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Abstract: This article describes the development of a coding scheme for analyzing mathematical tasks in Primary Source Projects (PSPs), curriculum materials based on primary historical sources designed for teaching standard topics from today's undergraduate mathematics curriculum. Our scheme attends to social-cultural aspects of mathematical learning while focusing on the student actions expected as they work tasks. We exemplify our scheme with tasks drawn from a diverse set of PSPs and report results from its application to these projects. We conclude with comments on how our work can assist instructors, curriculum developers, and researchers, including those who are interested in other types of curricular materials.

Keywords: task coding; mathematical tasks; Primary Source Projects; primary historical sources; undergraduate mathematics

1 Introduction

As mathematics instructors, we are accustomed to reviewing and selecting curriculum materials for use with our students. We attempt to gauge the alignment of available textbooks with the goals set out for our students by the programs in which we work, as well as our own (perhaps implicit) philosophies of learning. Alternatively, we may turn towards inquiry-oriented course notes as a means to promote more active student engagement than a typical textbook seems to allow. In some cases, we even develop our own materials after a futile search to find what we are looking for. Indeed, many of today's textbooks and Open Education Resources – including the materials described in this article – were born from such efforts.

However your own search for the right set of curriculum materials turns out, chances are that you will have taken a careful look at the student exercises or tasks provided. Educational researchers

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have also focused on the nature and type of tasks that students are asked to complete in mathematics courses (Breen & O'Shea, 2018; Mesa et al., 2012), and have developed various coding schemes that can be used to characterize the opportunities that these tasks provide for student learning (Stein et al., 2000). Despite their differences, these classification schemes tend to focus on some aspect of cognition. This might involve setting out a list of cognitive abilities (e.g., representation, symbolization), or characterizing the level of cognitive demand (e.g., memorization, procedures with connections), or otherwise describing the cognitive complexity of the tasks (e.g., one-step vs multiple steps). Of course, there is nothing intrinsically wrong with this type of approach – until one tries to apply it to curricular materials that lie somewhat outside the norm.

In this article, we offer a task coding scheme that attends to the social-cultural aspects of learning mathematics while focusing on students' mathematical activities or actions. We were led to develop this scheme by the nature of certain curricular materials that we use in our own teaching and research. These student-centered materials, known as Primary Source Projects (PSPs), place students in direct contact with the words and works of the mathematicians responsible for creating and shaping the mathematics taught in today's undergraduate courses through guided reading of original historical sources. Naturally, the specific nature of the non-standard qualities of PSPs has dictated the details of the coding scheme we have developed for the analysis we describe here. This scheme will thus not be directly applicable for the analysis of classroom materials that are not based on the use of primary sources. The narrative we share below of how the scheme was designed to highlight student action is thus meant also to inform others who wish to replicate our work in developing coding schemes for assessing the tasks embedded in other kinds of classroom materials.[4] In particular, while our work has focused on one specific type of student projects, our motivations for and approach to developing our coding scheme can inform the analysis and design of other types of curricular materials that break the mold of a traditional textbook presentation.

We begin by describing PSPs in more detail and our motivation for analyzing the tasks they contain. We then turn to a discussion of the methodology we followed to develop our task coding scheme and provide a narrative account of the three stages of formulation of these codes. We illustrate each of these stages with examples of tasks that represent various codes before sharing the complete

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[4] Of course, those who are interested in teaching with primary sources, either as PSPs or in some other form, will find our coding scheme of value in and of itself.

final version of the scheme (again illustrated with examples) and presenting an analysis of some sample coding results. This provides readers the opportunity to observe the slow crystallization of our coding scheme over the course of its development, from its initial design, through a series of iterations of application to specific PSP materials, to analyses of the information provided by the resulting coding profiles. We close with some remarks on the implications of the findings we report in this article for those who write and teach with PSPs, our plans for expanding our analysis efforts, and the applicability of our work to other types of curricular materials designed for use in the teaching of undergraduate mathematics.

2 Background, Motivation, and Goals

As noted above, our task coding work focuses on student projects (PSPs) that employ a guided reading approach to primary historical sources. At the core of each PSP are excerpts from primary historical sources that are selected for their connection to the emergence or subsequent development of a key concept in today's undergraduate mathematics curriculum. Context for that primary source material is provided via brief biographical information about the source author and historical commentary about the mathematical questions which that author set out to explore. PSPs also include secondary commentary that helps establish motivation for the ideas being presented, offers guidance to readers in interpreting the source material, and connects that material to contemporary standards of terminology and notation. The development of the mathematics within a PSP, however, is largely embedded within a series of tasks that seek to engage students with the ideas explored by the primary source author.[5]

As an illustration of this approach, we describe the PSP "Fermat's method of finding the maximum and minimum" (Monks, 2019, see Figure 1 for an excerpt). Designed for use in a first-semester calculus course,[6] this project focuses on one of the central concepts of calculus, optimization via derivatives, through excerpts taken from a short treatise entitled *Methodus ad disquirendam maximam et minimam* (*Method for the Study of Maxima and Minima)* composed in 1636 by Pierre de Fermat (1601–1665). This particular treatise (de Fermat 1679, pp. 63–73) was one of two works written by Fermat about his method of "adequality" for locating extrema that found their way to

[5] The evolution of this instructional approach is detailed in (Barnett, Lodder & Pengelley, 2014).
[6] Per the PSP author's suggestion (Monks, 2019, p. 11), classroom implementation of the full project can be accomplished via a combination of small-group and whole-class discussion in one 50-minute class period, assuming that the first section of the PSP is assigned as a class preparation assignment.

Marin Mersenne (1588–1648) in the latter part of 1637. Mersenne in turn passed Fermat's treatise along to his other mathematical correspondents, including René Descartes (1596–1650). Although initially dismissive of Fermat's presentation of his method, after reading some of his later works, Descartes eventually declared "I can reply to it in no other way than to say that (your method) is very good and that, if you had explained it in this manner at the outset, I would have not contradicted it at all" (as quoted in (Mahoney, 1994, p. 192)).

The secondary commentary in this particular PSP is fairly sparse, simply reminding students of the definition of the derivative and summarizing present-day procedures for finding and testing critical points, both of which are assumed as prerequisite knowledge for the project. After briefly describing the historical context of Fermat's work, the project presents three of Fermat's own examples of his method. As they read and analyze his instructions for working through these examples in the PSP tasks, students are continually prompted to compare Fermat's method with today's standard difference quotient definition of the derivative. Consequently, they are led to see how the more algebraic framework of Fermat's approach bears a strong family resemblance to the standard textbook approach of taking a derivative and setting it equal to zero. As the PSP author remarks in the project's "Notes to Instructors," the examples featured in these tasks were also "purposefully selected [from Fermat's treatise] for their similarity to the types of textbook optimization problems that are typically assigned in a first-semester calculus course" (Monks, 2019, p. 10). Students thus gain additional practice with today's approach to optimization while seeing the same problems approached by a different method that may serve to "break students out of recipe-thinking with regards to optimization" (Monks, 2019, p. 10).

This "guided reading" strategy for incorporating primary sources into the teaching of standard undergraduate mathematics courses emerged from pioneering work in the classroom use of primary sources that began in the 1990s. Since 2004, the United States National Science Foundation (NSF) has funded a series of three grants to support the development of PSPs based on this approach. The resulting cumulative collection of projects that have emerged from these three initiatives now includes over 120 freely-available PSPs on a range of mathematical topics.[7]

[7] Articles describing the design of specific PSPs include (Barnett, 2014, 2019; Barnett et al., 2011; Flagg, 2019; Lodder, 2014; Otero, 2019; Ruch, 2014). The use of PSPs to teach entire courses – for General Education and Discrete Mathematics, respectively – is described in (Barnett, Lodder & Pengelley, 2016) and (Barnett et al., 2016).

Figure 1: An excerpt from the PSP (Monks, 2019) that includes a primary source excerpt (Fermat, but in an English translation) together with secondary commentary by the PSP author, Monks. "Fermat's Theorem" refers to the statement that $f'(c) = 0$ whenever $f(c)$ is an extremum of the differentiable function f. Sample tasks from this PSP are given in Table 4.

1 Fermat's Method . . . and Descartes' Doubts!

Fermat's Theorem is so-called because it is traceable back to the ideas of Pierre de Fermat (1601–1665). Nonetheless, it is fascinating to consider how different his method looks from the modern method![a] Here we examine his 1636–1642 treatises, collectively titled *Maxima et minima*, found in [de Fermat, 1636–1642].

Let a be the desired unknown, whether it be a length, a plane region or a solid, depending on what the given magnitude equals, and let its maximum or minimum be found in terms of a, involving whatever degree. Replace this first quantity with $a + e$, and the maximum or minimum will be found in terms of a and e, with coefficients of whatever degree. These two representations of the maximum or minimum are adequated, to use Diophantus' term,[b] and the common terms are subtracted. Having done this, all terms from either part (affected by e or its powers) are divided each by e, or by a higher power of the same, until some term of one or the other of the expressions is altogether freed from being affected by e.

All terms involving e or one of its powers are then eliminated and the remaining terms are equated; or, should one of the expressions be left as nothing, then the positive terms are equated with the negatives, which reduces to the same thing. The solution to this last equation will yield the value of a, which will reveal knowledge of the maximum or minimum by referring again to the earlier solutions.

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[a] Part of this difference, of course, has to do with the passage of time and the evolution of how we are expected to write mathematics. However, part of it is also due to Fermat's unique personal style; he had a reputation for coming up with results in secret and then sending the result out into the mathematical community with no indication of how one might have come upon that, almost as a puzzle for the world to solve! Mathematics historian Victor Katz writes "In many cases it is not known what, if any, proofs Fermat constructed nor is there always a systematic account of certain parts of his work. Fermat often tantalized his correspondents with hints of his new methods for solving certain problems. He would sometimes provide outlines of these methods, but his promises to fill in gaps 'when leisure permits' frequently remained unfulfilled" (Katz, 1998, page 433).

[b] Diophantus (c. 200 CE–c. 284 CE) was a mathematician in the city of Alexandria who wrote in Greek. His word παρισότηζ (parisotes), meaning approximately equal, was translated into Latin as *adaequo* by the French mathematician Claude Gaspard Bachet de Méziriac (1581–1638). Fermat read Bachet's version of Diophantus' work (Katz et al., 2013).

The task coding scheme that we describe in this article is based on an analysis of eight PSPs from the larger collection, selected to represent a variety of PSP author styles, content topics, and levels of mathematical sophistication. Table 1 gives the titles, author names, and an overview of the mathematical content treated in each of these eight PSPs. Table 2 gives an overview of the different weights (measured as line counts within the project) given to the three design components (primary source text, tasks for students, and secondary commentary provided by the project author) that are central to our guided reading approach for each.[8] These weights provide a preliminary view of the variation that is possible in projects that adopt this approach, such as the percentage of the PSP that is dedicated to student tasks. Later in this article, after developing our task coding scheme, we also present an analysis of the commonalities and differences in the nature of those tasks.

