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**ABSTRACT:** The authors consider a conjecture by Chebyshev in 1853 on the distribution of odd primes among those that are one more than a multiple of four and those three more than a multiple of four—and use technology to explore the cardinality of these subsets. Generalizations are presented for student exploration along with several sources for more in-depth research.

**Keywords:** prime numbers, modular arithmetic, inquiry-based learning, computer science education

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# Introduction

Prime numbers have intrigued mathematicians for centuries and are essential in studying advanced mathematics topics such as abstract algebra and cryptography. Moreover, since primes are crucial building blocks of arithmetic, number theorists have compiled numerous prime facts over thousands of years (e.g., primes are infinite in number, the space between primes is arbitrary). Nonetheless, many properties of primes remain unknown or unproven, forming a robust area of inquiry and exploration for researchers and students.

This article explores Chebyshev's bias—a prime number topic first observed by Pafnuty Chebyshev, a 19th-century Russian mathematician. Specifically, Chebyshev noted that the odd prime numbers could be partitioned into two distinct subsets—those *one* more than a multiple of four (e.g., $13 = 4{\cdot}3{+}1$) and those *three* more than a multiple of four (e.g., $23 = 4 \cdot 5 + 3$). Moreover, he posited that there are more primes of the form $4k + 3$ than of the form $4k + 1$, up to the same limit. This phenomenon is commonly referred to as Chebyshev's Prime Bias Conjecture (Chebyshev, 1853). Over the years, many mathematicians have examined Chebyshev's bias from different perspectives; for example, see [ABG, FS, GM, Kim, RS].

In this paper, we discuss using GeoGebra to engage students in exploring Chebyshev's Prime Bias Conjecture. GeoGebra's visualization features and interactive applets enable students to investigate the distribution of odd primes between two subsets. Specifically, students can explore whether the odd prime numbers are evenly distributed between these two subsets using visual representations of the data. By analyzing patterns in the data and testing conjectures, students gain insights into the evenness of prime number distribution, similar to Chebyshev's work. The visualization capabilities and interactivity of GeoGebra applets make exploration more accessible.

We have organized this paper into four sections. Section 1 provides mathematical background, definitions, and notation for the activities demonstrated using GeoGebra. Key terms are provided in **boldface**. Readers familiar with set notation, modulo operations, congruence, and partitioning may safely skip this section. Section 2 provides a detailed overview of the odd primes activity and conjectures along with GeoGebra commands that could be used to look for patterns and generate conjectures. Section 3 provides a discussion of other potential explorations of prime number partitions, along with recommended sources for further research and teaching ideas. Finally, we state some conclusions in Section 4 and list our references. An appendix contains a list of the first 200 prime numbers.

# 1 Background

## 1.1 Partitions

In combinatorics, a **partition** of a nonempty set $S$ refers to the process of grouping the elements of $S$ into a finite union of nonempty, disjoint subsets in such a way that each element belongs to exactly one subset [Sta, p. 55]. For instance, the set $1, 2, 3, 4$ can be partitioned as the union of the two subsets $1, 4$ and $2, 3$, while $1 \cup 3 \cup 3, 4$ is another possible partition.

Infinite sets can be partitioned too, in fact, in infinitely many ways. For instance, the set of integers denoted $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ can be partitioned into positives, negatives, and zero. The set of natural numbers, $\mathbb{N} = \{1, 2, 3, 4, 5, 6, \dots\}$, is the set of evens, $\{2, 4, 6, 8, \dots\}$ and odds, $\{1, 3, 5, 7, \dots\}$. Furthermore, the set of odd numbers can be partitioned as:

$$\{1, 3, 5, 7, \dots\} = \{1, 5, 9, 13, 17, \dots\} \cup \{3, 7, 11, 15, \dots\}.$$

## 1.2 Prime Numbers

A **prime** number is a positive integer whose only divisors are 1 and itself. Let $\mathbb{P}$ denote the set of prime numbers. It is well-known that $\mathbb{P}$ is infinite [Hea]. The first ten prime numbers are $2, 3, 5, 7, 11, 13, 17, 19, 23$, and 29. A list of the first 200 primes are provided in Table 1 of the Appendix.

