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Attila Szabo

Ann-Sophie Tillnert

John Mattsson

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Attila Szabo[1], Ann-Sophie Tillnert[2], John Mattsson[2]

[1]Department of Teaching and Learning, Stockholm University, Sweden

[2]Sjöstadsskolan, Stockholm, Sweden

Abstract: When solving problems, mathematically gifted individuals tend to internalize intuitive ideas and approaches, and to shorten their reasoning. Consequently, for teachers it is difficult to observe gifted students' mathematical reasoning in the context of problem solving. In this paper we investigate nine gifted Swedish 9th grade students' mathematical reasoning during problem solving in small groups at vertical whiteboards. The data consists of 5 filmed group-activities, that were analysed according to a framework of collaborative problem-solving (Roschelle & Teasley, 1995). The analysis shows that every group solved proposed problems successfully within different socially negotiated Joint Problem Spaces (JPS) and, importantly, that students were able to verbalize and display their mathematical reasoning. Additionally, it is indicated that using vertical whiteboards facilitated considerably the exhibition of students' mathematical reasoning.
Keywords: gifted students; mathematical reasoning; problem solving; peer learning

1 Introduction

Mathematically gifted students differ in some pivotal respects from their non gifted peers. Not only they obtain and formalize mathematical information accurately from a given material and process it efficiently, but they also generalize mathematical structures swiftly and recall generalized relationships effectively (e.g., Krutetskii, 1976; Sheffield, 2002). Furthermore, gifted students enjoy solving difficult mathematical problems, adding creative elements to those problems, and exploring patterns in everyday situations (e.g., Krutetskii, 1976; Sheffield, 2002; Sriraman, 2003). During problem solving, these students strive after mathematical elegance and clarity, and – of importance for this paper – while looking for a comprehensive solution of a problem, instead of sequential steps, they tend to shorten their mathematical reasoning (e.g., Clements, 1984; Krutetskii, 1976; Sheffield, 2002). Consequently, a large part of gifted students' reasoning – both oral and in written form – is internalized and thereby not observable for their peers and teachers (e.g., Bloom, 1985; Clements, 1984; Krutetskii, 1976).

Taking into consideration that the present study was conducted in Sweden, it might be important to mention that gifted students have traditionally not been present in the Swedish educational discourse. Persson (2010) described that situation some years ago as: "It is likely that nowhere is

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resistance to assist gifted students in school stronger than in the Scandinavian countries for historical, cultural, and political reasons, particularly in Sweden" (Persson, 2010, pp. 536–537).

However, mentioned conditions have changed during the past decade. As for example, since 2015 the Swedish National Agency for Education (NAE) – the organization responsible for education in primary and secondary schools in Sweden – provides general support material about gifted students (Skolverket, 2015). Moreover. based on a governmental initiative, currently 24 primary schools are organizing programs for excelling students in grades 7-9 (ages 14-16). Nevertheless, even though governmental ambitions were in line with other OECD countries' excellence initiatives, the outcome of mentioned programs may not be regarded as unproblematic (Dodillet, 2019). An analysis performed by Dodillet (2019) highlights that the NAE diverged from governmental regulations by offering the planned elite education for students that are interested in school subjects, rather than students that are gifted in respective subjects. Further, it has also been stressed that the NAE's own evaluations of implemented elite education ignore aspects as students' talent and abilities, by focusing "on the students' gender, socio-economic status, and ethnic background" (Dodillet, 2019, p. 265) and by problematizing the fact that students are recruited from a homogenous group (Dodillet, 2019).

When it comes to mathematics education, the Swedish national curriculum provided by NAE (Skolverket, 2019) highlights some abilities that should be developed in all students. As for example, students should be able to formulate and solve problems, to evaluate selected strategies, to use and analyse mathematical concepts, to select and use appropriate mathematical methods and to reason mathematically. Additionally, in order to receive top-grades, students should be able "to talk about [mathematical] approaches in an appropriate and effective way", and "to reason mathematically and to follow mathematical reasoning" in "a way that moves the reasoning forward and deepens or broadens it" (Skolverket, 2019; our translation). Consequently, Swedish students' grades in mathematics are based on both written tests and on their oral communication, e.g., students should be able to explain, both orally and in written form, how they solved a given mathematical problem.

In the given context, it seems reasonable to assume that gifted students' shortened mathematical reasoning, which is remarkably quiet and internalized during the problem-solving process (e.g., Bloom, 1985; Clements, 1984; Krutetskii, 1976) places them in a precarious situation during both written and oral testing.

Accordingly, based on the difficulties associated to the displayed share of mathematical reasoning of gifted students, in combination with the requirements of the Swedish curriculum about communication in mathematics, the aim of the present study was to investigate methods that could facilitate and display the mathematical reasoning of gifted students during problem-solving. More precisely, we examined the effects of collaborative problem-solving, performed at vertically placed whiteboards, in the context of gifted students' mathematical reasoning.