Table 1: Summary of mathematical content in coded PSPs

PSP	Intended Course	Content Topics
Bolzano on continuity and the Intermediate Value Theorem (Ruch, 2017)	Real Analysis	ε-δ definition of continuity; proof of Intermediate Value Theorem
Cross cultural comparisons: The greatest common divisor (Flagg, 2020)	Mathematics for Elementary Teachers	Chinese rod numerals; algorithms for finding greatest common divisor; computation using physical manipulatives and written numerals
Fermat's method of finding the maximum and minimum (Monks, 2019)	Calculus I	optimization methods; definition of derivative as limit of difference quotient
Otto Hölder's formal christening of the quotient group concept (Barnett, 2018)	Abstract Algebra	cosets; quotient groups; homomorphisms; Fundamental Homomorphism Theorem
Quantifying certainty: The p-value (Klyve, 2017)	Introductory Statistics; Quantitative Reasoning	hypothesis testing; p-values
Seeing and understanding data (Bolch & Wood, 2018)	Introductory Statistics; Quantitative Reasoning; Mathematics courses for K-12 Teachers	graphical representation of data, including bar charts, pie charts, histograms, line charts, boxplots, and stem-and-leaf plots

[8] The coding results we report in this article are based on the versions of the PSPs that were posted on Digital Commons (https://digitalcommons.ursinus.edu/triumphs) at the time that we conducted our analyses; the access dates given in the reference list specify when these analyses were completed. The currently posted version may differ slightly due to revisions made by the author in the interim.

PSP	Intended Course	Content Topics
The closure operation as the foundation of topology (Scoville, 2017)	Topology	closure operation; equivalence of two different axiomatizations for a topology
The Pell Equation in Indian mathematics (Knudsen & Jones, 2017)	Number Theory	cyclic method for solving Pell Equation; linear Diophantine equations in two variables

Table 2: Line counts of PSP design elements in coded PSPs (with percentages relative to PSP total in parentheses)

PSP	Primary Source Excerpts	Student Tasks	Secondary Commentary		Total Line Counts
			Historical Context	*Other*	
Bolzano on continuity and the Intermediate Value Theorem (Ruch, 2017)	87 (36%)	88 (36%)	18 (7%)	52 (21%)	245 (100%)
Cross cultural comparisons: The greatest common divisor (Flagg, 2020)	113 (15%)	246 (33%)	92 (12%)	295 (40%)	746 (100%)
Fermat's method of finding the maximum and minimum (Monks, 2019)	57 (27%)	79 (38%)	4 (2%)	68 (33%)	208 (100%)
Otto Hölder's formal christening of the quotient group concept (Barnett, 2018)	244 (23%)	512 (48%)	47 (4%)	259 (24%)	1062 (~100%)
Quantifying certainty: The p-value (Klyve, 2017)	64 (27%)	51 (21%)	10 (4%)	116 (48%)	241 (100%)
Seeing and understanding data (Bolch & Wood, 2018)[a]	40 (17%)[a]	68 (28%)	15 (6%)	117 (49%)	240 (100%)
The closure operation as the foundation of topology (Scoville, 2017)	37 (40%)	11 (12%)	23 (25%)	21 (23%)	92 (100%)
The Pell Equation in Indian mathematics (Knudsen & Jones, 2017)	40 (13%)	160 (50%)	14 (4%)	105 (33%)	319 (100%)

Table Note:
[a] As most of the primary source excerpts in this PSP are in the form of figures, while our methodology focused on counting text lines, the presence of source excerpts is not fully represented by this percentage.

The impetus to develop our coding scheme emerged from the research component of the latest NSF grant effort (TRIUMPHS, 2023) to promote teaching and learning with PSPs. Specifically, the members of our working group (two of the grant's Principal Investigators and a then-graduate student research assistant) were asked to conduct a purposeful analysis of the pedagogical structure of individual PSPs. The initial intent of this analysis was to identify the opportunities for student reading, writing, and speaking about mathematics that each PSP might afford, with the aim of informing future efforts to investigate the potential effect of such opportunities on student learning and attitudes. We have since realized that our analysis also has the potential to inform authors' attempts to evaluate and refine the learning opportunities in their PSPs. We further envision that sharing the results of our analysis will assist authors and instructors in determining the extent to which a given PSP aligns with their learning goals for students and provide them with evidence that using PSPs in teaching mathematics does not necessitate sacrificing attention to the intended content. For instance, instructors could review a coding profile of a project to see whether its battery of tasks favor certain types of student work (or avoid others) and thereby contribute to serving the instructor's pedagogical goals, while authors designing new PSPs might consider building student tasks to aim towards a certain kind of coding profile.

3 Development of the Task Coding Scheme: Early Stages

Here and in the subsequent two sections, we describe our methodology for crafting code descriptions that captured characteristics of student tasks that were at the same time present in the PSPs and meaningful to us as indicators of student thinking. While doing this, we share sample tasks to exemplify how our task coding has changed throughout this process.

In its first meetings (early 2018), our working group considered some measures that might be useful for pointing out strengths and weaknesses in the design of PSPs. These included both quantitative data (number of pages; number of tasks; number of class periods in classroom implementation; proportion of PSP devoted to source texts, student tasks, and secondary commentary) and qualitative data (categorization of tasks, degree to which students are drawn into the source texts, style of integration of source texts with tasks and project narrative). After informally examining a small selection of PSPs from these various perspectives, we narrowed our focus to one of their chief features, namely, that the efficacy of a PSP for engaging and guiding student thinking rests on the ability of the project tasks to draw students into the mathematical discourse of the primary source author. Investigations of the literature (e.g., Breen & O'Shea, 2018; Glasnovic Gracin, 2018;

Mesa et al., 2012; Mkhatshwa & Doerr, 2016; Stein et al., 2000) made it apparent, however, that there was no existing framework that captured this aspect of student tasks.

We thus began development of our own coding scheme for PSP tasks by drawing on informal discussions about what we perceived as key characteristics of the kinds of activities that were expected of students who completed these tasks. This resulted in the creation of a preliminary coding scheme that included the nine codes displayed in Table 3.

Table 3: Preliminary version of task coding scheme

Code Number	Code Name and Description
I.	**Read and Interpret the Source** (What is the author trying to say?) These tasks help students derive meaning from the source text.
II.	**Read and React to the Source** (Comment on what you have read.) Such tasks generate thinking about the source, but on the student's terms.
III.	**Read and Respond to (or Reflect on) the Source** Uses the source to explore ideas.
IV.	**Reformulate in Modern Terms** (Rewrite what the author has done in symbolic form.) These tasks ask readers to make connections between ideas in source and modern approaches.
V.	**Check Details from the Source** (Verify some claim the source makes.) These tasks can be akin to standard exercises, but are helpful to students in navigating the source.
VI.	**"Standard" Exercises** Such tasks would be familiar from standard textbook presentations.
VII.	**Connections to Prerequisite Facts or Procedures** These tasks ask students to perform a computation or provide an explanation that connects the PSP content to their understanding of prerequisite material for the intended course.
VIII.	**Preparatory Explorations** Such tasks don't refer to the source, but get students thinking about concepts that are explicitly in the source.
IX.	**Tasks to Guide Students Toward Certain Thinking** These are "stage setting" exercises that prepare students to make sense of source material which they will read, but without necessarily having an explicit connection to what the source author has written.

As an initial illustration of the range of tasks that we wished to characterize with our coding scheme, as well as the diverse nature of the primary source material associated with these tasks, we here describe examples for two of these codes. Our first example is taken from (Knudsen & Jones, 2017), in which the primary source material is drawn from medieval Indian works that use somewhat dense and cryptic language. For instance, Figure 2 presents an extract from *Bijaganita*, a 12th-century work by the mathematician-priest Bhāskara II (1114–1185), that describes a computational procedure for solving a kind of equation known as the Pell equation. Immediately after displaying this extract in the PSP, the authors present the task shown below it in the figure. This, according to Table 3, is a Code II task, meant to draw out the reader's first impressions, without requiring those impressions to be directed toward any particular interpretation of the primary source text. In other words, it asks students to respond to the source material "on their own terms," in contrast to trying to place the student in the mindset of the historical author.[9]

Figure 2: An example of a task assigned Code II in preliminary version of coding scheme with associated primary source text, taken from (Knudsen & Jones, 2017, Task 29, p. 8).

Considering the lesser root, greater root and interpolator [of a square-nature] as the dividend, addend, and divisor [in a pulverizer], the [indeterminate] multiplier of [the pulverizer] should be so taken as will make the residue of the prakṛti diminished by the square of that multiplier or the latter minus the prakṛti [as the case may be] the least. That residue divided by the [original] interpolator is the interpolator [of a new auxiliary equation]; it should be reversed in sign in case of subtraction from the prakṛti. The quotient corresponding to that value of the multiplier is the [new] lesser root; thence the greater root. The same process should be followed repeatedly putting aside [each time] the previous roots and the interpolator. This process is called cakravāla [or 'The Cyclic Method']. By this method, there will appear two integral roots corresponding to an equation with ± 1, ± 2, or ± 4 as interpolator.

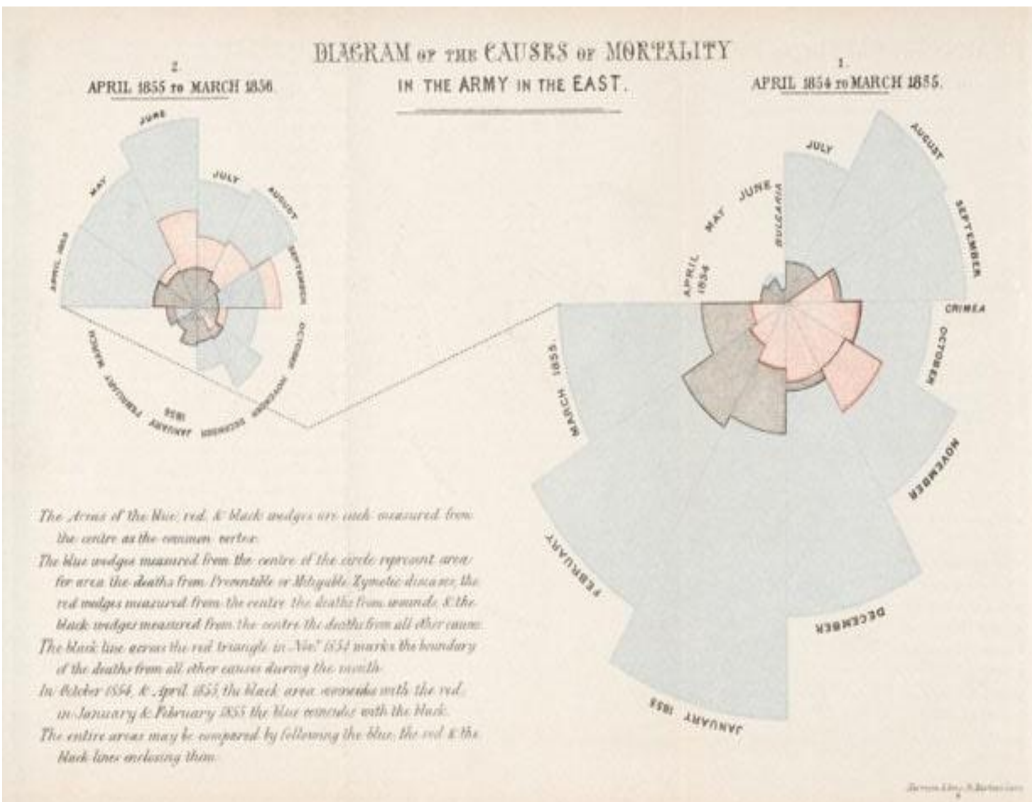
Task 29 — Write two or three questions or comments about Bhāskara's statement of the Cyclic Method, with at least one question and at least one comment.

