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From Zero to Epsilon: My Transformed Real Analysis Course

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ABSTRACT: In response to student evaluations I revised my undergraduate course in real analysis to a slides-and-worksheets model. This is the story of that revision, including why and how it was done, together with the results.

Keywords: Real analysis, slides, worksheets, student presentations

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1 Introduction

Would you be happy to receive this course evaluation?

> ...I felt as if I never truly understood the material. There needs to be more examples of the topics rather than confusing proofs and theorems ... expecting us to have unclear theorems memorized was too much, especially since the material was difficult already. I really didn't enjoy this class ... and I wish there was a better way to understand the course.

Neither would I. However, this was unfortunately one of the comments from my introductory course in real analysis in the spring of 2019. As disappointing as these remarks were, they were not altogether surprising. I had taught this subject many times, but had sensed of late that the course was beginning to fall flat. This student comment just served to further confirm my suspicions. It was becoming clear that my real analysis course had a problem; I needed to identify it and figure out how to fix it.

2 What was wrong?

I had always taught real analysis in a traditional manner, preparing my notes with great care and striving to present the material with enthusiasm and clarity. I collected and marked homework, met with students outside of class, and in general did everything that I thought a good professor should do. Further, the course had met with student approval. And yet now things were no longer working quite as they once had. Why? A closer look revealed two clear opportunities for improvement.

One was my tests, with which I had frankly always been dissatisfied; it was never clear to me just what to ask on a one-hour exam in a course of this nature. My existing tests had three sections, namely definitions, proofs of certain results we had demonstrated in class, and questions at large drawn from the course material. For both the definitions and the proofs, the students were given a list of items that were "fair game." This approach seemed reasonable enough; a central purpose of real analysis is to establish a body of theory, and so learning some of the proofs supporting that theory only made sense. However, in the case of the above student, at least, the result was just the opposite of what I was trying to achieve: "memorizing" the proofs of "unclear theorems" was if anything breeding confusion, not competence—and the experience was not enjoyable! Meanwhile the remaining portion of the test, the unscripted questions, posed its own set of problems. I found it difficult to write questions that were challenging enough to be worthwhile and yet accessible to the class as a whole within the time allotted.

A deeper issue, however, lay in the manner in which the course material was being introduced to begin with. Things were not getting across as intended, and the answer did not seem to lie simply in better exposition on the part of the instructor. The problem, rather, I came to believe, was that the students were watching *me* engage the subject in the classroom, rather than grapple with it themselves. It was as though I was giving them answers when they had not yet felt the weight of the questions. This is what I somehow needed to reverse. My students were good mathematics majors, but the format of the course made them effectively *spectators*, at least during class. Could I turn them instead into *researchers* who welcomed the theory of real analysis not as an alien world fraught with peril, but rather as the key to unlocking important problems which they had already tried to solve?

To pursue such a vision would doubtless entail wholesale changes in the course, and this in itself carried significant risk: I might pour countless hours into revising everything, only to see no improvement in my students' comprehension of real analysis. And yet, I could not shake the conviction that there was (to repurpose my student's phrase) "a better way to understand the course," a better way to get my young mathematicians from zero to epsilon.

3 Stealing a good idea

For some time I had noticed that a number of my colleagues at Baldwin Wallace University were using in-class worksheets as an integral part of their teaching. There were many variations, but the general idea was to use the board or slides to introduce a topic and then let the students take it from there, working collaboratively on the worksheets. The instructor was on hand to assist where needed. I knew these colleagues to be excellent teachers who got good results. Indeed, it was impossible not to admire the way in which their students left the classroom having gained first-hand experience with the material right on the spot; truth to tell, they even seemed to enjoy the process. But when it came to considering this approach for real analysis, I had many questions. Surely this way of doing things consumes a lot of class time; can we still make it through the syllabus? What exactly is the instructor's role under such a model? Where does one find time to generate all of the new materials needed? Further, the courses where I had seen this done were all calculus or below; could it work in an upper-division setting like real analysis?