2 Theory

2.1 Mathematical abilities of gifted students

The first attempts to investigate individuals' mathematical abilities were carried out at the end of the 19th century (Calkins, 1894) and since the pioneering years researchers have traditionally observed mathematical abilities in the context of problem solving (e.g., Krutetskii, 1976; Sheffield, 2002). Nevertheless, until the 1960's, the research field applied psychometric methods to examine abilities that can be characterized as mathematical, thereby considering abilities basically innate and only marginally developable by the individual (e.g., Krutetskii, 1976). However, the psychometric paradigm was unable to delimit the characteristics of the mathematically gifted, and since the last decades of the 20th century, the criticism towards psychometric approaches as predictors of intellectual performances have been accentuated (e.g., Stoeger, 2009).

A significant study, considerably divergent from mentioned psychometric approaches, performed by Hadamard (1945) – based on a mail correspondence with some outstanding scientists of the time – made a considerable contribution to the nature of mathematical thinking and problem solving. Hadamard (1945) suggests that prominent scientists, such as Albert Einstein, when working with difficult mathematical problems, experienced a relatively long, unconscious period of incubation before being able to transform intuitive thoughts into conscious, useful ideas. Remarkably, the period of incubation – described as internalized thinking – led frequently to a mathematical insight that could solve the given problem (e.g., Hadamard, 1945; Leikin, 2014). Thereby, of importance for the present study, it might be plausible to assume that the unconscious period of incubation of intuitive thoughts is a part of gifted problem-solvers mathematical reasoning.

Regarding the research on gifted students' mathematical abilities, relatively recent studies (e.g., Juter & Sriraman, 2011; Sheffield, 2002; Sriraman, 2003) confirm the findings of the pivotal work of a Soviet research team led by Krutetskii (Krutetskii, 1976). By observing the mathematical activities of around 200 individuals between 1955-1966, Krutetskii and his team concluded that mathematically gifted students are able to

- obtain and formalize mathematical information effectively,
- process mathematical information efficiently – by thinking logically and reversibly, by displaying a rapid and broad generalization of mathematical objects and relations, and by striving for clear and simple solutions,
- retain and recall generalized mathematical relationships efficiently, and
- display a general synthetic mathematical component (Krutetskii, 1976).

When describing above listed characteristics, it is emphasized that particular abilities are closely interrelated and mainly associated to the problem-solving process. It is also highlighted that gifted students tend to compensate less-developed abilities with more well-developed ones, and that their mathematical abilities form a "single integral system, a distinctive syndrome of mathematical giftedness, the mathematical cast of mind" (Krutetskii, 1976, p. 351). Important for the present context, "mathematical abilities are abilities to use mathematical material to form generalized, curtailed, flexible, and reversible associations and systems of them" (Krutetskii, 1976, p. 352). Particularly, gifted students' ability to form generalized, curtailed associations is related to the ability to process mathematical information, characterized as "rapid and broad generalization of mathematical objects, relations and operations, the ability to curtail the process of mathematical reasoning, flexibility in mental processes, striving for clarity and simplicity of solutions" (Krutetskii, 1976, p. 351).

As mentioned, studies indicate that the mathematically gifted tend to internalize and shorten their mathematical reasoning and that those characteristics are not easily discernible by external observers (e.g., Bloom, 1985; Clements, 1984; Krutetskii, 1976; Sheffield, 2002). Thus, it is not unreasonable to assume that teachers cannot observe gifted students' oral or written mathematical reasoning in accurate ways and, consequently, are not able to assess their factual mathematical skills.

2.2 Mathematical problem solving

Mathematical problems and the process of problem-solving are seen as central components of mathematics education (e.g., Kilpatrick, 2016; Mason, 2016; Pólya, 1966; Schoenfeld, 1994; Skolverket, 2019). Further, when addressing problem solving in the context of school mathematics, Halmos accentuates that "it is the duty of all teachers, and all teachers of mathematics in particular, to expose their students to problems much more than to facts" (Halmos, 1980, p. 523).