Our second example is taken from (Bolch & Wood, 2018), in which the primary source material consists entirely of visual displays. For instance, Figure 3 presents Florence Nightingale's (1820–

[9] Within the final version of our coding scheme (Table 4), Code II (Read and React to the Source) became Code SO-1 (Read and React to the Source) and was placed within the Subcategory SO (Source-dependent tasks that are *open* to student interpretation). We also elaborated the description of this code within the final version of the coding scheme in order to emphasize the role played by such tasks in activating the student's thinking about the source.

1910) famous "coxcombs" graph which summarizes data she collected on mortality during the Crimean War. Immediately after displaying this graph, the project authors present the task shown below it in the figure. This, according to Table 3, is a Code III task, meant to draw the student into "reading and responding to the source," in this case by identifying with Nightingale herself and speculating on how she might have produced the visual display at the center of the discussion.[10]

Figure 3: An example of a task assigned Code III in preliminary version of coding scheme with associated primary source material, taken from (Bolch & Wood, 2018, Task 9(b), p. 10).



Task 9(b)

What data did Nightingale need in order to calculate the areas to draw and shade? She did not have a computer to calculate or draw this illustration for her. Speculate on the tools she needed in order to draw these illustrations.

[10] Within the final version of our coding scheme (Table 4), Code III (Read and Respond to (or Reflect on) the Source) became Code SO-2 (Read and Respond to, Reflect on, or Evaluate the Source) and was placed within the Subcategory SO (Source-dependent tasks that are *open* to student interpretation). We also modified the description of this code slightly.

Naturally, the PSPs associated with these two examples include tasks that represent other codes from Table 3. We present summaries of the task distribution for both, as well as the other PSPs that we coded for this article, in the analysis of our coding scheme results that appears in the Section 7 of this article.

4 Development of the Task Coding Scheme: Next Steps

Over the course of some months, we applied the initial version of our coding scheme (Table 3) to three PSPs (Barnett, 2018; Bolch & Wood, 2018; Knudsen & Jones; 2017), assigning codes independently, then meeting to discuss our choices for each individual task. In cases where discrepancies arose in our independent code assignments, subsequent discussions revealed to us much about how well (or poorly) the codes fit the wide variety of styles of tasks designed by PSP authors, and we were able to more clearly define the code descriptions as we negotiated their meaning.

We also continued to modify the codes themselves, addressing gaps in the fit between the kinds of tasks we were coding and the current list of codes. For instance, due to the prevalence of tasks in which project authors prompted students to reflect on primary source material by way of some type of comparison, we added a new Code B:

> **B. Read, Compare and Contrast** (Compare and contrast what you read to … .)
>
> Task asks students to compare some piece of source text with another source, a reformulation of the content, or some other part of the source.

As illustrated by the example in Figure 4, the ideas that students are asked to compare in such tasks could involve primary source material written by two different historical authors (here, Otto Hölder (1859–1937) and Camille Jordan (1838–1922)) which describes essentially the same mathematical object (i.e., the quotient of a group G by a subgroup H). We comment on the further evolution of Code B within later versions of our coding scheme in the next section.

Figure 4: An example of a Code B (Compare and Contrast) task, taken from (Barnett, 2018, Task III.7, p. 44).

Task III.7	Describe carefully the type of objects in the set that Jordan denoted $\frac{G}{H}$. Notice that these are not cosets, as was the case for the objects that Hölder included in the quotient group G/H. How are the elements of Jordan's group $\frac{G}{H}$ related to those of Hölder's quotient group G/H?

Other modifications to our preliminary coding scheme included abandoning Code VIII (Preparatory Explorations) from Table 3 as we came to realize that the few tasks that we had assigned to this code could be captured with other existing codes. Since we had already realized the need for a reorganization of our coding scheme, we could easily capture the intention behind Code VIII within the reorganized coding scheme.

Simultaneously with these efforts, we began to compile (and continually update) a document of exemplars that listed a handful of sample tasks for each code, both as a reminder to us of the key features that each code was meant to capture and as an aid to help us better communicate these features to others. The current version of this exemplar document is provided in Appendix A.

As our list of codes began to stabilize, we turned our attention to the classification of codes into related categories in order to explore the inherent structure of our coding scheme. In our preliminary scheme, we had roughly ordered the codes into two main groups: those that identified tasks that made explicit reference to the source text (Codes I–V) and those that did not (Codes VI–IX). We thus introduced a more explicit labeling of these two broad categories, with a further refinement into subcategories – some of which we later discarded – that captured other commonalities:

Category S (Source-dependent): Tasks that require student-reading of the *source*.

- **Subcategory SO:** Source-dependent tasks that are *open* to student interpretation.
- **Subcategory SG:** Source-dependent tasks that *guide* student responses in some way/direction.

Category E (Exercise): Tasks that prompt students to *exercise* their mathematical understanding or skills in some way.

- **Subcategory ES**: Exercises that are motivated by the *source* text.
- **Subcategory EC**: Exercises that connect more directly with the curriculum of *current courses*.

In the description of Category E, we used the word "exercise" in the sense of something that is performed as a means of practice or training. Many of the tasks that fall into this category have the appearance of exercises drawn from standard textbooks, where the nature of what constitutes an exercise naturally depends on the course for which the PSP is intended. For instance, the example given in Figure 5 presents a typical proof exercise – showing the equivalence of two sets of axioms for defining a topology – that employs basic concepts and techniques for students taking an introductory topology course.

Figure 5: An example of a Code EC-2 (Connections to Current Curriculum) task,[11] taken from (Scoville, 2017, Task 1, p. 3).

Task 1

Prove that Hausdorff's Closed Sum and Intersection Axioms (CSIA) are equivalent to the Open Sum and Intersection Axioms (OSIA) that we use today. [Hint: Use De Morgan's laws.] When starting with CSIA, "closed" is an undefined term and $A \subseteq X$ is open if $X - A$ is closed. Similarly, when starting with OSIA, "open" is undefined and $A \subseteq X$ is closed if $X - A$ is open.?

While Category E (Exercise) tasks in PSPs written for lower division courses often have a more algorithmic flavor than the task shown in Figure 5, we also came to realize that not all algorithmic tasks should be coded as exercises. Importantly, this realization was a direct result of our efforts to construct categories that capture the essence of our materials. For instance, within the version of our coding scheme that we were applying at the time, the task shown in Figure 6 below was initially assigned a code that we were then calling "Practice Exercise: Mimicking the Source."

Figure 6: An example of a task initially assigned Code ES-2 (Practice Exercise: Mimicking the Source), taken from (Flagg, 2020, Task 12, p. 8).

Task 12

Problem 7 in the *Suan Shu Shu* used the fraction $\frac{162}{2016}$ to illustrate the rules for simplifying fractions. Find gcd(162, 2016) using the mutual subtraction algorithm. What arithmetic operation can we use to complete the repeated subtraction steps more efficiently?

[11] The EC-2 code number and name given here comes from the final version of our coding scheme, as this was one of the last PSPs that we coded for the purpose of this article.

In this task, students are prompted to complete a problem, drawn from a Chinese source (*Suan Shu Shu)*, that is algorithmic in nature after performing a series of tasks based on a "Method for Simplifying Fraction" that was presented in that same primary source. While its procedural aspect led us to initially place this task under what we were then calling the ES subcategory (Exercises motivated by the *source* text), the centrality of the source material within the task made us dissatisfied with this categorization. At the same time, we noted that this task (and others like it) encouraged students to move beyond a relatively strict use of the procedures presented in the primary sources (e.g., by reflecting on how to complete the repeated subtraction steps more efficiently), an action which the term "mimicking" failed to capture. Eventually, we changed both the code name and the category to which it belonged.[12]

Additional considerations of this kind over the following months led to a period of experimentation with the subcategories, during which we labored to identify the most important and useful features of the tasks we were encountering in these PSPs. Versions of these various subcategories that we considered – some of which we again later abandoned – included:

Subcategory SI: Source-dependent tasks that assist the student to *interpret* the source.

Subcategory ST: Source-dependent tasks that invite students to *transition* from open-ended explorations of the source to more determinable responses.

Subcategory SC: Source-dependent tasks that support and confirm statements of interpretation of the text or make *connections* with modern paradigms that explain its content.

Subcategory EP: Exercises that *prepare* the student's cognitive environment for engaging with the source text.

What ultimately guided our decision about which of these categories were useful enough to carry forward was a consideration of the nature of the curriculum materials, particularly their foundation in historical texts. One of the more important aspects of the work called for by students in their engagement with PSP tasks is their struggle to make sense of the language of the historical source author and to connect it to the contemporary mathematical discourse. It is in this struggle that the necessary connections to support student learning take place. Consequently, we redefined the EP

[12] Within the final version of our coding scheme (Table 4), the task in Figure 6 is assigned Code SI-1 (Carry out procedure), which belongs to the Subcategory SI (Source-dependent tasks that assist the student to *interpret* the source).

subcategory as shown in Table 4 below, and the category S (Source-dependent) codes were ultimately reorganized into three subcategories:

Subcategory SO: Source-dependent tasks for which student responses are *open* to interpretation by the student (and thus may vary).

Subcategory SI: Source-dependent tasks that ask the student to work with the source on its own terms, but in a fashion that guides them to a particular *interpretation* of the source.

Subcategory SD: Source-dependent tasks that prompt students to actively juxtapose or translate between two or more *mathematical discourses*.

With regard to this last subcategory, our use of the phrase "mathematical discourse" draws on the work of Sfard (2008), which views mathematics as a historically-situated way of communicating about mathematical objects that is practiced by an individual or group of individuals and an be characterized in part by a set of metarules that is recognized by those practitioners (perhaps only implicitly) as the appropriate way to carry out that communication. These metarules govern actions of discursants, such as determining what counts as a legitimate proof or definition, or as an appropriate topic for research. Tasks that fall under the SD code subcategory may then draw attention to differences between the metarules that govern the historical discourse of a primary source author and those that govern the discourse of current-day mathematics, For instance, the example shown in Figure 7 is one of a series of similar tasks from (Ruch, 2017) in which students are tasked with reformulating Bernard Bolzano's (1781–1848) proof of the Intermediate Value Theorem in keeping with modern standards in analysis, where those standards differ in certain ways (e.g., the explicit statement of lemmas) from those of Bolzano's discourse.

Figure 7: An example of task coded SD-2 (Reformulate in Modern Terms), taken from (Ruch, 2017, Task 23, p. 8).