All prime numbers except 2 are odd and can be expressed as either one more or three more than a multiple of four. For example, $5, 13, 17, 29$, and 37 are all one more than a multiple of four (i.e., $4k + 1$). In particular, $5 = 4(1) + 1$ and $37 = 4(9) + 1$.

In contrast, the primes $3, 7, 11$, and 19 are three more than a multiple of four, specifically, $3 = 4(0) + 3, 7 = 4(1) + 3$, and $11 = 4(2) + 3$ and can be expressed as $4k + 3$ for some $k \in \mathbb{N} \cup \{0\}$. In general, the odd prime numbers can be partitioned into two classes based on their remainder when divided by four.

Specific to the current discussion, the odd prime numbers can be partitioned into two sets:

$$\mathbb{P} \setminus \{2\} = P_{1,4} \cup P_{3,4} \tag{1.1}$$

where $P_{1,4} = \{5, 13, 17, 29, 37, \dots\}$ and $P_{3,4} = \{3, 7, 11, 31, \dots\}$.

### 1.3 Congruence Modulo $n$

Let $x$ and $y$ be integers, and $m$ a positive integer. We say that $x$ is **congruent** to $y$ **modulo** $m$, and write

$$x \equiv y \pmod{m} \tag{1.2}$$

whenever $x - y$ is divisible by $m$ [Big]. For example, if $m = 4$, then $17 \equiv 1 \pmod{4}$ because $17 - 1 = 16$, which is divisible by 4. Similarly, $23 \equiv 3 \pmod{4}$. Using the notation in (1.1), we have

$$P_{1,4} = \{p \in \mathbb{P} \mid p \equiv 1 \pmod{4}\}$$

and

$$P_{3,4} = \{p \in \mathbb{P} \mid p \equiv 3 \pmod{4}\}.$$

### 1.4 GeoGebra

GeoGebra is a mathematical software that combines geometry and algebra in one platform. It allows users to create, manipulate and visualize mathematical objects such as points, lines, curves, and surfaces. GeoGebra is available on multiple platforms, including desktop, mobile, and web-based applications, making it accessible to users on various devices.

GeoGebra is a freely downloadable software. Its user-friendly interface makes it a good choice when working with students in entry-level courses. As a dynamic geometry software, GeoGebra enables users to explore relationships among mathematical objects in real time as they change variables, drag sliders, or other objects on the screen. This functionality promotes problem-posing and conjecturing among students. The software is versatile. Students can use it to create geometric structures, plot functions and data, resolve equations, and investigate mathematical ideas using simulations and animations. The GeoGebra user and developer community shares content and activities in multiple languages on their Resources page at `https://www.geogebra.org/materials`. We proceed to investigate the distribution of odd primes using a GeoGebra sketch that we've uploaded to the site.

### 1.5 Statement of the Problem

Are the odd prime numbers uniformly distributed among the partitions $P_{1,4}$ and $P_{3,4}$? In other words, for a given positive integer $n$, do the two partitions contain an equal number of odd primes? We will investigate this question with the aid of GeoGebra.

## 2 Methods

First, we define $N_1(n)$ as the number of odd primes $\leq n$ in the set $P_{1,4}$. Similarly, we define $N_3(n)$ as the number of odd primes $\leq n$ in set $P_{3,4}$. Lastly, we define the difference between these two values, $D(n) = N_3(n) - N_1(n)$. For example, $P_{1,4}$ contains only one element less than or equal to 11 (i.e., 5); $P_{3,4}$ contains three elements less than or equal to 11 (i.e., $3, 7$, and 11) making the difference of the types $D(11) = 3 - 1 = 2$.

## 2.1 Construction Protocols for Dynamic Sketch in GeoGebra

The steps used to construct the GeoGebra sketch we'll use for this exploration are as follows:

1. Hide all viewing windows except for Algebra view and Graphics view. Drag the Graphics view below the Algebra view and make the Input Bar visible.

