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Importance of Understanding the Physical System in Selecting Separation of Variables Based Methods to Solve the Heat Conduction Partial Differential Equation

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Abstract: Separation of variables is a common method for producing an analytical based solution to partial differential equations. Despite the wide application of this method, often the physical phenomena described by the differential equations are not adequately involved in the discourse over the appropriate methods to solve a given problem, particularly in mathematics curricula. However, as mathematics is the tool to better understanding of the physical world, the meaning of the differential equation, boundary conditions, and initial conditions cannot be detached from the methods used to solve the differential equations. Failure to recognize the physical conditions being studied can lead to solution methods that are invalid or unphysical. This paper demonstrates how awareness of the physical nature of the system being investigated and its relationship to the mathematics can guide the selection of the relevant solution methods. To illustrate the importance of the comprehension of the physical meaning behind the mathematical equations and representations and the need to avoid rote application a solution technique, the logic behind the selection of the appropriate solution techniques for the one-dimensional transient heat conduction equation is considered under different imposed conditions which lead to different trends in system operation.

Keywords: Separation of variables; superposition; heat conduction; diffusion; partial differential equations

1 Introduction

A major topic in analytical based solutions of partial differential equations is the use of separation of variables. The conditions under which separation of variables can be implemented are limited by the form of the differential equation, the form of the boundary conditions, and the physical domain (Kreyzsig, 2011). Several methods to manipulate the system so that separation of variables can be used as part of the solution process for a given problem are widely known (Arpaci, 1966). The selection of the solution method combination best suited for a given problem must rely upon knowledge of both the mathematics and the physical phenomena being modeled. Without considering this interdependence, the mathematical solution reached may lead to unphysical predictions despite the ability to formulate a solution or the attempted solution method may reveal inconsistencies making a solution through the method impossible. The need for comprehension of the physical meaning behind the differential equation, boundary

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conditions, and initial conditions in the determination of the solution techniques appropriate for a given problem is demonstrated through the discussion of the solution approaches for a set of examples of the heat conduction or diffusion equation using separation of variables-based techniques for various boundary and source conditions.

When the needed conditions for solving a differential equation using separation of variables are not met, often the method of superposition is implemented, dividing the problem into sub-problems typically involving a sub-set of the independent variables and then a sub-problem that involves a solution function of all of the independent variables where separation of variables can then be applied. For many transient problems, the division commonly involves a steady state and a transient component of the solution. However, not all systems achieve a steady state condition. The boundary conditions and form of the differential equation, when understood in a physical context, dictate whether a specific superposition approach, including this common steady plus transient component solution, can generate a viable solution. Hence, this superposition method cannot be applied in a rote manner and the selection of the solution method and form must be informed by knowledge of the physical phenomena being represented by the mathematical formulation. In this work, logical and critical thinking is used to link together the math and the physical system and assess the techniques appropriate for solving problems with superposition and separation of variables.

2 Background

In the discussion of the importance of the physical understanding of the system in yielding physically meaningful results, the diffusion or heat conduction equation as in Equation 1 is utilized merely as an example.

$$\nabla^2 u + Q(x, y, z, t) = \frac{1}{c^2}\frac{\partial u}{\partial t} \qquad \text{Eq. (1)}$$

The general assumptions in this work are constant properties and a linear form of the differential equation. Further, to focus on the solution method and the effect of the boundary conditions and the physical system, a one-dimensional transient rectangular system of a finite length is considered with a cross sectional area that does not vary along the length of the domain. Thus, the reduced differential equation studied is of the form in Equation 2:

$$\frac{\partial^2 u}{\partial x^2} + Q(x, t) = \frac{1}{c^2}\frac{\partial u}{\partial t} \qquad \text{Eq. (2)}$$

Separation of variables assumes that the solution can be written as a product of functions of the independent variables or, for the one-dimensional transient problem:

$$u(x,t) = X(x)T(t) \qquad \text{Eq. (3)}$$

As a result, several limitations are placed on the systems to which this method can be applied (Kreyzsig, 2011).

1. The differential equation must be linear and homogeneous and must fall into the Sturm-Liouville format.
2. For a transient problem, the boundary conditions must be homogenous and be of the acceptable form of the Sturm-Liouville Problem boundary conditions, namely the homogeneous forms of the Dirichlet, Neumann, and Robin or mixed boundary conditions.
3. The system domain must follow the coordinate axes. For a system with independent variables x and t, the domain must be linear along the x axis such that the cross-sectional area cannot vary.