Task 23	Rewrite Bolzano's claim that "the result $f(\alpha + i - \omega) > \varphi(\alpha + i - \omega)$ follows from the assumption $f(\alpha + i) > \varphi(\alpha + i)$, as long as ω is taken to be small enough" using modern terminology and call this Lemma 2.

Within the SD code subcategory, we also placed tasks that we coded as "Compare and Contrast," such as the example given in Figure 4 above in which students are asked to compare two different discourses (e.g., the discourses of Hölder and Jordan on the quotient group concept). Within the final version of our coding scheme (Table 4), the code name for these "Compare and Contrast" tasks was thus changed from "Code B" to "Code SD-1" and the code description was modified in order to emphasize the discourse aspect of these tasks.

5 Development of the Task Coding Scheme: Final Refinements

As we worked to create categories that captured the inherent structure of the codes, we engaged in further rounds of analysis that involved coding the remaining five PSPs in Table 1. Then, in an effort to test the robustness of the coding scheme and its potential benefits for informing task design for future PSP authors, we invited the authors of some of these to apply the coding scheme to their own projects (Flagg, 2020; Klyve, 2017; Monks, 2019; Ruch, 2017). After sharing with them the profile of codes that we had generated for these same PSPs, we held individual discussions with each of the four authors to review these applications of the coding scheme. This allowed us to further triangulate our methodology by gauging the reactions of the individual authors to these coding profiles, thereby checking the consistency of our coding scheme and allowing us to more fully capture the complexity of the PSP tasks that they had designed. Indeed, our discussions with project authors Klyve and Monks led us to create a new code for a type of student task encountered in their PSPs:

> **SI-4: Make Sense of the Socio-historical Context of the Source**
> Task prompts students to interpret or explain some aspect of the general socio-historical context in which the source text was written as a means to deepen their understanding of what the source author is communicating to the reader.

Figure 8 displays an example of a task to which we applied this new code, in which students are asked to consider data that was collected from church baptismal records by the physician John Arbuthnot (1637–1735) from a socio-cultural (rather than mathematical) perspective.

Figure 8: An example of a task coded SI-4, taken from (Klyve, 2016, Task 4, p. 2).

Task 4	How similar do you think the baptismal records that Arbuthnot collected are to the actual birth numbers? What might cause these to be different.

Our discussion with Ruch further suggested that we should redefine the code that was originally numbered I in our preliminary coding scheme (Table 3) and eventually became SI-3 (Make Sense of the Mathematics in the Source) in the final version of that scheme (Table 4). While the types of actions that students may be asked to undertake in order to make sense of the mathematics in a source vary considerably, Figure 9 displays a particularly interesting instance. Based on an example that Bolzano gave in connection with his proof of the Intermediate Value Theorem – an example which, perplexingly, fails to satisfy the hypotheses of that theorem – this task was inspired by the first set of students who used the project and "were rather critical of Bolzano's footnote function" (Ruch, 2017, p. 11).

Figure 9: An example of a task coded SI-3 (Make Sense of the Mathematics in the Source) with associated primary source material, taken from (Ruch, 2017, Task 5, p. 3). In this task, the "Preface proposition that Bolzano was discussing" is the Intermediate Value Theorem.

∗ *Bolzano's footnote*: There are functions that are continuously variable for all values of their argument, e.g., αx + βx. But there are also others that vary according to the laws of continuity only within or beyond certain limiting values of their argument. For instance, $x + \sqrt{(1 - x)\,(2 - x)}$ varies continuously only for all values x that are < +1 or > +2, and not for those values that lie between +1 and +2.

Task 5

Consider the function Bolzano discussed in his footnote.

(a) Sketch a graph of this function on the interval $[0, 3]$.

(b) Based on the Preface proposition that Bolzano was discussing, why is this an interesting example?

(c) How could you adjust the function to make it better fit the issues surrounding the Preface proposition?

Following our discussion with Flagg, we also came to see the need for refinement of the two EC codes in the final version of our coding scheme (Table 4) that correspond to the codes numbered VI and VII in our preliminary scheme (Table 3). Examples of both these codes and all others in the final version of our coding scheme are given in the next section.

6 Task Coding Scheme: Final Version and Additional Sample Tasks

We developed our coding scheme to help us identify mathematical activities or actions expected from learners while working on a task. In Table 4, we share brief descriptions of the final version of our codes and categories; in brackets next to codes from the source-dependent (S) category, we provide action statements that we have found helpful in understanding the essence of what a task assigned to that code asks a student to do. We also provide sample tasks for each code, drawn primarily from the PSP "Fermat's method of finding the maximum and minimum" (Monks, 2019), described earlier in this paper.

For the three codes that do not appear in that PSP, we provide an example from another PSP, "Cross-Cultural Comparisons: The Art of Computing the Greatest Common Divisor" (Flagg, 2020). This PSP is specifically aimed at prospective elementary school teachers and explores the "repeated subtraction" method for finding the greatest common divisor through excerpts from two ancient texts, the Chinese *Nine Chapters* (c. 100 CE) and Euclid's *Elements* (c. 30 0 BCE). Students are led to discover that the Chinese method is equivalent to the Euclidean algorithm, to compare that method to others that are typically taught in K-12 mathematics, and to reflect on issues related to their teaching.

Table 4: Final version of task coding scheme, with examples (*in italics*) in the second column

Source-Dependent Category. Tasks in this category student reading of the source.

Subcategory SO: Source-dependent tasks for which student responses are open to interpretation by the student (and thus may vary).

SO-1 Read and React to the Source (Comment on what you have just read.)	
The task asks students to write a question or comment about the source, but without directing their attention to anything specific. Such tasks activate student thinking about the source, but on the student's terms and unfiltered by any secondary commentary or preliminary task work.	*Read the definitions below and write down at least three questions and three comments you have about them as you read.* (Flagg, 2020 p. 11, Task 21)
SO-2 Read and Respond to, Reflect on, or Evaluate the Source (Comment on whether you find what you have read convincing/correct/etc.)	
The task asks students to evaluate the text and its claims in light of their engagement with the PSP. An *open-ended* task that asks the student to offer their own evaluation of specific assertions made by the source author.	*Do you agree with Liu Hui when he said that all of the remainders of the subtraction algorithm are simply 'overlaps of the gcd'? Explain.* (Flagg. 2020, p. 9, Task 16d)

Subcategory SI: Source-dependent tasks that ask the student to work with the source on its own terms, but is intended to guide them to a particular interpretation of the source.

SI-1 Carry out a Procedure or Calculation found in the Source (Do what the source author has done, but with this other data.)	
The task asks students to perform some mathematical computation or procedure that is explicitly included in the source (either in general terms or by way of a specific example), but using different input information than the source (if an example is provided).	*Use Fermat's method to find the maximum of the quantity $a + \sqrt{a - a^2}$. That is, set up the adequality*[a] *between $a + \sqrt{a - a^2}$ and the same expression with $a + e$ substituted for a. Then continue to follow the steps in Fermat's method!* (Monks, 2019, p. 8, Task 8a)
SI-2 Provide Details or Verify a Result within the Source (Verify a claim made in the source.)	
The task directs the students to verify computations or deductions that are referenced by the source author in the excerpt. The computation / deduction may or may not be re-stated in the task itself, so that reference back to the source excerpt may be needed in order to proceed with the task.	*Check Fermat's work in the example above,*[b] *filling in the details of the algebra that he glossed over. Can you confirm each of his steps?* (Monks, 2019, p. 6, Task 5a)
SI-3 Make Sense of the Mathematics in the Source (Explain what the author means by ….)	
The task prompts students to interpret or explain some specific mathematical aspect of the source in a fashion that focuses their attention on making sense of what the source author is saying in the text. (Applying this code to tasks that are preceded by a secondary commentary which seeks to provide an already-formulated interpretation of the source may not be appropriate.)	*Let us observe Fermat's results regarding "all solids" by actually looking at a few solids! First, notice that when he said "all solids," he was not talking about solids like balls, tetrahedra, etc. What kinds of solids was he restricting his attention to? How can you tell?* (Monks, 2019, p. 7, Task 6a)
SI-4 Make Sense of the Socio-historical Context of the Source (Reflect on the extra-mathematical context of the author's writing.)	
The task prompts students to interpret or explain some aspect of the general context in which the source was written as a means to deepen their understanding of what the source author is saying in the text.	*Why does it make sense that Fermat's method would have had to rely more on algebra and less on analysis than the modern method? (For a hint, consider the year in which he was working! Do a bit of research and see if you can find who came up with our modern definitions of limits and derivatives, and when that happened!)* (Monks, 2019, p. 8, Task 9b)

Subcategory SD: Source-dependent tasks that prompt students to actively juxtapose or translate between two or more mathematical discourses.

SD-1 Compare and Contrast (Compare and contrast the author's ideas to ….)

The task asks students to compare some part of the source text with another source, a reformulation of the content, or some other part of the source.	*Compare and contrast the modern method with Fermat's method. Can you find three similarities between them? Can you find three differences between them?* (Monks, 2019, p. 3, Task 2)

SD-2 Reformulate or Translate the Source into Modern Terms (Rewrite the source text using ....)

The task asks students to rewrite what the author has done in the source using modern terminology / notation / concepts, or in keeping with standards for logic / rigor / writing conventions that are currently in place, but differ from those that were in place at the time of the source author's work.	*First, we solve the same problem using the modern method. Denote by b the fixed total length of AC (just as Fermat did). Then denote by x the length of AE, which implies $b - x$ is the length of EC. With the above notation, what is the function $f(x)$ that we are trying to maximize? What interval of x values are we considering?* (Monks, 2019, p. 4, Task 3a)

Exercise Category. Tasks in this category prompt students to exercise their mathematical understanding or skills in some way that does not directly refer to the source.

Subcategory EP: Exercises that prepare the student for engaging with the source text.

EP Exercise – Getting Ready for the Source

The task asks the student to give an explanation or make a calculation that establishes a context or prepares the way for some concept explored within the source.	*Briefly explain how Fermat's Theorem serves as the basis for the optimization algorithm described above.*[c] (Monks, 2019, p. 1, Task 1)

Subcategory EC: Exercises that explore ideas contained in the PSP but connect more directly with the current course curriculum or the curriculum of a prerequisite course.

EC-1 Exercise – Connections to Prerequisite Facts or Procedures

The task asks students to give an explanation or perform a computation that draws on some prerequisite material for the intended course.	*Explain what simplifying fractions means in your own words.* (Flagg, 2020, p. 1, Task 1)

EC-2 Exercise – Connections to Current Curriculum

The task focuses on mathematical content from the curriculum of the intended course; completion of the task does not rely on or elucidate the primary source material in any direct way, even though the PSP contains ideas that are needed to complete it.	*Verify that Fermat's result matches what is produced by the modern method. Specifically, maximize the function $f(x) = (b - x)x^2$ on the interval $(0, b)$.* (Monks, 2019, p. 6, Task 5b)

Table Notes:

[a] The term "adequality" refers to an approximate equality that holds between two expressions; in this case, between $f(a)$ and $f(a + e)$.

[b] In the example referenced in Task 5a of (Monks, 2019), Fermat solved the geometric problem of maximizing the volume of a certain solid via his method of adequality; this example is equivalent to finding the maximum of the function defined by $f(x) = (b - x)x^2$.