2. Create a slider (number), $n$, ranging from 1 to 1000.

3. Define a sequence of prime numbers by entering the following command into GeoGebra's Input Bar:
```
Primes=RemoveUndefined[Sequence[If[IsPrime[k],k],k,3,n]]
```

4. Determine the length of the list of primes by entering the following command:
```
P_{ALL}=Length[Primes]
```

5. Next, generate a list of primes of Type 1 by entering the following command:
```
P_1=RemoveUndefined[Sequence[If[Mod[Element[Primes,k],4]==1,
  Element[Primes, k]],k,1,n]]
```

6. Similarly, enter the following command to generate a list of primes of Type 3:
```
P_3=RemoveUndefined[Sequence[If[Mod[Element[Primes,k],4]==3,
  Element[Primes, k]],k,1,n]]
```

7. Define the length of each of the sequences. `N_1=Length[T1]` and `N_3=Length[T3]`

Readers can create a sketch from scratch by implementing the above steps. Alternatively, a completed sketch is available at `https://tinyurl.com/primebiasorig`.

Figure 1 illustrates a finished sketch with slider, $n$, set to 76. In the sketch, $P_1$ and $P_3$ are two partitions with order $N_1$ and $N_3$, respectively. Note that the value of $n$ may be changed by dragging directly on the slider or typing a value directly into the text box immediately to the right.

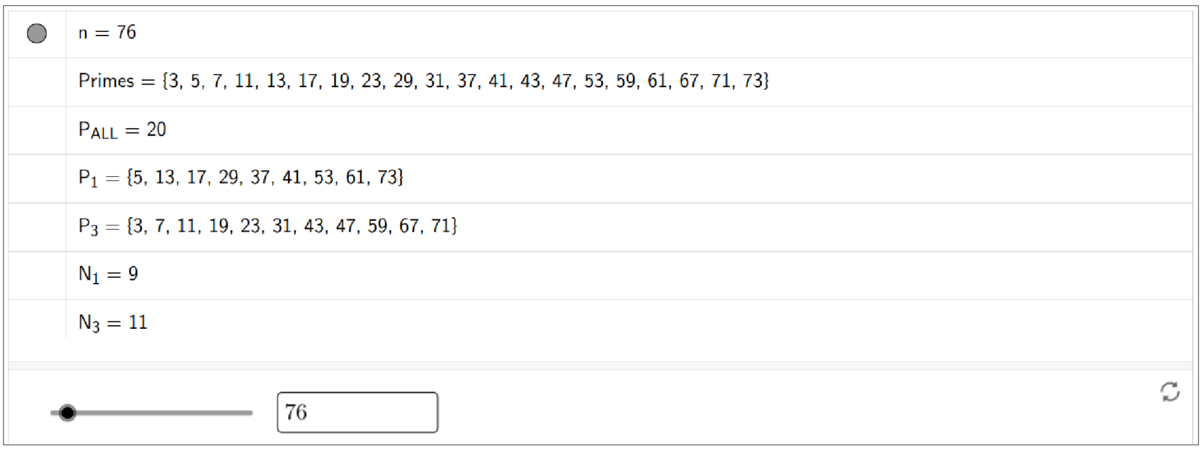


Figure 1: Prime bias GeoGebra sketch (available at `https://tinyurl.com/primebiasorig`).

## 2.2 A Curious Observation

With a dynamic sketch such as that provided in Figure 1, it is natural to explore various instances of $N_1$ and $N_3$, varying $n$ by dragging on the slider as suggested in Figure 2. In Figure 3, we share an alternate sketch we created to provide a more visual depiction of the same data. Students enter $n$ in the upper left corner. The numbers of elements in $P_1$ and $P_3$ are depicted as separate bars and as lists of values.

Note that for each instance of $n$ provided in Figures 1–3 (i.e., $76, 91, 145$), $P_3 > P_1$. It is natural to wonder if this is *always* so, and—moreover—*why* this appears to be the case.