For some systems that do not meet these criteria, methods can be applied to manipulate the problem so that separation of variables can be used to solve a sub-problem that meets the separation of variables criteria. To appropriately divide the problem, the physical system or events being modeled must be well understood. Among possible alternative methods are the following:

1. A shift in the level of u, the dependent variable, may be used to remove a non-homogeneous Dirichlet type boundary condition. This method is beneficial if a flux boundary condition or the same boundary condition value is applied at the other boundary of a one-dimensional system, as the shift may yield a homogeneous boundary condition.

$$u^*(x,t) = u(x,t) - u_{ref} \qquad \text{Eq. (4)}$$

2. Non-homogeneous boundary conditions of any of the three allowed types can be handled using the principle of superposition. For each non-homogeneous boundary condition, the initial problem is divided into simpler problems, each involving a steady state problem with only one non-homogeneous boundary condition, and then, if applicable, a transient problem with all homogeneous boundary conditions can be solved. For example, for a two-dimensional transient problem with two non-homogeneous boundary conditions, two steady problems, each carrying one of the

non-homogeneous boundary conditions can be formulated and potentially solved using separation of variables, leaving a transient problem with all homogenous boundary conditions that can be solved using separation of variables. Implicit in this form is the existence of a steady state solution due to the exponential decay in the transient term.

$$u(x,y,t) = u_1(x,y) + u_2(x,y) + u_3(x,y,t) \qquad \text{Eq. (5)}$$

3. When a non-homogeneous term appears in the differential equation, in some cases, superposition in different forms may be implemented, where the linear equation and/or boundary conditions are broken into a series of simpler problems, at least one of which can be solved using separation of variables. The sum of these problems returns the original differential equation and boundary conditions. The specific form of the superposition is dependent upon the boundary conditions, differential equation, and an understanding of the physical system being modeled.
4. If transient boundary conditions or sources are applied or sources or functions of multi-dimensions that cannot be separated, then eigenfunction expansion methods might be used to find the solutions.

The specific method implemented must be adequate to describe the system operation, both spatially and as a function of time, and be consistent with the differential equation, boundary conditions, initial conditions, and the physical phenomena being modeled. The critical role knowledge of the physical system being explored plays in the selection of the appropriate solution method can be demonstrated by examining the effects of changing the boundary conditions or source terms on the solution of a simple one-dimensional transient heat conduction problem.

3 Solution form across different conditions

3.1 Standard solution type: Solution with a function of x, t only with one Dirichlet and one Neumann boundary condition

The one dimensional heat conduction equation, with the boundary and initial conditions shown below, can be solved using the principle of superposition for a domain where x[0,*a*].

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t} \qquad \text{Eq. (6a)}$$

where

$$u(x,0) = f(x); u(0,t) = u_o \; and \; \frac{\partial u}{\partial x}(a,t) = 0 \qquad \text{Eq. (6b)}$$

Because of the non-homogeneous boundary condition, separation of variables cannot be used to solve the problem directly, but a division of the solution of this linear differential equation and the linear boundary conditions into a function of x and a function of x and t can be made. Applying superposition:

$$u(x,t) = u_1(x) + u_2(x,t) \qquad \text{Eq. (7)}$$

When examining the transient form of the solution that results, this solution form implies a steady state solution to the problem exists or that the time variation of u will diminish after a given duration. For this particular problem, no source terms are present in the differential equation, so the source terms cannot drive a continual change in the value of u. The boundary conditions are not transient. While a Neumann boundary condition is present, indicating a flux of u through a boundary, or a transfer of energy if u represents temperature, any continual change in the system condition that this flux might cause is tempered by the Dirichlet boundary condition at x=0, which limits the value of u. As a result, the boundary conditions and differential equation, taken in a physical context, indicate the u value must remain bounded with time and the u function must have a steady state solution. The assumed form of the solution is consistent with the expected characteristics of the physical system.