[c] "Fermat's Theorem" refers to the statement that $f'(c) = 0$ whenever $f(c)$ is an extremum of the differentiable function f. This fact is found in every standard calculus textbook and is repeated in the introduction of the PSP as a means to set the project's mathematical stage. This is the first task in the project and appears prior to any primary source material. Although it draws on students' prerequisite knowledge of today's optimization procedure, its main purpose in the PSP is to prepare students to fruitfully compare that procedure to the one they will encounter in the excerpts from Fermat's treatise.

7 Task Coding Scheme: Analysis

In the process of developing our task coding scheme, we coded several PSPs as described in earlier sections of this article. As the coding structure evolved over time, we updated coding profiles for the tasks in the eight PSPs that were coded during this process using the final version of the task coding scheme. Results are presented below in Table 5.

Table 5: Frequency of individual codes across 8 coded PSPs. PSPs are listed here in the same order as they appear in Tables 1 and 2.

PSP	**SO-1**	**SO-2**	**SI-1**	**SI-2**	**SI-3**	**SI-4**	**SD-1**	**SD-2**	**EP**	**EC-1**	**EC-2**	**Total**
Ruch (2017)		1		1	8			9		1	12	32
Flagg (2020)	2	3	15		26		4	1	1	3	20	75
Monks (2019)			1	6	2	1	4	1	3		2	20
Barnett (2018)		3		11	11		2	6	6	1	37	77
Klyve (2017)		5			7	1	1	3	5		2	24
Bolch & Wood (2018)		2	1		7		3				4	17
Scoville (2017)				3	1						2	6
Knudsen & Jones (2017)	1	1	1		14			3	5		23	48
Total	2	15	19	21	76	2	14	23	20	5	102	299

Presenting results for multiple PSPs in a table such as this provides fruitful information about the affordances that PSPs offer in terms of activities that promote students' mathematical learning. To illustrate this point, we note that EC-2 (Connections to Current Curriculum) is one of two codes (along with SI-3, Make Sense of the Mathematics in the Source) that appear in all the coded PSPs. The presence of this code in each of these projects indicates that none of these PSPs fails to connect to the modern curriculum in an explicit way. Indeed, given that almost a third of the coded tasks in our sample were coded as EC-2, we infer that PSPs devote a significant amount of attention to directly engaging students with the current curriculum. Similarly, the presence of the SI-3 code (Make Sense of the Mathematics in the Source) in all the coded PSPs indicates the authors of these PSPs paved a way toward making sense of the mathematics in the sources in the context of the current curriculum by engaging students in active readings of primary sources. Yet another key feature of PSPs revealed by this table relates to the relatively frequent presence of source-dependent "open to interpretation" SO-category tasks (primarily SO-2, Read and Respond to, Reflect on, or Evaluate the Source) in the coded PSPs, where the prompts for these tasks offer instructors ready-made opportunities for more student-centered teaching practices.

In addition to the tabular format of Table 5, we developed "heat maps," as displayed in Appendix B, as an alternative mode of presenting task coding results for specific PSPs. In these maps, the codes for all tasks are listed in the order in which the tasks appear in the PSP, with the seven source dependent codes illustrated via (seven) different shades of blue and the three exercise codes illustrated via (three) different shades of orange. This mode of presentation visually reveals the sequencing of tasks that utilize primary sources (blue) and those that form bridges with current curriculum (orange) in a particular PSP. For instance, a pattern we have noticed in certain PSPs is a tendency to use source-based codes (blue) at the beginning of the project, illustrating how students are asked to first think about the mathematical content of the source, and more exercise-based codes (orange) near the end, as explicit connections are made to the modern curriculum. In the heat map for (Monks, 2019) shown in Figure 10, however, we observe the reverse. Partly, this is because some of the early codes are tasks that prepare students to engage with the source by having them recall items familiar to them from their first calculus course by way of exercises of a standard type. Additionally, the source-based codes near the end – in particular the SD ("juxtaposition of discourses") and SI-2 (Provide Details or Verify a Result within the Sources) codes – identify tasks in which the project author prompted students to either compare the modern discourse to that of Fermat in the source text, or to verify results in Fermat's examples using today's optimization procedure.

Figure 10: Heat map for (Monks, 2019) using codes defined in Table 5.

EP	Exercise - Getting Ready for the Source
SD-1	Compare and Contrast
SD-2	Reformulate or Translate the Source into Modern Terms
EC-2	Exercise - Connections to Current Curriculum
SI-2	Provide Details or Verify a Result within the Source
EP	Exercise - Getting Ready for the Source
EP	Exercise - Getting Ready for the Source
SD-1	Compare and Contrast
SI-2	Provide Details or Verify a Result within the Source
EC-2	Exercise - Connections to Current Curriculum
SD-1	Compare and Contrast
SI-3	Make Sense of the Mathematics in the Source
SI-3	Make Sense of the Mathematics in the Source
SI-2	Provide Details or Verify a Result within the Source
SI-2	Provide Details or Verify a Result within the Source
SI-2	Provide Details or Verify a Result within the Source
SI-1	Carry out a Procedure or Calculation in the Source
SI-2	Provide Details or Verify a Result within the Source
SD-1	Compare and Contrast
SI-4	Make Sense of the Socio-historical Context of the Source

8 Future Applications of and Plans for our Coding Scheme

Our task coding scheme, along with the task coding results that we have shared, reveal meaningful properties of PSPs for instructors and their classroom implementations. Indeed, this article has provided empirical evidence of an explicit connection to the current mathematics curriculum in all the coded PSPs, addressing a potential concern instructors may have about how the introduction of historical context in PSPs might detract from their intended course goals. Information such as that provided in Table 5 and the heat maps in Appendix B can also equip instructors with data for making a more informed decision about which particular PSPs they may want to use in their teaching and about how to implement a given PSP in their classroom.

The development of our coding scheme has further made us more aware of the variety and nature of the mathematical actions that students are expected to complete as they engage with a PSP. The scheme identifies specific mathematical practices resulting from engagement with tasks that require the reading of primary historical sources. The range of these practices not only indicates a rich mathematical learning experience for students, but also provides opportunities for instructors to create a more student-centered learning environment that focuses on meaningful inquiries into the mathematical ideas of the course.

This information is clearly useful to PSP authors, both current and future. Our coding scheme highlights a wide variety of modes in which students can be engaged in mathematical thinking and writing while analyzing a primary source, from open-ended reactions to a first reading of a text, to the mimicking of a computation found therein, to a request for recasting the original form of some mathematical statement into more modern terms. This diversity of forms provides authors with a palette for designing suitably scaffolded tasks that allow students to negotiate the mathematical ideas embedded in the source text and the accompanying secondary commentary. Furthermore, by sharing our findings with PSP authors (and instructors), the choices made for what sorts of student tasks they include in their materials (or implement in their classrooms) now become more explicit. This can in turn lead to an increase in the diversity of the kinds of experiences made available to students. For instance, the limited use of tasks coded SI-4 (Make Sense of the Socio-historical Context of the Source) in the PSPs we have analyzed to date points to missed opportunities for bringing this kind of thinking into the classroom.

Even though our coding scheme has taken a relatively stable form, as evidenced by our ability to apply it reliably to a number of PSPs by different authors investigating mathematical topics at different levels, we are aware that it cannot capture a description of all possible student tasks that might appear in a PSP. Consequently, our future plans with regard to PSP analysis efforts are two-fold. First, we will extend the application of our task coding scheme to a broader range of projects from the existing PSP collection and modify our coding scheme as appropriate based on those results. We will also continue to examine the cumulative coding results for patterns that allow us to categorize PSPs in a meaningful way. Here, we expect that heat maps of coded PSPs may be especially useful. Additionally, we plan to expand our analysis to include other dimensions of PSPs (e.g., the nature of the secondary commentary) and to involve other theoretical frameworks (e.g., ritual-enabling versus exploration-requiring opportunities-to-learn (Nachlieli & Tabach, 2019)) to allow for more nuanced coding of the essential features of PSPs. These efforts will allow us to also explore the relationships between the aspects of students' mathematical experience revealed by our task coding and aspects of those experiences that other frameworks have highlighted.

9 Concluding Remarks on the Development of Task Coding Schemes

Of course, classroom materials that do not employ primary sources also use a variety of kinds of student tasks aimed at engaging students in some form of mathematical activity. While certain features of our coding scheme and associated findings are specific to the nature of the activities embodied in the foundational role that historical texts play in the design of PSPs, we propose that similar insights into other types of curricular materials could be gained through the development of a task coding scheme specific to the types of student actions found in those materials. The narrative summary of our experience in developing this coding scheme was offered in part as a guide to those who wish to develop their own schemes. For readers interested in adapting the methodology described here to work with other kinds of classroom materials, we offer the following additional comments.

Naturally, the process of developing a workable coding scheme should include testing drafts of the scheme against a reasonable array of actual materials in order to verify the ability of the scheme to accomplish its intended purpose. Experimentation with applying the scheme to multiple examples of these materials is necessary to prove that the scheme accurately highlights the key characteristics of the materials that researchers want to focus on, and is robust enough to do this across the range of materials under consideration. The use of individual coding by team members followed by inter-team discussions is also a standard step in qualitative coding. In our own work, we found it highly valuable to discuss our individually assigned codes for *every single* task in *every single* PSP that we coded during the development process. This hermeneutic of negotiating the definitions of each code based on our individual interpretations helped us not only to improve those definitions, but also to further refine and clarify what the codes were meant to bring to light.

Whereas curriculum analysis efforts are often carried out by individuals who are not part of the team that developed the materials, it was valuable for us that two of the authors of this article are also PSP authors and came to this work with first-hand experience in crafting activities that attempt to guide student thinking about the mathematics found in the source texts that lie at the center of our PSPs. Our discussions with other PSP authors further provided us a deeper understanding of their intended pedagogical goals and, in some cases, suggested new features of these tasks that were useful in modifying the coding scheme. As previously noted, this triangulation of our methodology allowed us to both check the consistency of our coding scheme and to more fully capture the complexity of the PSP tasks that they had designed. A variation of this idea (which we have not yet implemented) is to hold similar discussions with instructors who served as site testers (but not authors) of coded PSPs to give us a better perspective on how coding information might guide classroom practice.

Finally, we propose that the most distinguishing feature of our development process and the resulting coding scheme lies in the analysis of student prompts that ultimately served as our focus of attention; namely, the specific mathematical activities or actions expected from learners while working on a particular task. Naturally, the precise nature of those actions will depend integrally on the defining characteristics of the materials (e.g., use of primary sources, mathematical modeling, inquiry-based discovery). However, by attending to the nature of those student actions – what students are explicitly asked to do (or not do) as a reflection of those defining characteristics – individuals involved in the development of classroom materials may gain insights into the design of classroom materials that lead to their improvement. Similarly, task coding results for those materials can help instructors foresee the kinds of experiences students will have when using these materials. Educational researchers who study the effects of curriculum materials may also benefit from consideration of such coding profiles, regardless of the specific learning theory they use to explore students' mathematical experiences. As a result, all three groups can become better informed about student learning.

