical one. Pestalozzi emphasized the role of psychology and advocated adapting teaching content to the student's abilities. In particular, for arithmetic he requested a move away from the earlier teaching of content as a (closed) set of rules that were meant to be learned by heart to a more experience-based one, as well as shifting the focus from the numerals towards the numbers themselves. In his opinion, the objective of learning reckoning was not the proficiency itself, but the "strengthening of mental strength". Influenced by Pestalozzi, the first arithmetic textbooks of a new style appeared, starting directly with reckoning with small integers (for example [Til, Hae]), without introductory chapters devoted solely to numeration. Corresponding methodological instructions were published too, advocating starting the arithmetic instruction from small numbers and gradually increasing them for example, Wilhelm von Türk's *Leitfaden für den Rechenunterricht* (1816). While Pestalozzi's ideas essentially moved away from calculations with digits towards mental arithmetic, somewhat later Adolph Diesterweg found a balanced pedagogical approach, combining Pestalozzi's subjective style and the objective style of earlier times [Ung, Sau, Ster, KS].

All of these tendencies reflected upon a greater variety of styles and approaches of arithmetic textbooks being published (see table 4). Even if by the end of the 19th century the introductory chapters of the majority of reckoning textbooks were still devoted to numeration, the presentation in most cases became more systematic, with more examples building up from one digit to large numbers, but generally not surpassing 15 digits, for example [Schü, Dav, Sche, Ber, Cor]. In some cases even if the content remained

Year	Book	Largest number
1800	[Gou, p. 8]	987654321
1806	[Til]	–
1815	[Spr, p. 8]	448765000078987734
1815	[Sche, p. 9]	53693579864208457968
1818	[Dew, p. 15]	393829759751235871297473918651437256
1819	[Let, p. 5]	12457044530256
1825	[Kei, p. 5]	123456123456123456123456578371123875
1828	[Schü, p. 9]	590003800675094760964700306
1831	[Con, p. 15]	946258713395173826143697495213191216493121
1835	[Die, p. 15]	34907808065432
1840	[Ano2]	–
1840	[Col]	–
1844	[Sco, p. 7]	73458946537053096
1845	[Ohm, p. 8]	7654321
1845	[Gre, p. 9]	123456789123456123456123123456789789323456-
		-789712333345789123137890711716371712456711
1847	[Dav, p. 15]	920323842768319675
1848	[Ada, p. 14]	3082715203174592837463512
1849	[Ber, p. 6]	34211893514
1852	[Ste, p. 7]	668435627412727154364332222222
1855	[Cor, p. 9]	8274169325
1858	[Egg, p. 69]	946758432967895
1865	[Hae]	–
1873	[Qua, p. 17]	123400789000
1879	[Ish]	–
1883	[Whi, p. 69]	683417998
1886	[Bos]	–

Table 4: Selected arithmetic textbooks from 19th century, and the largest integer given as example of the principle of numeration

fairly old-fashioned, it's presentation became simpler and clearer, for example the Scottish [Ste, p. 7] gives just the following simplified description of reading a large integer:

$$\underbrace{668{,}435.}_{\text{Quadrillions.5th.}}\ \underbrace{627{,}412.}_{\text{Trillions.4th.}}\ \underbrace{727{,}154.}_{\text{Billions.3rd.}}\ \underbrace{364{,}332.}_{\text{Millions.2nd.}}\ \underbrace{222{,}222}_{\text{Units.1st Period.}}\ .$$

Other arithmetic textbooks, for example [Til, Ano2, Col], chose the more modern approach, starting with reckoning with small numbers (sometimes first purely verbally), and systematically building the content from smaller numbers up to larger ones. For example, [Egg] explicitly organizes the material so that numeration and reckoning with large numbers appears in the fourth chapter, intended for the fourth year of schooling, and similarly [Whi] does not completely avoid large numbers, but postpones their description after basic reckoning with smaller integers is fully described. But, even if the general trend in basic arithmetic education moved towards clarity and simplification, there still remained a number of textbooks concurring to the tradition of large integers, even to the point of using ridiculously large number examples, for example [Gre] contains an incredible 84-digit example, Fig. 5, right before going on to classic elementary reckoning (not surpassing results larger than 10-digit numbers). Such textbooks are an exception to the trend, and by the end of the 19th century the tradition of large integers gradually disappeared to be replaced with more modern introductions to arithmetic.