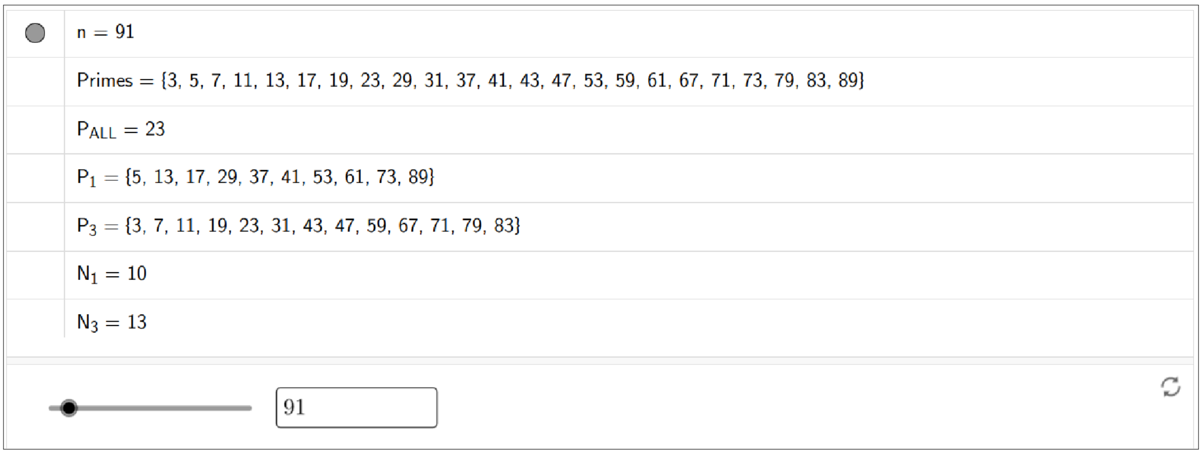


Figure 2: Exploring $P_1$ and $P_3$ with $n = 91$.

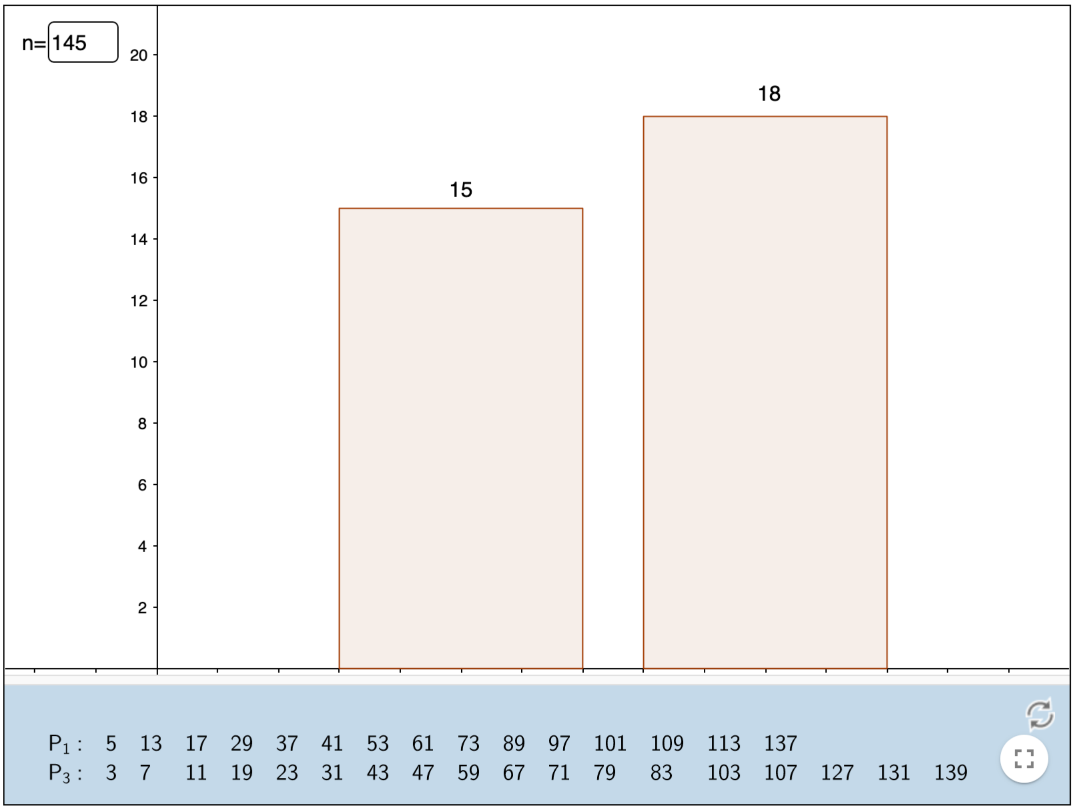


Figure 3: Exploring $P_1$ and $P_3$ with $n = 145$ (available at `https://tinyurl.com/primebiasvisual`).