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Appendix A: Exemplar Document Task Coding Scheme for Primary Source Projects

The coding structure used for these examples is based on two main categories, each of which is then divided into 1–3 subcategories. The main categories are:

- **Category S**: Tasks that require student reading of the source.
- **Category E**: Tasks that prompt students to exercise their mathematical understanding or skills in some way

The exemplars in this document include tasks from PSPs that have not yet been fully coded. Links to the online versions of all PSPs from which an exemplar has been drawn are included in the References.

Category S: Tasks that require student reading of the source.

Subcategory SO: Source-dependent tasks for which student responses are open to student interpretation (and thus may vary).

- **Code SO-1: Read and React to the Source**
  [Comment on what you have just read.]

  Code Description
  Such tasks activate student thinking about the source, but on the student's terms and unfiltered by any secondary commentary or preliminary task work. The task asks students to write a question or comment about the source, but without directing their attention to anything specific.

  Code Examples

  ✠ Read the definitions [from Euclid's *Elements*] below and write down at least three questions and three comments you have about them as you read.

  Cross Cultural Comparisons: The Greatest Common Divisor [Flagg, 2020, Task 21]

  ✠ After reading Euler's description of the "method for determining the ratio of the vanishing increments that any functions take on when the variable, of which they are functions, is given a vanishing increment":

  (a) What do you think Euler's goal was in this excerpt?
  (b) Write at least one comment and one question that you have about what Euler was doing here.

  The Derivatives of the Sine and Cosine Functions [Klyve, 2017, Task 2]

Category S: Tasks that require student reading of the source.

Subcategory SO: Source-dependent tasks for which student responses are open to student interpretation (and thus may vary).

- **Code SO-2: Read and Respond to, Reflect on, or Evaluate the Source**
  [Comment on whether you find what you have read convincing/correct/etc.]

  Code Description
  The task asks students to evaluate the text and its claims in the light of their engagement with the PSP. An open-ended task that asks the student to offer their own evaluation of specific assertions made by the source author.

  Code Examples

  ✠ The first and the third rule [given for simplifying fractions in a 2nd-century BCE Chinese text] appear to be describing the same procedure. Which do you find most enlightening?

  Greatest Common Divisor: Algorithm and Proof [Flagg 2019, Task 3]

  ✠ Are these reasons [given by statistician Ronald Fisher] strong enough that you believe we should always choose 0.05 as a guide to what is significant? Why or why not?

  Quantifying Certainty: The p-value [Klyve, 2017, Task 18]

  ✠ Does Liu Hui give us good advice [about the importance of thinking before blindly starting to calculate]? Have you experienced problems for which this advice would have helped make solving easier?

  Solving a System of Linear Equations Using Ancient Chinese Methods [Flagg 2017, Task 43]

Category S: Tasks that require student reading of the source.

Subcategory SI: Source-dependent tasks that ask the student to work with the source on its own terms, but in a fashion that guides them to a particular interpretation of the source.

- **Code SI-1: Mimic a Procedure or Calculation in the Source**
  [Do what the source author has done, but with this other data.]

  Code Description
  The task asks students to mimic some mathematical computation or procedure that is explicitly included in the source but using different input information.

  Code Examples:

  ✠ Use Fermat's method to find the maximum of the quantity $a + \sqrt{a - a^2}$. That is, set up the adequality between $a + \sqrt{a - a^2}$ and the same expression with $a + e$ substituted for a. Then continue to follow the steps in Fermat's method!

  Fermat's Method of Maximizing and Minimizing [Monks, 2019, Task 8(a)]

  ✠ After studying a 17th-century graph of the distance from Toledo, Spain to Rome, Italy:

  As a class, choose two cities near your college or university that you would like to estimate the miles between. Then, have everyone in the class write their estimate between the two cities on a piece of paper (rounding to the nearest tenth of a mile) along with their name. Have one or two students write all the estimates on the board as students call out their guesses.
  In groups of three to four students, create a graph similar to van Langren's using the class estimates for the distance from the chosen city A to chosen city B.

  Seeing and Understanding Data [Bolch & Woods, 2018, Task 4(a)]

  ✠ Verify directly that $31 \cdot 72 + 2 = 392$ and $31 \cdot 52 + 9 = 282$. Use the Principle of Composition [given by Brahmagupta] with these auxiliary equations to find two solutions to the equation $31x^2 + 18 = y^2$.

  The Pell Equation in India [Knudsen & Jones, 2017, Task 8]

Category S: Tasks that require student reading of the source.

Subcategory SI: Source-dependent tasks that ask the student to work with the source on its own terms, but in a fashion that guides them to a particular interpretation of the source.

- **Code SI-2: Provide Details or Verify a Result within the Source**
  [Verify a claim made in the source.]

  Code Description

  The task directs the reader to verify computations or straightforward deductions that are referenced by the source author in the excerpt. The computation/deduction may or may not be re-stated in the task itself, so that reference back to the source excerpt may be needed in order to proceed with the task.

  Code Examples

  ✠ Justify Abel's claim in his Theorem III proof that "quantity r will clearly be less than Bc_0".

  Abel and Cauchy on a Rigorous Approach to Infinite Series [Ruch 2017, Task 49]

  ✠ Following his discussion of cubic roots of unity in the preceding excerpt, Lagrange next considered the case $m = 5$, where $\alpha = \cos(\frac{2\pi}{5}) + i\sin(\frac{2\pi}{5}) = e^{\frac{2\pi i}{5}}$ and the five fifth roots of unity are $\alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5 = 1$.

  Complete the following to prove that α^2, α^3, and α^4 are also primitive fifth roots of unity.

  (a) Find the first five powers of α^2, and show that these are the same as the five original roots rearranged in the following order: $\alpha^2, \alpha^4, \alpha, \alpha^3, \alpha^5$.

  (b) Find the first five powers of α^3, and show that these generate the five original roots rearranged in the following order: $\alpha^3, \alpha, \alpha^4, \alpha^2, \alpha^5$.

  The Roots of Early Group Theory in the Works of Lagrange [Barnett 2017, Task 6(a,b)]

Category S: Tasks that require student reading of the source.

Subcategory SI: Source-dependent tasks that ask the student to work with the source on its own terms, but in a fashion that guides them to a particular interpretation of the source.

- **Code SI-3: Make Sense of the Mathematics in the Source**
  [Explain what the author means by . . . .]

  Code Description
  These tasks prompt students to interpret or explain some specific mathematical aspect of the source in a fashion that focuses their attention on making sense of what the source author is saying in the text. (Applying this code to tasks that are preceded by a secondary commentary which seeks to provide an already-formulated interpretation of the source may not be appropriate.)

  Code Examples

  ✠ Find a few values of x less than π which, substituted into $\left[\frac{1}{2} = \cos x - \cos 2x + \cos 3x - \text{ etc.}\right]$, produce strange results and support Abel's contention that the series is divergent.

  Abel and Cauchy on a Rigorous Approach to Infinite Series [Ruch 2017, Task 1]

  ✠ Abel claimed [above] that the series $\left[\frac{x}{2} = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right]$ is valid "for all values of x smaller than π." Do you think he meant x itself or $|x|$?

  Abel and Cauchy on a Rigorous Approach to Infinite Series [Ruch 2017, Task 2]

  ✠ In what dimension are the figurate numbers that Pascal refers to as "numbers of the second order"? Is Pascal's use of the word "order" the same as our use of the word "dimension"?

  Pascal's Triangle and Induction [Lodder, 2017, Task 1.3]

  ✠ The Sign Rule [given in the *Nine Chapters*] explains how to add or subtract positive and negative numbers, yet it makes no explicit mention of the sign of the answer. Why do you think this is? (Hint: Liu's *Commentary* gives us the clue that the rule was meant for rod arithmetic.)

  Solving a System of Linear Equations Using Ancient Chinese Methods [Flagg 2017, Task 3]

Category S: Tasks that require student reading of the source.

Subcategory SI: Source-dependent tasks that ask the student to work with the source on its own terms, but in a fashion that guides them to a particular interpretation of the source.

- **Code SI-4: Make Sense of the Socio-historical Context of the Source**
  [Reflect on the extra-mathematical context of the author's writing.]

  Code Description
  These tasks prompt students to interpret or explain some aspect of the general context in which the source was written as a means to deepen their understanding of what the source author is saying in the text.

  Code Examples

  ✠ Why does it make sense that Fermat's method would have had to rely more on algebra and less on analysis than the modern method? (For a hint, consider the year in which he was working! Do a bit of research and see if you can find who came up with our modern definitions of limits and derivatives, and when that happened!)

  Fermat's Method of Finding the Maxima and Minima [Monks, 2019, Task 9(b)]

  ✠ How similar do you think the [Church of England] baptismal records that Arbuthnot collected [in 1710] are to the actual birth numbers [for England at that time]? What might cause these to be different?

  Quantifying Certainty: The p-value [Klyve, 2017, Task 4]

Category S: Tasks that require student reading of the source.

Subcategory SD: Source-dependent tasks that prompt students to actively juxtapose or translate between two or more mathematical discourses.

- **Code SD-1: Compare and Contrast**
  [Compare and contrast the author's ideas to . . .]

  Code Description
  The task asks students to compare some part of the source text with another source, a reformulation of the content, or some other part of the source.

  Code Examples

  ✠ Explain why Fermat's method and the modern method are essentially equivalent. Where do they differ?

  Fermat's Method of Maximizing and Minimizing [Monks, 2019, Task 9(a)]

  ✠ Describe carefully the type of objects in the set that Jordan denoted $\frac{G}{H}$. Notice that these are not cosets, as was the case for the objects that Hölder included in the quotient group G/H. How are the elements of Jordan's group $\frac{G}{H}$ related to those of Hölder's quotient group G/H?

  Otto Hölder's Formal Christening of the Quotient Group Concept [Barnett, 2018, Task III.7]

  ✠ Compare and contrast this bar graph [published by William Playfair in 1786] with the bar graphs we use today in the newspaper and other media. What do you see that is different between [Playfair's] bar graph compared to a bar graph today? Does how we interpret [Playfair's] bar graph differ from how we interpret bar graphs made today?

  Seeing and Understanding Data [Bolch & Woods, 2018, Task 5]

Category S: Tasks that require student reading of the source.

Subcategory SD: Source-dependent tasks that prompt students to actively juxtapose or translate between two or more mathematical discourses.

- **Code SD-2: Reformulate or Translate the Source into Modern Terms**
  [Rewrite the source text using . . . .]

  Code Description
  The task asks students to rewrite what the author has done in the source using modern terminology / notation / concepts, or in keeping with standards for logic / rigor / writing conventions that are currently in place, but differ from those that were in place at the time of the source author's work.

  Code Examples

  ✠ Carefully reread Cauchy's sentence beginning with "In other words ..." and notice that he is making two separate equivalence claims, from a modern viewpoint. Rewrite each equivalence claim in Cauchy's sentence with modern $\epsilon - N$ terminology.