The differential equations for each of the solution component functions and the associated boundary conditions/initial conditions can be formulated. For the steady state portion, the differential equation is shown in Equation 8a:

$$\frac{\partial^2 u_1}{\partial x^2} = 0 \qquad \text{Eq. (8a)}$$

subject to:

$$u_1(0) = u_o \; and \; \frac{\partial u_1}{\partial x}(a) = 0 \qquad \text{Eq. (8b)}$$

For the transient portion, the governing differential equation is shown in Equation 9a:

$$\frac{\partial^2 u_2}{\partial x^2} = \frac{1}{c^2}\frac{\partial u_2}{\partial t} \qquad \text{Eq. (9a)}$$

subject to:

$$u_2(x,0) = f(x) - u_1(x); u_2(0,t) = 0 \; and \; \frac{\partial u_2}{\partial x}(a,t) = 0 \qquad \text{Eq. (9b)}$$

First, the solution to the steady portion can be found. Integrating Equation 8a twice with respect to x and applying the two boundary conditions in Equation 8b, the solution for $u_1(x)$ is:

$$u_1(x) = u_o \qquad \text{Eq. (10)}$$

Finally, the transient part of the solution needs to be determined. Using separation of variables, for the solution for u_2 in the form of Eq. 3, the product of a function of x, X(x), and a function of t, T(t), and noting that the boundary conditions indicate the eigenvalues, α^2, are greater than zero, the two separated ordinary differential equations are:

$$X'' + \alpha^2 X = 0; \qquad T' + (\alpha c)^2 T = 0 \qquad \text{Eq. (11a)}$$

where

$$X(0) = 0; X'(a) = 0 \qquad \text{Eq. (11b)}$$

As zero is not an eigenvalue, the solution functions become:

$$X_n = \sin(\alpha_n x)\,;\; \alpha_n = \frac{(2n+1)\pi}{2a}; n = 0,1,\dots; T_n = exp(-(\alpha_n c)^2 t) \qquad \text{Eq. (12)}$$

Applying the initial condition, the solution must take the form of a series:

$$u_2(x,t) = \sum_{n=0}^{\infty} B_n \sin(\alpha_n x) exp(-(\alpha_n c)^2 t) \qquad \text{Eq. (13a)}$$

where

$$B_n = \frac{\int_0^a \sin(\alpha_n x)(f(x) - u_1(x))dx}{\int_0^a \sin^2(\alpha_n x)dx} \qquad \text{Eq. (13b)}$$

The final solution is then:

$$u(x,t) = u_o + \sum_{n=0}^{\infty} B_n \sin(\alpha_n x) exp(-(\alpha_n c)^2 t) \qquad \text{Eq. (14)}$$

The separation of variables solution for this differential equation, with positive, non-zero eigenvalues will yield a time varying T(t) that decays over time. Examining the form of the transient part of the solution, clearly, as time increases this transient solution diminishes, leaving $u_1(x)$, the steady state solution. Hence, this solution form effectively assumes that a steady state solution exists.

Not all system boundaries and differential equations yield steady state conditions as time advances. For example, for the one-dimensional heat condition problem, unequal heat fluxes at the boundaries for the system with a constant cross-sectional area, a system with an energy source, or a system with transient boundary conditions may generate conditions where the incoming energy flow and outgoing energy flow are not equal. Hence, the system will remain time varying, and a steady state condition is not physically realistic. Under such conditions, the solution methods must change from the standard steady and transient combination typically assumed. Instead, the solution approach must be reconciled with the physical conditions present.

3.2 Attempt at solution with a function of x and t only with two unequal Neumann Boundary conditions, one of which is non-zero

To illustrate how the system conditions and solution approach are interconnected, the problem is now modified by altering the boundary conditions to two Neumann boundary conditions, one of which is non-zero:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t} \qquad \text{Eq. (15a)}$$

where

$$u(x,0) = f(x); \frac{\partial u}{\partial x}(0,t) = q_o \ and \ \frac{\partial u}{\partial x}(a,t) = 0 \qquad \text{Eq. (15b)}$$

Initially, the same solution method is attempted in the form of a steady and a transient solution, without regard to the physical meaning of the boundary conditions, so the solution form is taken as:

$$u(x,t) = u_1(x) + u_2(x,t) \qquad \text{Eq. (16)}$$

The first issue encountered is in the solution of $u_1(x)$. The differential equation for $u_1(x)$ is:

$$\frac{\partial^2 u_1}{\partial x^2} = 0 \qquad \text{Eq. (17a)}$$

The boundary conditions for u_1 are:

$$\frac{\partial u_1}{\partial x}(0) = q_o \ and \ \frac{\partial u_1}{\partial x}(a) = 0 \qquad \text{Eq. (17b)}$$

Integrating Equation 17a once, the first derivative of u_1 with respect to x is a constant. This requirement is inconsistent with the boundary conditions as no constant can satisfy both boundary conditions. This issue stems from the inconsistency between the assumed form of the solution and the physical nature of the system with the prescribed boundary conditions.