### 2.3 Recording Data to a Spreadsheet

We revised our original GeoGebra sketch to explore such questions. Whenever the user clicks the "Record to Spreadsheet" button, values of $n$, $P_1$, and $P_3$ are recorded to a built-in GeoGebra spreadsheet. Then, users can drag the slider or type the desired $n$ into the text box to explore specific cases. For instance, in Figure 4, we see 12 primes in partition $P_1$ and 13 primes in partition $P_3$ when $n$ is set to 101.

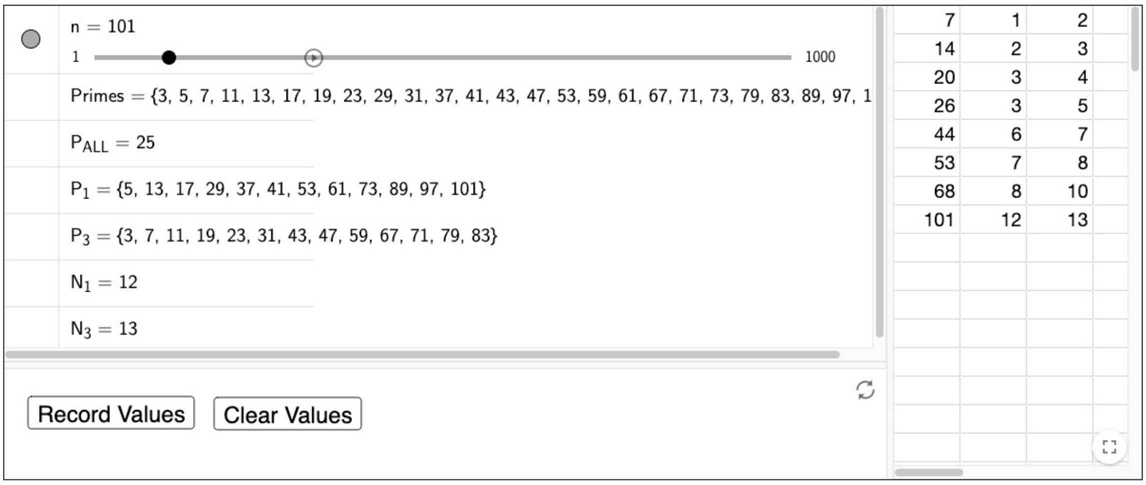


Figure 4: A revised GeoGebra sketch with values of $n$, (see `https://tinyurl.com/primebiasrevised`).

# 3 Discussion and Further Investigations

Chebyshev's Prime Bias Conjecture provides numerous opportunities for investigation by students (and researchers). For instance, *how often* is "more often than not?" For which values of $n$ does the Prime Bias Conjecture *not* hold? How common are these instances?

Several lines of inquiry are accessible to entry-level students. For example, the same question can be asked with another modulus, e.g., modulo 6 instead of modulo 4. Here are several other questions that we have explored with our students.

1. Is there an $n$ for which $n(P_{1,4}) = n(P_{3,4})$; for which $n(P_{1,4}) > n(P_{3,4})$?
2. Is there an $n$ with a gap distance of 7 (or $6, 15, 22, \ldots$)?
3. What is the distribution of gap distance as a function of $n$?
4. Is there an upper bound on the gap distance? Are there arbitrary gap distances, or are there never any primes modulo 4 of one type more than a given distance, say $x$, away?
5. What are the results modulo 6? Note: odd primes must be either $1, 3$ or $5 \pmod 6$.
6. Does the modulus have to be even, such as 4 or 6? Could it be odd, like 7 or 9?