  Abel and Cauchy on a Rigorous Approach to Infinite Series [Ruch 2017, Task 14]

  ✠ In this part of [his paper], Cauchy gave a proof that the geometric series is convergent for $|x| < 1$, using the new Cauchy criterion for series convergence that you put into modern form in Task 21. Notice that he was a bit cavalier for the negative x case when stating that terms are "all contained between the limits...". Write a careful modern version of his proof using the modern form of the Cauchy criterion for series convergence.

  Abel and Cauchy on a Rigorous Approach to Infinite Series [Ruch 2017, Task 24]

  ✠ After reading a passage from al-Biruni's *Treatise on Shadows*:
  (a) Translate the second paragraph in the last passage above into a formula relating $\cot t$ and $\csc t$, adopting the convention that the gnomon has unit length.
  (b) Translate the third paragraph in the passage above into a formula relating $\cos t$ with $\sin t$, $\cot t$ and $\csc t$, again assuming that the gnomon has unit length.

  A Genetic Context for Understanding the Trigonometric Functions [Otero, 2017, Task 27]

  ✠ Restate and summarize Arbuthnot's explanation [for why more boys being born than girls was good for humanity] using more modern terms.

  Quantifying Certainty: The p-value [Klyve, 2017, Task 8]

Category E: Tasks that prompt students to exercise their mathematical understanding or skills in some way.

Subcategory EP: Exercises that prepare the student for engaging with the source text.

- **Code EP: Exercise – Getting Ready for the Source**

  Code Description
  These tasks are "stage setting" exercises that prepare students to make sense of source material, but without necessarily having an explicit connection to what the source author has written.

  Code Examples

  ✠ Compute the following ratios of adjacent cells in the tenth base of Pascal's table. Can you identify a pattern?

  $\frac{E_{9,2}}{E_{10,1}} =$ $\frac{E_{8,3}}{E_{9,2}} =$ $\frac{E_{7,4}}{E_{8,3}} =$

  $\frac{E_{6,5}}{E_{7,4}} =$ $\frac{E_{5,6}}{E_{6,5}} =$ $\frac{E_{4,7}}{E_{5,6}} =$

  $\frac{E_{3,8}}{E_{4,7}} =$ $\frac{E_{2,9}}{E_{3,8}} =$ $\frac{E_{1,10}}{E_{2,9}} =$

  Pascal's Triangle and Induction [Lodder, 2017, Task 6.2]

  ✠ Suppose your friend ... pulls out a suspicious-looking coin, and proceeds to flip heads 20 times in a row. Would you believe that the coin is "fair"? That is, would you believe that the coin will, in the long run, come up as "heads" half of the time? Why or why not?

  Quantifying Certainty: The p-value [Klyve, 2017, Task 2]

  ✠ How unlikely would something have to be before you were willing, in practice, to assume that it won't happen? Come up with a specific value and explain why you chose that.

  Quantifying Certainty: The p-value [Klyve, 2017, Task 13]

Category E: Tasks that prompt students to exercise their mathematical understanding or skills in some way.

Subcategory EC: Exercises that explore ideas contained in the PSP but connect more directly with the current course curriculum or the curriculum of a prerequisite course.

- **EC-1: Exercise – Connections to Prerequisite Facts or Procedures**

  Code Description

  The task asks students to give an explanation or perform a computation that requires an understanding of prerequisite material for the intended course.

  Code Examples

  ✠ Use Theorem 2 [on the convergence of alternating series] to prove the following series converge, or explain why the theorem cannot be applied to the particular series.

  (a) $\sum_{k=0}^{\infty} (-1)^k \dfrac{1}{k+1}$

  (b) $\dfrac{2}{1} - \dfrac{3}{2} + \dfrac{4}{3} - \dfrac{5}{4} + \cdots$

  (c) $1 - \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{4}} + \dfrac{1}{\sqrt{5}} - \cdots$

  Abel and Cauchy on a Rigorous Approach to Infinite Series [Ruch 2017, Task 53]

  ✠ Give an example of a function which is discontinuous but integrable.

  The Cantor Set before Cantor [Scoville, 2016, Task 2]

Category E: Tasks that prompt students to exercise their mathematical understanding or skills in some way.

Subcategory EC: Exercises that explore ideas contained in the PSP but connect more directly with the current course curriculum or the curriculum of a prerequisite course.

- **EC-2: Exercise – Connections to Current Curriculum**

  Code Description
  The task focuses on mathematical content from the curriculum of the intended course; completion of the task does not rely on or elucidate the primary source material in any direct way, even though the PSP contains ideas that are needed to complete it.

  Code Examples

  ✠ For a generic natural number n, let T_n denote the nth triangular number. Let's find an equation relating T_n to the preceding triangular number T_{n-1}.

  Fill in the blank with a natural number so that $T_n = T_{n-1} + \square$.

  Pascal's Triangle and Induction [Lodder, 2017, Task 1.5]

  ✠ Suppose series $\sum |x_k|$ converges. Use the Cauchy criterion to prove that $\sum x_k$ must converge.

  Abel and Cauchy on a Rigorous Approach to Infinite Series [Ruch 2017, Task 23]

  ✠ Cauchy has shown us that the nth order determinant of a "symmetric system," what today we would call the determinant of an $n \times n$ matrix $A = (a_{ij})$, is found by forming the sum of signed products of entries of the matrix, terms of the form

  $$\pm a_{\alpha 1} a_{\beta 2} \cdots a_{\zeta n}, \tag{4}$$

  where this sum includes a single term for every possible permutation $\begin{pmatrix} 1.2.3.\ldots n \\ \alpha.\beta.\gamma.\ldots\zeta \end{pmatrix}$ of the indices $\{1, 2, ..., n\}$, with signs determined by a well-specified procedure.

  Suppose that A is upper triangular. Since many of the entries of such a matrix equal 0, lots of the terms (4) in Cauchy's expansion of the determinant will vanish, leaving only those for which every one of the entries $a_{\alpha 1}, a_{\beta 2}, \ldots, a_{\zeta n}$ have their first index less than or equal to their second. Consider the possible values for the indices α, β, etc., in turn that will ensure that none of these entries appears below the principal diagonal of A; remember that since $\alpha\beta\gamma\ldots\zeta$ is a permutation of $123\cdots n$, none of the values of $\alpha, \beta, \gamma, \ldots, \zeta$ may repeat. Conclude that Cauchy's formula simplifies to a single term, and write out this simplified formula for $\det A$.

  Determining the Determinant [Otero, 2018, Task 17(a)]

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Appendix B: Heat Maps of Task Codes in Selected Primary Source Projects

Color Codes

- SO - 1 Read and React to the Source
- Reflect on or Evaluate the Source
- SI - 1 Carry out a Procedure or Calculation in the Source
- SI - 2 Provide Details or Verify a Result within the Source
- SI - 3 Make Sense of the Mathematics in the Source
- historical Context of the Source
- SD - 1 Compare and Contrast
- Translate the Source into Modern Terms
- EP - Exercise - Getting Ready for the Source
- to Prerequisite Facts or Procedures
- EC - 2 Exercise - Connections to Current Curriculum