3 Discussion and Conclusion

The ubiquitous presence of unwieldy large integers as examples and exercises for writing and reading of (decimal) numbers in historic elementary arithmetic textbooks, represents an intriguing example of

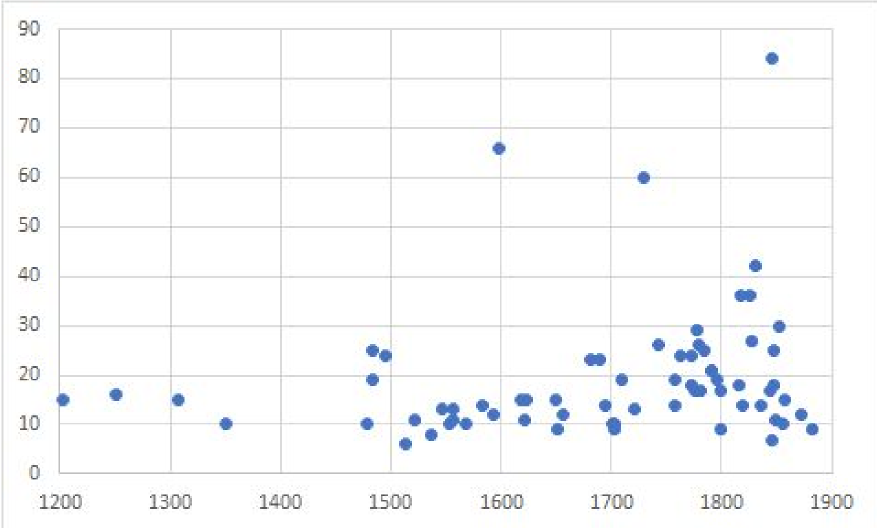


Figure 6: Number of digits of largest integers in textbooks vs. year of publication

an evolutionary dead-end in development of teaching of mathematics. The earliest appearances of such examples served a definite purpose: to introduce the Hindu-Arabic decimal position system, familiarise the reader with it, and exemplify its superiority over the earlier systems by showing how any number, no matter how large, can be written with ease. However, as the Hindu-Arabic notation became the standard in Europe, the necessity of such an introduction may have waned, but the custom of starting an introductory arithmetic textbook with it remained. As the time progressed, the span of the sizes of the numbers used greatly increased (Fig. 6) — while the majority of early authors (up to and including the 16th century) were content to provide examples of 10–15 digit numbers, in later centuries some authors increase this to quite absurd sizes. The reason for this inflation does not seem to be advance in mathematics education — there does not seem to be much educational methodological justification for forcing a beginner student to read a 30-digit integer out loud. Also, such large large numbers are not introduced because of the necessity of understanding of the later subject matter, because the majority of the calculations which appear later in these books use significantly smaller numbers, which is reasonable, as they were elementary arithmetic books intended for the introduction (or sometimes repetition) of basic reckoning in primary (and later also secondary) education and not for advanced mathematics calculations needed in science and engineering. Rather, this may be a case of series of attempts by the authors to distinguish themselves from their predecessors (the authors of earlier textbooks they have based theirs on), by a kind of one-upmanship: If earlier textbooks have no examples larger than, say, 24 digits, I will add a 27-digit one! Of course, not all authors have succumbed to this fashion, and numerous textbooks throughout the 16th–19th centuries still keep their examples in the classical 10 to 15-digit range. Indeed, by the end of the 18th century some authors began advocating dropping this practice [Pac]. The 19th century marked the zenith of large integers in elementary arithmetic textbooks — while on the one hand the examples grew to the utterly ridiculous (such as the 84-digit monstrosity from [Gre]), on the other increasingly more of these textbooks dropped this practice entirely as times went by, so that by the end of the century it finally, and well deservedly, went more or less extinct.

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