However, the differentiation of routine tasks from mathematical problems cannot be considered a trivial task. In that respect, it seems that "not so much a function of various task variables as it is of the characteristics of the problem solver" (Lester, 1994, p. 664) that qualifies a given task to be perceived as a mathematical problem. That is, the challenge experienced by the individual when solving the task seems to be pivotal when differentiating mathematical problems from routine tasks (e.g., Carlson & Bloom, 2005; Pólya, 1966). Consequently, a mathematical problem should entail complexities that cannot be directly addressed with the methods or algorithms that are in the immediate possession of the solver (e.g., Blum & Niss, 1991). By building on Pólya's (1957) seminal work, Kilpatrick (2016) indicates that mathematical problem-solving is a process of "getting from where you are to where you want to be through successive reformulations of the problem until it becomes something you can manage" while replacing "unsuccessful efforts by successful ones through a heuristic inquiry process" (Kilpatrick, 2016, p. 69). Nevertheless, given the aspect of novelty that is associated to the concept, it is indicated that a non-routine task, that is perceived as a mathematical problem at a given time – after internalizing the associated general problem-solving method – might be regarded as a routine task at another time by the solver (Arcavi & Friedlander, 2007).

In a similar vein, a series of studies scrutinizing students' attempts to solve mathematical tasks and problems, labelled their mathematical reasoning in terms of imitative and creative, respectively (Lithner, 2008). In the context, reasoning is described as "the line of thought adopted to produce assertions and reach conclusions in task solving" (Lithner, 2017, p. 939). While students´ imitative reasoning was mostly characterized by selecting and applying algorithms, able students displayed creative mathematical reasoning when solving non-routine tasks (Lithner, 2017). Importantly, drawing on Lithner's (2008) ideas, Granberg and Olsson (2015) indicate that creative mathematical reasoning contains suggestions of problem-solving strategies as well as mathematical argumentations for suggested strategies.

In summary, it seems that a mathematical task has to represent both a considerable challenge and a substantial aspect of novelty for the solver in order to be perceived as a mathematical problem. Of particular importance for this paper, it should also be stressed that gifted students' mathematical abilities are closely linked to the process of problem-solving (e.g., Krutetskii, 1976; Sheffield, 2002; Sriraman, 2003).

2.3 Collaborative problem-solving

By differentiating collaborative from cooperative problem-solving, Roschelle and Teasley (1995) suggest that students' collaborative problem solving involves a "mutual engagement of participants in a coordinated effort" to solve a given problem, characterized by "a coordinated, synchronous activity that is the result of a continued attempt to construct and maintain a common perception of a problem" (Roschelle & Teasley, 1995, p. 70). By analyzing students' communication and activities, Roschelle and Teasley (1995) indicate that students' social interaction during collaborative problem-solving is mainly associated to a philosophically delimited Joint Problem Space (JPS). In the context, JPS is described as a shared and socially negotiated knowledge structure that supports a collaborative problem-solving process, by integrating "a) goals; b) descriptions of the current problem state; c) awareness of available problem solving actions, and d) associations that relate goals, features of the current problem state, and available actions" (Roschelle & Teasley, 1995, p. 70). However, the occurrence of JPS during problem-solving is not trivial, that is, in order to build a well-functioning JPS participants have to actively collaborate with respect to the mathematical content, to monitor group activities that indicate divergences, and to repair divergences that hinder the collaborative process (Roschelle & Teasley, 1995).

Even though the notion of JPS offers an applicable framework for understanding students' interaction during mathematical problem-solving, the oral interaction between students contains some discourse elements that should be addressed (Roschelle & Teasley, 1995). According to the limitations of this paper, we are not able to offer a description of the terminology associated to discourse analysis. Regardless, we would like to highlight that a conversation between individuals is carried out according to a well-specified model and it is characterized by turn-taking (Schegloff et al., 1977). In that respect, turn-taking is a pivotal concept of the interplay between the participants in the conversation, and can be exemplified by a situation in which one participant begins a sentence that another participant builds on or completes with a subsequent statement. Questions and acceptances as well as disagreements and corrections are additional concepts that are used to describe the content and structure of the conversation – mainly by indicating how well participants understand each other (Clark & Schaefer, 1989).

In order to be able to observe the mathematical reasoning of students, we also paid attention to studies that analyzed the physical conditions of high school students' collaboration during problem-solving. In that respect, Liljedahl (2016) highlights, that students working in groups at vertical whiteboards not only demonstrated more thinking and persistence, but also discussed and participated more actively in the problem-solving process, compared to groups which used horizontal whiteboards, vertical worksheets, horizontal worksheets, and notebooks. Moreover, it is indicated that the non-permanent nature and the upright position of the vertical whiteboards were of importance in facilitating students' participation, discussion, and persistence during problem-solving (Liljedahl, 2016). The non-permanence of writing and drawing on the whiteboard seemed to increase the non-linear work within the groups and students´ willingness of taking risks during problem-solving. In addition, the vertical placement of the whiteboards – by encouraging students to stand up and to be physically active – strengthened students' collaboration (Liljedahl, 2016).