Scoville, 2017 6 tasks	Bolch & Woods, 2018 17 tasks	Monks, 2019 20 tasks	Klyve, 2017 24 tasks	Ruch, 2017 32 tasks	Jones & Knudsen, 2017 48 tasks	Flagg, 2020 75 tasks	Barnett, 2018 77 tasks*
EC - 2 Exercise - Connections to Current Curriculum	SI - 3 Make Sense of the Mathematics in the Source	EP - Exercise - Getting Ready for the Source	EP - Exercise - Getting Ready for the Source	SO - 2 Read and Respond to Reflect on or Evaluate the Source	SI - 3 Make Sense of the Mathematics in the Source	EC - 1 Exercise - Connections to Prerequisite Facts or Procedures	SI - 3 Make Sense of the Mathematics in the Source
SI - 3 Make Sense of the Mathematics in the Source	SI - 3 Make Sense of the Mathematics in the Source	SD - 1 Compare and Contrast	EP - Exercise - Getting Ready for the Source	SD - 2 Reformulate or Translate the Source into Modern Terms	EP - Exercise - Getting Ready for the Source	SI - 3 Make Sense of the Mathematics in the Source	SI - 2 Provide Details or Verify a Result within the Source
SI - 2 Provide Details or Verify a Result within the Source	SI - 3 Make Sense of the Mathematics in the Source	SD - 2 Reformulate or Translate the Source into Modern Terms	EP - Exercise - Getting Ready for the Source	SD - 2 Reformulate or Translate the Source into Modern Terms	EP - Exercise - Getting Ready for the Source	SI - 1 Carry out a Procedure or Calculation in the Source	SI - 2 Provide Details or Verify a Result within the Source
SI - 2 Provide Details or Verify a Result within the Source	SI - 1 Carry out a Procedure or Calculation in the Source	EC - 2 Exercise - Connections to Current Curriculum	SI - 4 Make Sense of the Socio-historical Context of the Source	EC - 2 Exercise - Connections to Current Curriculum	EP - Exercise - Getting Ready for the Source	SI - 1 Carry out a Procedure or Calculation in the Source	EC - 2 Exercise - Connections to Current Curriculum
SI - 2 Provide Details or Verify a Result within the Source	EC - 2 Exercise - Connections to Current Curriculum	SI - 2 Provide Details or Verify a Result within the Source	SI - 3 Make Sense of the Mathematics in the Source	SI - 3 Make Sense of the Mathematics in the Source	EP - Exercise - Getting Ready for the Source	EC - 2 Exercise - Connections to Current Curriculum	SI - 2 Provide Details or Verify a Result within the Source
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	SD - 1 Compare and Contrast	SD - 1 Compare and Contrast	SD - 2 Reformulate or Translate the Source into Modern Terms	EC - 2 Exercise - Connections to Current Curriculum	SI - 3 Make Sense of the Mathematics in the Source	SI - 1 Carry out a Procedure or Calculation in the Source	EC - 2 Exercise - Connections to Current Curriculum
	SI - 3 Make Sense of the Mathematics in the Source	SI - 2 Provide Details or Verify a Result within the Source	SD - 2 Reformulate or Translate the Source into Modern Terms	EC - 2 Exercise - Connections to Current Curriculum	SI - 3 Make Sense of the Mathematics in the Source	SI - 3 Make Sense of the Mathematics in the Source	EP - Exercise - Getting Ready for the Source
	SO - 2 Read and Respond to Reflect on or Evaluate the Source	EC - 2 Exercise - Connections to Current Curriculum	SD - 2 Reformulate or Translate the Source into Modern Terms	EC - 2 Exercise - Connections to Current Curriculum	SI - 3 Make Sense of the Mathematics in the Source	SI - 1 Carry out a Procedure or Calculation in the Source	EC - 2 Exercise - Connections to Current Curriculum
	SI - 3 Make Sense of the Mathematics in the Source	SD - 1 Compare and Contrast	SI - 3 Make Sense of the Mathematics in the Source	EC - 2 Exercise - Connections to Current Curriculum	SI - 1 Carry out a Procedure or Calculation in the Source	SI - 1 Carry out a Procedure or Calculation in the Source	SI - 2 Provide Details or Verify a Result within the Source
	SO - 2 Read and Respond to Reflect on or Evaluate the Source	SI - 3 Make Sense of the Mathematics in the Source	EP - Exercise - Getting Ready for the Source	EC - 2 Exercise - Connections to Current Curriculum	EC - 2 Exercise - Connections to Current Curriculum	SI - 1 Carry out a Procedure or Calculation in the Source	SO - 2 Read and Respond to Reflect on or Evaluate the Source
	SD - 1 Compare and Contrast	SI - 3 Make Sense of the Mathematics in the Source	EP - Exercise - Getting Ready for the Source	EC - 2 Exercise - Connections to Current Curriculum	EC - 2 Exercise - Connections to Current Curriculum	EC - 2 Exercise - Connections to Current Curriculum	EC - 2 Exercise - Connections to Current Curriculum
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	EC - 2 Exercise - Connections to Current Curriculum	SI - 2 Provide Details or Verify a Result within the Source	SO - 2 Read and Respond to Reflect on or Evaluate the Source	SI - 3 Make Sense of the Mathematics in the Source	EC - 2 Exercise - Connections to Current Curriculum	EC - 2 Exercise - Connections to Current Curriculum	SD - 2 Reformulate or Translate the Source into Modern Terms
	EC - 2 Exercise - Connections to Current Curriculum	SI - 2 Provide Details or Verify a Result within the Source	SI - 3 Make Sense of the Mathematics in the Source	SD - 2 Reformulate or Translate the Source into Modern Terms	SI - 3 Make Sense of the Mathematics in the Source	SI - 3 Make Sense of the Mathematics in the Source	EC - 2 Exercise - Connections to Current Curriculum
	EC - 2 Exercise - Connections to Current Curriculum	SI - 1 Carry out a Procedure or Calculation in the Source	SI - 3 Make Sense of the Mathematics in the Source	SI - 3 Make Sense of the Mathematics in the Source	EC - 2 Exercise - Connections to Current Curriculum	SI - 1 Carry out a Procedure or Calculation in the Source	EC - 2 Exercise - Connections to Current Curriculum
		SI - 2 Provide Details or Verify a Result within the Source	SO - 2 Read and Respond to Reflect on or Evaluate the Source	SI - 3 Make Sense of the Mathematics in the Source	SI - 3 Make Sense of the Mathematics in the Source	SI - 3 Make Sense of the Mathematics in the Source	SI - 3 Make Sense of the Mathematics in the Source
		SD - 1 Compare and Contrast	SI - 3 Make Sense of the Mathematics in the Source	SD - 2 Reformulate or Translate the Source into Modern Terms	SI - 3 Make Sense of the Mathematics in the Source	SI - 3 Make Sense of the Mathematics in the Source	SI - 3 Make Sense of the Mathematics in the Source
		SI - 4 Make Sense of the Socio-historical Context of the Source	SD - 1 Compare and Contrast	SD - 2 Reformulate or Translate the Source into Modern Terms	SI - 3 Make Sense of the Mathematics in the Source	SO - 2 Read and Respond to Reflect on or Evaluate the Source	SI - 2 Provide Details or Verify a Result within the Source
			SI - 3 Make Sense of the Mathematics in the Source	SD - 2 Reformulate or Translate the Source into Modern Terms	EC - 2 Exercise - Connections to Current Curriculum	SI - 3 Make Sense of the Mathematics in the Source	SD - 2 Reformulate or Translate the Source into Modern Terms
			EC - 2 Exercise - Connections to Current Curriculum	SI - 3 Make Sense of the Mathematics in the Source	EC - 2 Exercise - Connections to Current Curriculum	EC - 2 Exercise - Connections to Current Curriculum	SI - 3 Make Sense of the Mathematics in the Source
			EC - 2 Exercise - Connections to Current Curriculum	SI - 3 Make Sense of the Mathematics in the Source	EC - 2 Exercise - Connections to Current Curriculum	EC - 2 Exercise - Connections to Current Curriculum	SI - 2 Provide Details or Verify a Result within the Source
			SO - 2 Read and Respond to Reflect on or Evaluate the Source	SD - 2 Reformulate or Translate the Source into Modern Terms	EC - 2 Exercise - Connections to Current Curriculum	EC - 2 Exercise - Connections to Current Curriculum	SO - 2 Read and Respond to Reflect on or Evaluate the Source
				SD - 2 Reformulate or Translate the Source into Modern Terms	EC - 2 Exercise - Connections to Current Curriculum	EC - 2 Exercise - Connections to Current Curriculum	SI - 3 Make Sense of the Mathematics in the Source
				SI - 2 Provide Details or Verify a Result within the Source	EC - 2 Exercise - Connections to Current Curriculum	SO - 1 Read and React to the Source	EC - 2 Exercise - Connections to Current Curriculum
				EC - 1 Exercise - Connections to Prerequisite Facts or Procedures	EC - 2 Exercise - Connections to Current Curriculum	SI - 3 Make Sense of the Mathematics in the Source	EC - 2 Exercise - Connections to Current Curriculum
				SI - 3 Make Sense of the Mathematics in the Source	EC - 2 Exercise - Connections to Current Curriculum	SI - 3 Make Sense of the Mathematics in the Source	EC - 2 Exercise - Connections to Current Curriculum
				SD - 2 Reformulate or Translate the Source into Modern Terms	SI - 3 Make Sense of the Mathematics in the Source	EP - Exercise - Getting Ready for the Source	EC - 2 Exercise - Connections to Current Curriculum
				EC - 2 Exercise - Connections to Current Curriculum	EC - 2 Exercise - Connections to Current Curriculum	EP - Exercise - Getting Ready for the Source	SD - 2 Reformulate or Translate the Source into Modern Terms
				EC - 2 Exercise - Connections to Current Curriculum	SI - 3 Make Sense of the Mathematics in the Source	SI - 3 Make Sense of the Mathematics in the Source	SD - 2 Reformulate or Translate the Source into Modern Terms
				EC - 2 Exercise - Connections to Current Curriculum	EC - 2 Exercise - Connections to Current Curriculum	EP - Exercise - Getting Ready for the Source	SI - 3 Make Sense of the Mathematics in the Source
					EC - 2 Exercise - Connections to Current Curriculum	SI - 3 Make Sense of the Mathematics in the Source	EC - 2 Exercise - Connections to Current Curriculum
					SO - 1 Read and React to the Source	SI - 3 Make Sense of the Mathematics in the Source	EC - 2 Exercise - Connections to Current Curriculum
					SD - 2 Reformulate or Translate the Source into Modern Terms	SD - 2 Reformulate or Translate the Source into Modern Terms	EC - 2 Exercise - Connections to Current Curriculum
					SO - 2 Read and Respond to Reflect on or Evaluate the Source	EC - 2 Exercise - Connections to Current Curriculum	EC - 2 Exercise - Connections to Current Curriculum
					SI - 3 Make Sense of the Mathematics in the Source	SD - 1 Compare and Contrast	EC - 2 Exercise - Connections to Current Curriculum
					SD - 2 Reformulate or Translate the Source into Modern Terms	EC - 1 Exercise - Connections to Prerequisite Facts or Procedures	EC - 2 Exercise - Connections to Current Curriculum
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					EC - 2 Exercise - Connections to Current Curriculum	SO - 2 Read and Respond to Reflect on or Evaluate the Source	EC - 2 Exercise - Connections to Current Curriculum
					EC - 2 Exercise - Connections to Current Curriculum	SI - 1 Carry out a Procedure or Calculation in the Source	EP - Exercise - Getting Ready for the Source
					EC - 2 Exercise - Connections to Current Curriculum	SI - 3 Make Sense of the Mathematics in the Source	EP - Exercise - Getting Ready for the Source
					EC - 2 Exercise - Connections to Current Curriculum	SI - 1 Carry out a Procedure or Calculation in the Source	EP - Exercise - Getting Ready for the Source
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						SI - 1 Carry out a Procedure or Calculation in the Source	SI - 2 Provide Details or Verify a Result within the Source
						SD - 1 Compare and Contrast	EC - 2 Exercise - Connections to Current Curriculum
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						SI - 1 Carry out a Procedure or Calculation in the Source	EC - 2 Exercise - Connections to Current Curriculum
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						SI - 1 Carry out a Procedure or Calculation in the Source	EC - 2 Exercise - Connections to Current Curriculum
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						SI - 3 Make Sense of the Mathematics in the Source	EC - 1 Exercise - Connections to Prerequisite Facts or Procedures
						SO - 2 Read and Respond to Reflect on or Evaluate the Source	SD - 2 Reformulate or Translate the Source into Modern Terms
						SI - 3 Make Sense of the Mathematics in the Source	SI - 3 Make Sense of the Mathematics in the Source
						SI - 3 Make Sense of the Mathematics in the Source	SI - 2 Provide Details or Verify a Result within the Source
						EC - 2 Exercise - Connections to Current Curriculum	SI - 2 Provide Details or Verify a Result within the Source
						SI - 3 Make Sense of the Mathematics in the Source	SO - 2 Read and Respond to Reflect on or Evaluate the Source
						EC - 2 Exercise - Connections to Current Curriculum	**EC - 2 Exercise - Connections to Current Curriculum**
						SD - 1 Compare and Contrast	**SI - 3 Make Sense of the Mathematics in the Source**
						EC - 2 Exercise - Connections to Current Curriculum	**EC - 2 Exercise - Connections to Current Curriculum**
						EC - 2 Exercise - Connections to Current Curriculum	**EC - 2 Exercise - Connections to Current Curriculum**
						EC - 2 Exercise - Connections to Current Curriculum	**SI - 3 Make Sense of the Mathematics in the Source**
						EC - 2 Exercise - Connections to Current Curriculum	**SI - 2 Provide Details or Verify a Result within the Source**
						EC - 2 Exercise - Connections to Current Curriculum	**SD - 2 Reformulate or Translate the Source into Modern Terms**
						EC - 2 Exercise - Connections to Current Curriculum	**SD - 1 Compare and Contrast**
						EC - 2 Exercise - Connections to Current Curriculum	**SI - 3 Make Sense of the Mathematics in the Source**
						EC - 2 Exercise - Connections to Current Curriculum	**SD - 1 Compare and Contrast**
							EC - 2 Exercise - Connections to Current Curriculum
							EC - 2 Exercise - Connections to Current Curriculum

* Tasks in bold border appear in PSP Appendices

Author, Year	Course	PSP Title
Scoville, 2017	Topology	The Closure Operation as the Foundation of Topology
Bolch & Woods, 2018	Statistics / General & Elementary Education	Seeing and Understanding Data
Monks, 2019	Calculus 1	Fermat's Method of Finding the Maximum and Minimum
Klyve, 2017	Statistics	Quantifying Certainty: the *p*-value
Ruch, 2017	Introductory Analysis	Bolzano on Continuity and the Intermediate Value Theorem
[ones & Knudsen, 2017	Number Theory	The Pell Equation in Indian Mathematics
Flagg, 2020	Elementary Education	Cross Cultural Comparisons: The Greatest Common Divisor
Barnett, 2018	Abstract Algebra	Otto Holder's Formal Christening of the Quotient Group Concept