The failure of the assumed solution form to represent the physical system can be understood from the boundary conditions. The boundary conditions indicate a non-zero u flux at the x=0 boundary and zero flux at the x=a boundary. If u represents the temperature and the flux represents heat flow, then energy is carried into the system at the x=0 boundary, is not removed at x=a. Net energy is input into the system and the system cannot reach a steady state solution, but will continually change with time. This condition contradicts the basic form of the solution selected as the sum of a function of x and a function of x and t. Therefore, an alternative solution form must be sought.

3.3 Modified approach for a physically feasible solution

For the constant flux boundary conditions of this type, a solution form as shown in Equation 18, which that involves separate functions for x and t summed with a separation of variables-based function of x and t, is adequate to find the solution for u.

$$u(x,t) = u_1(x) + u_2(t) + u_3(x,t) \qquad \text{Eq. (18)}$$

For the $u_3(x,t)$, the governing equations and boundary and initial conditions become:

$$\frac{\partial^2 u_3}{\partial x^2} = \frac{1}{c^2}\frac{\partial u_3}{\partial t} \qquad \text{Eq. (19a)}$$

where in order for separation of variables to be applied,

$$u_3(x,0) = f(x) - u_1(x) - u_2(0);\ \frac{\partial u_3}{\partial x}(0,t) = 0\ and\ \frac{\partial u_3}{\partial x}(a,t) = 0 \qquad \text{Eq. (19b)}$$

The remaining differential equation is then:

$$\frac{\partial^2 u_1}{\partial x^2} = \frac{1}{c^2}\frac{\partial u_2}{\partial t} \qquad \text{Eq. (20a)}$$

with the consistent boundary or initial conditions:

$$u_2(0) = f(x) - u_1(x) - u_3(x,0); \frac{\partial u_1}{\partial x}(0) = q_o\ and\ \frac{\partial u_1}{\partial x}(a) = 0 \qquad \text{Eq. (20b)}$$

Now, in Equation 20a, a function of x is equal to a function t and so these functions must be equal to a constant called B. From this, the flux of u_1 must vary at most linearly in the system if B is non-zero.

$$\frac{\partial^2 u_1}{\partial x^2} = \frac{1}{c^2}\frac{\partial u_2}{\partial t} = B\ \ and\ \ \frac{\partial u_1}{\partial x} = Bx + D;\ \frac{1}{c^2}\frac{\partial u_2}{\partial t} = B \qquad \text{Eq. (21a)}$$

Physically, for the constant cross-section conditions, if the flux of u at the two boundaries is not equal, or B is not equal to zero, then the solution must be transient for all times as net energy flow into the system occurs as seen in Equation 21a. The solutions for u_1 and u_2 are therefore of the form:

$$u_1(x) = \frac{1}{2}Bx^2 + C_1x + C_2;\ \ u_2(t) = c^2Bt + C_3 \qquad \text{Eq.(21b)}$$

Now, applying the boundary conditions to $u_1(x)$, independently yields B and C_1 so that:

$$u_1(x) = -\frac{1}{2}\frac{q_o}{a}x^2 + q_ox + C_2;\ \ u_2(t) = -c^2\frac{q_o}{a}t + C_3 \qquad \text{Eq. (21c)}$$

Because of the boundary condition type, this separate sum of the function of x and function solely of t is adequate to describe the system operation. Note, the two constants C_2 and C_3 remain to be determined.

Proceeding to the $u_3(x,t)$ solution, because of the two zero flux boundary conditions (Equation 19b), zero is an eigenvalue for this problem. Additionally, an infinite set of positive eigenvalues is present. The solution form is:

$$X'' + \alpha^2X = 0; \qquad T' + (\alpha c)^2T = 0 \qquad \text{Eq. (22a)}$$

where

$$X'(0) = 0; X'(a) = 0 \qquad \text{Eq. (22b)}$$

As zero is an eigenvalue, the solution becomes:

$$X_n = \cos(\alpha_nx)\,;\ \alpha_n = n\pi; n = 0,1,\dots; T = exp(-(\alpha_nc)^2t) \qquad \text{Eq. (23)}$$

Then applying the initial condition:

$$u_3(x,t) = A_o + \sum_{n=1}^{\infty} A_n\cos(\alpha_nx)exp(-(\alpha_nc)^2t) \qquad \text{Eq. (24a)}$$

where

$$A_n = \frac{\int_0^a \cos(\alpha_nx)(f(x) - u_1(x) - u_2(0))dx}{\int_0^a \cos^2(\alpha_nx)dx} \qquad \text{Eq. (24b)}$$