3 Methods

3.1 Participants

In total, nine students participated in the study. Participants were 9th grade students (15-16 years old) exhibiting pivotal abilities associated to mathematical giftedness (e.g., Krutetskii, 1976). The participation in the study was optional. The participants have been taught by the second author of this paper at a regular primary school in Sweden, both in their ordinary classes and at extracurricular mathematical activities during two years prior to the present study. Important for the context, during those two years, participants solved mathematical problems once a week in groups of 2-3 by using whiteboards. Thus, when entering the study, the participants had considerable experience of collaborative problem-solving in front of vertical whiteboards.

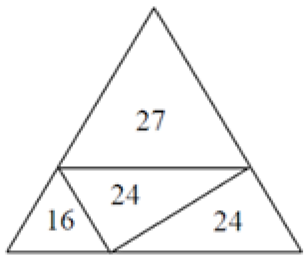
3.2 The selected problems

When selecting tasks for the participants, some main characteristics of mathematical problems were taken into consideration (e.g., Lester, 1994; Carlson & Bloom, 2005; Lithner, 2008; Pólya, 1957). That is, based on the experience of working with the participants for two years, the second author of this paper assessed the novelty and non-routine character, as well as the degree of difficulty, of selected tasks in order to classify them as mathematical problems. Moreover, actual and former textbooks used by participants were examined, and their previous mathematics teachers were consulted. When presenting the selected problems, participants were asked if they have encountered the problems previously. Based on their answers, we could assume that the selected problems were new to them. In addition, we tried to select problems that would facilitate students' creative reasoning (Lithner, 2017).

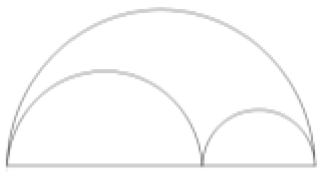
Accordingly, two mathematical (geometrical) problems were proposed. Problem 1 was selected from Pythagoras Quest (2009), which is a Swedish mathematics competition for students aged 16-19 years. Problem 2 has been used in previous studies on students' mathematical abilities associated to problem solving (Szabo, 2017).

Next, we are going to display the problems which were presented for the participants.

Problem 1: The numbers in the figure display the perimeter of each one of the four triangles. How large is the perimeter of the outer triangle?



Problem 2: In a semicircle, two additional semicircles are drawn, as shown in the figure. Is the perimeter of the large semicircle longer, shorter, or equal to the sum of the perimeters of the two inscribed semicircles? Motivate your answer.



3.3 The study

The observations were carried out by the second and third authors of this paper, during two separate occasions one month apart. At the first occasion, when working with Problem 1, six students were present, at the second occasion, when working with Problem 2, nine students. Every group was constituted by three students. In order to avoid the impact of rigid group-compositions on the problem-solving process, different groups were formed on each occasion.

The problem-solving activities were carried out in similar ways. Each activity began by handing out worksheets with the respective problem and reading them loudly. Then, participants had to think quietly for 2-3 minutes. Next, the groups solved the problems collaboratively.

As mentioned, the participants were familiar with collaborative problem-solving. Thus, the groups worked at separate whiteboards, by writing or verbalizing their thoughts, calculations and drawings. Moreover, in order to offer autonomy to the groups and to stimulate the problem-solving process – based on indications from previous studies (Liljedahl, 2016) – the observers were very restrictive in their communication with the students.

The activities of each group, i.e., communication and gesturing, as well as writings and drawings on whiteboards were filmed and audio recorded. Accordingly, five different problem-solving activities were documented in mentioned ways, two activities associated to Problem 1, and three to Problem 2. Afterwards, all communication between participants was transcribed.

3.4 Analysis

The transcribed data and the video-recorded activities resulted in accurate linear reproductions of the participants' communication and actions during problem solving. To classify the communication and activities of participants, based on concepts presented in the theoretical background section, we decided to perform qualitative content analysis of the empirical material (e.g., Elo & Kyngäs, 2008). Such analyses are typically either theory-driven or data-driven, depending on the goals of the actual study (Kvale & Brinkmann, 2009). Due to the intricate nature of the displayed reasoning of gifted pupils, in order to achieve an appropriate level of reliability of the study, we performed two different analyses. First, we performed a deductive, theory-driven analysis of the material. Afterwards, we performed a subsequent inductive, data-driven analysis.

The analytical framework

In the first phase, we analyzed the communication between students during problem-solving through a theory-driven approach, by focusing on aspects that are characteristic for collaborative problem solving. Consequently, drawing on the complex concept of JPS (Roschelle & Teasley, 1995) – in order to investigate the shared and socially negotiated problem-solving process of the participants – we outlined the following analytical framework:

(I): Understanding the problem and creating a goal, that is, initiating the reasoning within the group and thereby a JPS,
(S): Creating, implementing and evaluating problem-solving strategies,
(N): Observing and negotiating misconceptions as well as deviations from the goal,
(G): Solving the problem and thereby reaching the goal.