An interesting question to consider relates to the prime factorization of numbers. We can designate numbers as either an "odd type" or "even type" based on the number of factors (including multiplicities). For instance, $12 = 2 \times 2 \times 3$, is "odd" since it has 3 non-distinct factors, whereas $24 = 2 \times 2 \times 2 \times 3$ is "even" since it has 4 non-distinct factors. Using this alternative definition of "odd" and "even," we can ask many of the same questions. For instance, is there an even distribution of "even/odd" types, or is there a bias toward one or the other type? Can you find a run of even types? Will it eventually go odd? Where does this occur? Can there be arbitrary length odd and even runs?

GeoGebra's `PrimeFactors` and `Dimension` commands are helpful when exploring our new designations. `PrimeFactors(n)` returns the list of primes whose product is equal to $n$. `Dimension(<list>)` returns the number of elements in a list. Combining the two commands, `Dimension(PrimeFactors(n))`, returns the number of factors of $n$ (including multiplicities). Figure 5 illustrates the command in action for our earlier examples (i.e., 12 and 24).

```
a = Dimension(PrimeFactors(12) → 3
b = Dimension(PrimeFactors(24) → 4
```

Figure 5: `Dimension` and `PrimeFactors` commands in GeoGebra.

## 4 Conclusion

Undergraduate research is a high-impact educational practice that promotes student achievement, advances intellectual growth, enhances problem-solving and communication skills, and increases retention rates among underrepresented groups [OK, PTS]. This is particularly relevant for mathematics, where open problems such as the prime bias conjecture provide opportunities for students to explore and engage in mathematical research. The prime bias conjecture is also an excellent topic for an entry-level computer science class project incorporating loops, conditional statements, arrays, and the modulo function to partition prime numbers and count their cardinality. The goal of such a project is not to produce a solution but to provide an impetus for further questioning, problem-posing, and conjecturing. Additionally, dynamic mathematics technologies like GeoGebra make mathematical research accessible to novice mathematicians, enabling them to explore authentic tasks [BOR].

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| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 |
| 31 | 37 | 41 | 43 | 47 | 53 | 59 | 61 | 67 | 71 |
| 73 | 79 | 83 | 89 | 97 | 101 | 103 | 107 | 109 | 113 |
| 127 | 131 | 137 | 139 | 149 | 151 | 157 | 163 | 167 | 173 |
| 179 | 181 | 191 | 193 | 197 | 199 | 211 | 223 | 227 | 229 |
| 233 | 239 | 241 | 251 | 257 | 263 | 269 | 271 | 277 | 281 |
| 283 | 293 | 307 | 311 | 313 | 317 | 331 | 337 | 347 | 349 |
| 353 | 359 | 367 | 373 | 379 | 383 | 389 | 397 | 401 | 409 |
| 419 | 421 | 431 | 422 | 439 | 443 | 449 | 457 | 461 | 463 |
| 467 | 479 | 487 | 491 | 499 | 503 | 509 | 521 | 523 | 541 |
| 547 | 557 | 563 | 569 | 571 | 577 | 587 | 593 | 599 | 601 |
| 607 | 613 | 617 | 619 | 631 | 641 | 643 | 647 | 653 | 659 |
| 661 | 673 | 677 | 683 | 691 | 701 | 709 | 719 | 727 | 733 |
| 739 | 743 | 751 | 757 | 761 | 769 | 773 | 787 | 797 | 809 |
| 811 | 821 | 823 | 827 | 829 | 839 | 853 | 857 | 859 | 863 |
| 877 | 881 | 883 | 887 | 907 | 911 | 919 | 929 | 937 | 941 |
| 947 | 953 | 967 | 971 | 977 | 983 | 991 | 997 | 1009 | 1013 |
| 1019 | 1021 | 1031 | 1033 | 1039 | 1049 | 1051 | 1061 | 1063 | 1069 |
| 1087 | 1091 | 1093 | 1097 | 1103 | 1109 | 1117 | 1123 | 1129 | 1151 |
| 1153 | 1163 | 1171 | 1181 | 1187 | 1193 | 1201 | 1213 | 1217 | 1223 |

Table 1: First 200 prime numbers listed left to right, top to bottom.

## Appendix - List of first 200 Prime Numbers

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