$u_1(x)$ and $u_2(t)$ still contain the unknown C_2 and C_3 constants. A constant is also present in $u_3(x,t)$ in the A_o. Thus, the two constants in $u_1(x)$ and $u_2(t)$ are set equal to zero and absorbed into the A_o. These functions become:

$$u_1(x) = -\frac{1}{2}\frac{q_o}{a}x^2 + q_o x;\;\; u_2(t) = -c^2\frac{q_o}{a}t \qquad \text{Eq. (25)}$$

Then, A_o can be found ($u_2(0)=0$), recognizing the orthogonality of the eigenfunctions:

$$A_o = \frac{\int_0^a (1)\left(f(x) - u_1(x)\right)dx}{\int_0^a 1dx} \qquad \text{Eq. (26)}$$

The final solution is:

$$u(x,t) = A_o + \sum_{n=0}^{\infty} A_n \cos(\alpha_n x) exp(-(\alpha_n c)^2 t) - \frac{1}{2}\frac{q_o}{a}x^2 + q_o x - c^2\frac{q_o}{a}t \qquad \text{Eq. (27)}$$

The form of u(x,t) in Equation 27 allows for the continual time variation of u and is appropriate for the boundary condition type. Again, the special form of the boundary condition type is adequately satisfied with independent functions of x and t added to the separation of variables-based solution series. This solution form, however, may not be adequate for any general type of source, boundary, or initial condition combination where net energy input might be present in a system and a transient solution that does not decay with time is needed. Two additional examples are presented.

3.4 Constant source term with flux boundaries

With at least one Dirichlet boundary condition, a limit on the u value will persist even with a constant source term type of problem and so a steady state solution. However, other boundary conditions such as two flux boundary conditions can result in sustained net energy input over time. Here, a two-flux boundary condition set with a constant source term is considered. A steady state is not possible for such a system unless the heat flux and the source properly balance. Without such a balance, the problem can be treated in a manner similar to the two-flux boundary condition with a minor modification. Suppose a system has a constant energy source following the differential equation below with the two-flux boundary condition so the solution will remain transient.

$$\frac{\partial^2 u}{\partial x^2} + Q = \frac{1}{c^2}\frac{\partial u}{\partial t} \qquad \text{Eq. (28)}$$

Assume a solution of the form:

$$u(x,t) = u_1(x) + u_2(t) + u_3(x,t) \qquad \text{Eq. (29)}$$

For the u_3(x,t), the governing equations and boundary or initial conditions become:

$$\frac{\partial^2 u_3}{\partial x^2} = \frac{1}{c^2}\frac{\partial u_3}{\partial t} \qquad \text{Eq. (30a)}$$

where

$$u_3(x,0) = f(x) - u_1(x) - u_2(0);\ \ \frac{\partial u_3}{\partial x}(0,t) = 0\ and\ \frac{\partial u_3}{\partial x}(a,t) = 0 \qquad \text{Eq. (30b)}$$

The remaining equation is then:

$$\frac{\partial^2 u_1}{\partial x^2} = \frac{1}{c^2}\frac{\partial u_2}{\partial t} - Q \qquad \text{Eq. (31a)}$$

where the remaining consistent boundary or initial conditions are:

$$u_2(0) = f(x) - u_1(x) - u_3(x,0); \frac{\partial u_1}{\partial x}(0) = q_o\ and\ \frac{\partial u_1}{\partial x}(a) = 0 \qquad \text{Eq. (31b)}$$

With Q as a constant, the left and right sides of equation 31a must be equal to a constant. As before, the flux of u_1 with x may vary at most linearly in the system.

$$\frac{\partial^2 u_1}{\partial x^2} = \frac{1}{c^2}\frac{\partial u_2}{\partial t} = B\ \ and\ \ \frac{\partial u_1}{\partial x} = Bx + C_1;\ \frac{1}{c^2}\frac{\partial u_2}{\partial t} - Q = B \qquad \text{Eq. (33a)}$$

Now,

$$u_1(x) = \frac{1}{2}Bx^2 + C_1 x + C_2;\ \ u_2(t) = c^2(B+Q)t + C_3 \qquad \text{Eq.(33b)}$$

Applying the boundary conditions to $u_1(x)$, yields:

$$u_1(x) = -\frac{1}{2}\frac{q_o}{a}x^2 + q_o x + C_2;\ \ u_2(t) = c^2\left(-\frac{q_o}{a} + Q\right)t + C_3 \qquad \text{Eq. (33c)}$$

Clearly Q cannot be a function of x or a function of t and be treated in the manner presented in Equation 29. For a source that is a function of x alone, the time variation of u_2 must be zero and the source must then balance with the energy flow from the boundaries. If Q were a function of time, then the second derivative of the u_1 would be zero, two different fluxes at the boundaries could not be maintained, and the separation of variables method could not be used since the transient term would be present in the boundary conditions.