Using the analytical framework

To illustrate the coding process, we display a part of a group-activity that occurred while the participants were working with Problem 1. That is, we provide an overview of the sequential development of the problem-solving activity of group A, and the JPS-phases identified during the process.

Table 1: Group A working with Problem 1

Part	Statement	Action	JPS Code
1	Liam: I am starting to draw the figure then. Sara: Um.	Liam**:** Steps to the whiteboard.	I
2	Liam: It´s ok? Sara**:** It's ok.	 Sara: Places the task on the whiteboard.	I

As seen above, students started to work by drawing a figure (1) and by placing the worksheet with the task on the whiteboard (2). This episode occurred after they have been thinking quietly, at the very beginning of their collaborative activity. Based on the above episode, but also on the content of their communication during following episodes, we assumed that their actions and communication initiated the establishment of a JPS (I).

Next, we present the episode that occurred directly afterwards.

Table 2: Group A working with Problem 1

Part	Statement	Action	JPS Code
4	Liam: Eh… I would argue that the whole triangle is similar… Sara: Um.		S
5	Liam: And the top triangle is similar. Sara: Um.		S
6	Liam: And this one is similar. Sara: Um. Liam: Or equilateral, it is said. And then they are similar with each other.	Liam: Points to the triangle with the perimeter 16.	S
7	Sara: And also, those two, aren't they?	Sara: Points into the air.	S
8	Liam: That they are similar? Sara: Thus, these two.	 Sara: Points to the figure.	S

In the above displayed interaction, the statements "I would argue that the whole triangle is similar" and "… the top triangle is similar and this one is similar.", even though the use of singular form in the context of similarity is not mathematically stringent, indicate that Liam starts talking about similar triangles in the figure. Next, he states, more stringently, that the triangles "are similar with each other". Given that the students indicate that they try to use similar triangles to solve the problem, it is reasonable to assume that Liam initiates a strategy for a possible solution (S). The reply from Sara "And also those two, aren't they?" shows that she participates in the strategical reasoning (S), thereby indicating that the group starts to deal with the preconditions for a solution strategy (S).

After displaying the strategy (S) above, in the next episode, the group tested the efficiency of the strategy by questioning the similarities between the triangles observed in the figure.

Table 3: Group A working with Problem 1

Part	Statement	Action	JPS Code
9	Tom: It is equilateral.		N
10	Liam: Yes, yea, they ... No, I do not think so.		N
11	Sara: Yes, or they… Tom: That one. Sara: They are similar, but in different ways.		N

As seen above, after Tom's statement "It is equilateral" – which is his first contribution to the process – Liam displays a trace of doubt about the similarities of triangles, by replying "Yes, yea, they ... No, I do not think so". This is followed by Sara's reply "Yes, or they… They are similar, but in different ways." indicating that neither Sara is convinced that the observed triangles are similar. Accordingly, we assumed that participants started to discuss misconceptions and different opinions associated to the selected strategy by negotiating with each other (N).

The episode displayed below occurred after the participants continued to negotiate the selected strategy based on similar triangles (N), doubting its stringency.

Table 4: Group A working with Problem 1

Part	Statement	Action	JPS Code
21	Liam: But we can… Because I think, if you now… We can make an assumption that it is equilateral, and, in that case, it is very easy to calculate. Tom: Um.		S
22	Liam: Eh ... But … And based on that, maybe we can prove that it is equilateral, as it hopefully is. Otherwise, we have to rethink.		S

As seen above, after the negotiation-phase (N) displayed in parts 9-10-11, Liam proposes another strategy (S) “But we can… We can make an assumption that it is equilateral”. Next, after Tom’s affirmative “Um”, Liam explains the modified strategy (S) “… based on that, maybe we can prove that it is equilateral, as it hopefully is”.

Next, we display the episode when group A reached a solution to the problem. This occurred after the group implemented the modified strategy (S) by calculating perimeters of respective triangles.

Table 5: Group A working with Problem 1

Part	Statement	Action	JPS Code
40	Liam: Um. But that must be correct.		S
41	Sara: And what will that be? It will be…		S
42	Tom: Forty-three.		G
43	Liam: If we take this, it should be around fourteen something…		G
44	Sara: Fourteen times three…		G
45	Liam: Fourteen times three is around…		G
46	Sara: Fifteen times three, forty-five.		G
47	Liam: Yes.		G
48	Sara: Also, minus three, then forty-two, type. So, that both those that we have seen gave the same result.		G

In the above episode, Liam’s affirmation “But that must be correct.”, followed by Sara thinking out loud “And what will that be? It will be…” while calculating the result are parts of the selected strategy (S). However, when Tom suddenly delivers the result “Forty-three” the group solve the problem to a great extent (G). Afterwards, Sara and Liam calculate the perimeter of the outer triangle in a different way, thereby reaching the goal from another perspective (G). Finally, Sara´s statement “both those that we have seen gave the same result” indicates that the group arrived at a complete solution of the problem (G).

The inductive analysis

After performing the theory-driven analysis, we performed an inductive, data-driven examination of the empirical material. As mentioned, in this respect as well, we performed a qualitative content analysis (e.g., Elo & Kyngäs, 2008).

The aim of the inductive analysis was to display those aspects of participants' conversations and actions, that developed their shortened mathematical reasoning (e.g., Clements, 1984; Krutetskii, 1976; Sheffield, 2002) into more observable, creative mathematical reasoning (Lithner, 2017). Particularly, we examined participants turn-taking (Schegloff et al., 1977), during which they posed questions and expressed acceptances or disagreements (Clark & Schaefer, 1989) in the context of suggested problem-solving strategies and associated mathematical arguments (Granberg & Olsson, 2015).

Firstly, we identified phases that displayed characteristics of shortened mathematical reasoning i.e., mathematically correct statements or calculations that lacked explanations or mathematically grounded motivations. Secondly, and conversely, in order to identify participants´ displayed mathematical reasoning, we scrutinized the material for phases that were explanatory or clarifying and mathematically stringent. In addition, we analysed participants´ physical actions at the whiteboard in these sequences. Accordingly, corresponding phases were coded by SR (shortened reasoning) respectively with DR (displayed reasoning).

Next, we display some episodes when Group A developed phases of SR into DR.

Table 6: Group A working with Problem 1

Part	Statement	Action	Reasoning Code
29	Tom: Take only twenty-seven plus sixteen plus twenty-four. No, not twenty-four… twenty-seven plus sixteen.	Tom: Writes 27 + 16 + 24 on the whiteboard and then erases 24.	SR

Above, Tom verbalizes the calculation which he is performing. However, the statement "No, not twenty-four… twenty-seven plus sixteen." offers no explanation of why he removes 24 from the sum to be calculated. He just informs the group that he erases the number 24.

In the episode below, following the previously displayed phase, the communication between participants displays characteristics of SR.

Table 7: Group A working with Problem 1

Part	Statement	Action	Reasoning Code
33	Sara: Motivate.		SR
34	Tom: Um.	Tom: Points with his finger to the top triangle in the figure and then points out the two upper sides of the top triangle that are parts of the large triangle's sides.	SR
35	Tom: Twenty-seven. So, all have a side against twenty-four.	Tom: Points out every triangle that is around the triangle in the middle, with perimeter of 24. Afterwards, he points to the perimeter of the large triangle.	SR

As displayed above, even though Sara expresses a need for clarification, Tom continues to point at the whiteboard, by uttering only one sentence: "Twenty-seven. So, all have a side against twenty-four." That is, the reasoning within the group is shortened (SR), without offering mathematical explanations.

However, Tom's actions lead to the immediately following episode, displayed below, when Liam and Tom developed SR into DR.

Table 8: Group A working with Problem 1

Part	Statement	Action	Reasoning Code
38	Liam: So, if we then think that twenty-seven… That the whole triangle is that side plus that side. These are the two sides. It will be twenty-seven. And then we all have… and then we have… that side is sixteen ... and that side is sixteen. Then we have the bottom side of the twenty-seven here. And then we have the right side in the sixteen square ... eh, the triangle there.	Liam: Starts to draw a new triangle, using the sides that are not a base in the upper triangle with perimeter of 27. Points to the same sides in the old figure. Continues to draw the new triangle by using the sides that are not a base in the triangle with perimeter of 16. Extends the new triangle's base to the right corner.	DR

		Draws the remaining lower part of the right side in the new triangle.	
39	Tom: Um. Because each has one point to this, that is, to the perimeter of the large triangle. And then one that points to this triangle.	Tom: Points to the large outer triangle in the figure on the worksheet. Points to the triangle in the middle of the figure on the worksheet.	DR

In the episode above, by saying "That the whole triangle is that side plus that side. These are the two sides.", "then we have… that side is sixteen ... and that side is sixteen. Then we have the bottom side of the twenty-seven here" and by highlighting "Because everyone has that point to this, that is, to the perimeter of the large triangle" Liam seems to understand the mathematics in Tom's shortened reasoning (SR). Consequently, he offers an explanatory, mathematically well-grounded displayed reasoning (DR) for the group. Importantly, while talking, Liam demonstrates his DR by drawing the large triangle step by step on the whiteboard.