3.5 Transient boundary conditions or an x and t varying source

If transient boundary conditions or a transient source term are applied, the general separation of variables techniques cannot be implemented since the solution will necessitate functions where the x and t dependencies cannot be separated either through the separation of variables or through the sum of separate functions of the independent variables. In such cases, eigenfunction expansion may be a viable means of determining the solutions for u. Here, a flux boundary condition will be assigned at each of the system boundaries, but other boundary conditions can utilize such a method.

A solution of the form below assigns transient boundary conditions in a manner that will shift the transient portions into a source term. Let

$$u(x,t) = v(x,t) + w(x,t) \qquad \text{Eq. (34a)}$$

where the boundary condition at x=0 is $v_x(0,t) = A(t)$ and at x=L, $v_x(L,t) = B(t)$.

$$\frac{\partial v}{\partial x}(x,t) = B(t)\frac{x}{L} + \left(1 - \frac{x}{L}\right)A(t) \qquad \text{Eq. (34a)}$$

Then,

$$v(x,t) = B(t)\frac{x^2}{2L} + \left(x - \frac{x^2}{2L}\right)A(t) + D(t) \qquad \text{Eq. (34b)}$$

D(t) will be taken up by w(x,t), so D(t) can be set to zero. With this, the governing differential equation becomes:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} + Q(x,t) = \frac{1}{c^2}\frac{\partial w}{\partial t} + \frac{1}{c^2}\frac{\partial v}{\partial t} \qquad \text{Eq. (35a)}$$

Making substitutions, where D(t)=0, yields:

$$\frac{B(t)}{L} - \frac{A(t)}{L} + \frac{\partial^2 w}{\partial x^2} + Q(x,t) = \frac{1}{c^2}\frac{\partial w}{\partial t} + \frac{1}{c^2}\left(\frac{\partial B}{\partial t}\frac{x^2}{2L} + \left(x - \frac{x^2}{2L}\right)\frac{\partial A}{\partial t}\right) \qquad \text{Eq. (35b)}$$

Now, grouping the non-w(x,t) terms under one source term:

$$\frac{\partial^2 w}{\partial x^2} + Q^*(x,t) = \frac{1}{c^2}\frac{\partial w}{\partial t} \qquad \text{Eq. (36a)}$$

where:

$$Q^*(x,t) = Q(x,t) + \frac{B(t)}{L} - \frac{A(t)}{L} - \frac{1}{c^2}\left(\frac{\partial B}{\partial t}\frac{x^2}{2L} + \left(x - \frac{x^2}{2L}\right)\frac{\partial A}{\partial t}\right) \qquad \text{Eq. (36b)}$$

The boundary and initial conditions for w(x,t) are then:

$$w_x(0,t) = 0;\ w_x(L,t) = 0; w(x,0) = u(x,0) - v(x,0); \qquad \text{Eq. (37)}$$

Eigenfunction expansion with the solution of the form below can then be implemented where the $X_n(x)$ functions are the eigenfunctions of the homogeneous form of the differential equation in Equation 36a.

$$w(x,t) = \sum_{n=0}^{\infty} b_n(t) X_n(x) \qquad \text{Eq. (38)}$$

Hence, again, knowledge of the physical system and the appropriate mathematical representation of the system are both needed in order to acquire a mathematically and physically feasible solution to the differential equation.

4 CONCLUSIONS

The set of examples presented associated with the solution of the heat conduction or diffusion equation for various types of boundary conditions and sources conditions has shown that the solution approach selection must consider the physical meaning of the differential equations and boundary conditions applied to a system. The ability to connect the physical meaning to the mathematical formulations and the development of the appropriate mathematical solution techniques is crucial to solving differential equations and is particularly critical to teaching the solution of differential equations in applied mathematics courses. Solution methods cannot be universally applied but must be appropriate for the specific problem considered.

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