3.5 The reliability and validity of the study

The validity and reliability of the study was tested according to the qualitative paradigm (e.g., Golafshani, 2003). When discussing validity and reliability in the given context, it is indicated that "the most important test of any qualitative study is its quality" (Golafshani, 2003, p. 601). Also, it is recommended that reliability and validity in qualitative research should be addressed in terms of trustworthiness, rigor, and quality (e.g., Golafshani, 2003).

Accordingly, we tested the validity of the present study in terms of the reproducibility and the rigor of observations and obtained results (Golafshani, 2003; Stenbacka, 2001). Therefore, we have documented and are displaying every phase of our study – i.e., design, selection of analytical frameworks, intervention, and analysis of empirical data – accurately.

The reliability of the study was addressed in collaboration between all authors of this paper. In the first phase, the anonymized empirical data was analyzed independently. When comparing the findings of respective authors, it turned out that around 90% of analyzed episodes emerged in congruent conclusions. Divergent conclusions were discussed further, in order to find plausible, common ways of interpretation. Accordingly, the relatively high proportion of similar conclusions in the first phase, and the subsequent scrutinization of divergent conclusions, indicates that the analysis holds an acceptable level of reliability.

4 RESULTS

4.1 Collaboration during problem-solving with respect to the JPS

The analysis shows that each group of students created a shared and socially negotiated JPS (Roschelle & Teasley, 1995). Moreover, all phases of JPS that were included in our analytical framework – i.e., understanding the problem and initiating the reasoning (I); creating and implementing problem-solving strategies (S); observing and negotiating misconceptions (N) and G reaching the goal by solving the problem (G) – were present in every group's problem solving. However, the displayed time of different JPS-phases and the sequential order between those were different in respective problem-solving activities. Thus, in these aspects, the present study cannot offer well-grounded conclusions.

Nevertheless, every group started their collaboration with a phase of understanding the problem and initiating a solution (I). In the analysis section, we displayed the start of JPS for Group A, next we will display another example of initiating solutions (I) in the JPS:

Table 9: Group D working with Problem 2

Part	Statement	Action	JPS Code
1	Ron: Shall we? Sara: Um, yes.	Ron: Places the paper with the task on the whiteboard. Ron: Reads Problem 2 from the paper.	I
2	Ron: I think … that this is equal with… Because they have the same diameter.	Ron: Draws two semicircles in one large semicircle on the whiteboard.	I

Furthermore, the analysis indicates that every group completed respective JPS by solving proposed problems (G). After displaying the actions of group A in the analysis section, here, we display an additional example of G from group D:

Table 10: Group D working with Problem 2

Part	Statement	Action	JPS Code
101	Ron: So, this is a better calculation. We can write it like this … If we don't take …	Ron: Writes $B_d \cdot \pi/2 + C_d \cdot \pi/2 = A_d \cdot \pi/2$	G

102	Sara: This is the final solution.		G

As mentioned, even though the groups started and finished their problem-solving process in similar ways – with a phase of I and a phase of G, respectively – between those two phases, the groups altered at different extent the phases of implementing problem-solving strategies (S) and negotiating misconceptions (N). In other words, the present study could not identify a common pattern for the variation of phases S and N at observed problem-solving activities.

4.2 The displayed mathematical reasoning of gifted students during collaborative problem-solving

The main goal of the present study was to investigate if there was possible to display gifted students´ mathematical reasoning, despite their tendency to internalize and shorten their mathematical reasoning during problem solving (e.g., Bloom, 1985; Clements, 1984; Krutetskii, 1976; Sheffield, 2002). The analysis shows that every group developed their shortened mathematical reasoning (SR) into displayed reasoning (DR) during collaborative problem-solving at vertical whiteboards.

After displaying this phenomenon emerging at group A in the analysis section, below, we display another example. The following phases occurred during the activity of group C's initial phase (I), while drawing figures corresponding to Problem 2 on the whiteboard.

Table 11: Group C working with Problem 2

Part	Statement	Action	Reasoning Code
1	 David: If you, like, move this one … here. And take away this little, then this… … should be exactly the same large like this.	David: Draws the figure with the semicircles. Points to the tangent point of the two semicircles. Points to the middle-sized semicircle drawn on the whiteboard. Points to the large semicircle.	SR
2	Adam: How? Tom: Can you explain?		SR

3	Adam: Of course, if you take away this. If this edge… … was instead here. Then, this arch… …would be identical with this. Because, the thing is, that in a circle the edge is always the same distance from each other. This segment … … is the same segment as to this. David: Yes.	Adam: Points to the smallest semicircle. Points to the common point of the two inscribed semicircles. Points to the outmost right point on the large semicircle's diameter. Points to the middle-sized semicircle. Points to the large semicircle. Draws a radius on the large semicircle's diameter, from the middle of the diameter. Points to the radius. Shows the diameter.	DR

4.3 The role of vertical whiteboards in the context of gifted students' mathematical reasoning during problem-solving

The analysis indicates that the whiteboard as a common work-surface has a pivotal role in the collaboration within the groups. Moreover, it seems that the vertical whiteboards facilitated considerably the development of SR into DR within respective groups.

As seen in Table 11, David drew figures on the whiteboard and briefly justified his actions, which lead his peers in group C to question David's SR. Then, by using the figures on the whiteboard and by building on David's reasoning, Adam developed David's SR, step by step, into a more intelligible mathematical reasoning (DR) that was understood by the group. Next, David, confirmed that his SR correspond with the DR offered by Adam.

Another example that demonstrates the role of vertical whiteboards in the context, is the below phase from group B, which occurred after drawing the figures associated to Problem 2.

Table 12: Group B working with Problem 1

Part	Statement	Action	Reasoning Code

30	David: You add them. Because there is…	David: Points to the figure drawn on the whiteboard.	SR
	If one understood that this side…	Points to the left side of the lower triangle in the middle.	SR
	… and this side … have the same length.	Points to the right side of the lower triangle in the left.	SR
	It should be, because if this is equilateral …	Points to the upper triangle.	DR
	… then this segment should be straight. Like this … that …	Points to the base of the upper triangle.	DR
	These angles are similar.	Points to two lower angles of the upper triangle and nearby angles of lower triangles to left and to right.	DR
	Then those two should be … then those two are similar.	Points to the base of the upper triangle.	DR

In the episode above, in ways similar to those shown in Table 11, group B have also drawn figures on the whiteboard. In this case, David – who was in group C during the second observation – uses the whiteboard to develop a DR. That is, at every step of his reasoning, he points to corresponding triangles, sides, and angles. In that way, David uses both the figure and his statements from a SR to develop DR.

Similar situations occurred in every group's problem-solving activity. Thus, we would like to highlight three concluding observations:

a) while collaborating at vertical whiteboards, every group developed their SR into DR,
b) when developing SR into DR, every group used the vertical whiteboards actively, by writing calculations, drawing figures, and using mathematical connections presented on the whiteboards, and
c) the vertical whiteboards facilitated the understanding and monitoring of the problem-solving process within respective groups.

5 Discussion and conclusions

Based on difficulties in observing and understanding the mathematical reasoning of the gifted, in combination with the requirement of the Swedish curriculum regarding oral communication during problem-solving, the main goal of the present study was to investigate methods that could display the mathematical reasoning of gifted students. Consequently, we examined gifted students mathematical reasoning during collaborative problem-solving at vertical whiteboards.

The analysis shows that every group was able to create a JPS, with all essential components. That is, every group collaborated actively when solving proposed problems. In that regard, the present study indicates that gifted students are able to collaborate in structured and meaningful ways when solving mathematical problems. This result is also supported by the fact that participants were parts of different groups during the observations. Another result indicates that every group displayed its mathematical reasoning (DR) during collaborative problem-solving. In that respect, even though we only observed a restricted number of participants in the Swedish context, we found it promising that all groups developed SR – a frequently addressed problem in the context of the mathematically gifted – into a more intelligible DR.

Consequently, even though the results of the present study should not be interpreted in a more general perspective or transposed to a context different from the Swedish educational system, it does not seem unreasonable to assume that gifted students´ collaborative problem-solving at vertical whiteboards have the potential to facilitate their interaction during problem solving and to display their mathematical reasoning.

In addition, we would like to discuss the role of the vertical whiteboards in the process. As mentioned, when developing DR, the students stood in front of whiteboards and used the whiteboards actively. They drew figures and wrote calculations on whiteboards when initiating the JPS, and subsequently used the drawings and calculations when developing SR into DR. Thus, it seems that some attributes of the vertical whiteboards, that were highlighted in previous studies (Liljedahl, 2016) had a pivotal effect on students' collaboration. Consequently, despite that our study did not compare the effects of different working surfaces during problem-solving, it seems that standing up and acting physically in front of vertical whiteboards was beneficial in initiating and maintaining a JPS. Also, it seems that drawing and writing on the whiteboard – an easily accessible and rewritable surface – considerably facilitated the verbalization of participants´ thoughts in the interaction with their peers. Thereby, the present study indicates that the recommendations of previous studies on mixed-ability students (Liljedahl, 2016) seem to be relevant and appliable also for the mathematically gifted.

Finally, we would like to stress that there is a need for additional studies which investigate methods aimed to observe gifted students' internalized and shortened mathematical reasoning. In that way, the mathematical skills of the gifted would probably be assessed more objectively in a school context. As for example, it should be of interest to observe mathematically gifted students in earlier school years in circumstances similar to those outlined in the present study.

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