Upon reflection, however, I began to see that (with one exception) each of these concerns actually presented an opportunity. For example, it occurred to me that often it was *I* who was guilty of wasting class time with my long explanations; perhaps those minutes could be better spent with the students at work on a well-chosen problem that gets the point across. And as the instructor, I would always play a central role in presenting the material, regardless of how the class was structured; after all, few of us are likely to independently formulate the Riemann integral or prove Taylor's theorem. But there are other important things that instructors do, such as address students' questions and help them solve problems, and in truth I had seldom fulfilled those aspects of the role to any great extent during class. Moreover, perhaps real analysis was in fact an ideal course for an approach like this; it could provide a perfect opportunity to mentor my budding mathematicians during class in a way that the traditional format would not allow. So one by one my objections seemed to fade and I decided to give the slides-and-worksheets model a try. (The exception: I was right to worry about the time required to create the requisite class materials. Fortunately, it need be done only once!)

4 Assembling the materials

Now it was time to lay out a detailed plan for the course and to get busy creating everything that would be needed. The only existing features of the course that I decided to leave in place were the textbook—Steven Lay's *Analysis with an Introduction to Proof*, whose exposition and approach I still admired—and the homework problem sets. All else would have to be built from scratch.

4.1 Slides and worksheets

The slides and worksheets used in class had to dovetail. In most cases I would use the slides to introduce a topic and then pass the baton to the students, who would develop the idea further *via* the worksheet. However, the reverse approach could also be effective: often, with the right questions, the worksheet could itself serve as the doorway into a new area. Further, the worksheets offered an opportunity to model mathematical exposition for my students, provided I published solutions for them. And no more handwritten notes on legal pads; instead, to set a professional example I used Beamer and LaTeX for everything. Thus, for any given section of the text I typically had three documents open at once, the slides, worksheet, and worksheet solution, all moving forward together. The result was a complete unit that would not only carry us through a section of the text but also serve as reference for my students.

When summed over the entire syllabus this was no small task, but it had the great virtue of forcing me to lay the whole course out on the table and think it through piece by piece. Exactly what did I want the next bullet point to say? Is this the right juncture at which to turn to the worksheet? What should I ask on the worksheet to best serve the purpose here? How should I break down this larger question into bite-sized pieces that will be accessible to the students? Very often, the act of writing out the solution to a question led me to revise the question itself: no, that's actually not what I wanted to ask, or the way in which I wanted to ask it; it should be this instead. And so on, idea by idea, point by point, throughout the course. Never, perhaps, had I passed introductory real analysis through such a fine sieve, nor thought this carefully about what would be going through the student's mind as each topic was developed.

The notoriously difficult topic of compactness furnishes a good test case. Powerful results such as the Heine-Borel, Bolzano-Weierstrass, and Nested Intervals Theorems are the supporting pillars of real analysis; on them rest such crucial results as the Extreme Value Theorem, the Mean Value Theorem (and thus the "obvious" fact that functions with positive derivatives are increasing), Taylor's Theorem, the Fundamental Theorem of Calculus, and more. Yet beginners find the scaffolding that these pillars require, namely the open cover definition of compactness, forbidding to say the least. How should I deploy my slides and worksheets to introduce my students to this strange definition? I decided on a simple opening slide, with just four points:

- Let S be a subset of $\mathbb{R}$.
- A family $\mathscr{F}$ of open sets whose union contains S is called an *open cover* of S.
- A family $\mathscr{G} \subseteq \mathscr{F}$ of open sets whose union also contains S is called a *subcover* of S.
- If the family $\mathscr{G}$ happens to contain only *finitely many* open sets, then $\mathscr{G}$ is called a *finite subcover* of S.

With that, we turned to the worksheet, where we could put these definitions immediately into practice. Here are the first few questions:

1. The notion of compactness turns on whether, for a given subset S of $\mathbb{R}$, *every* open cover of S contains a finite subcover. In the case where S is the open interval $(0, 1)$, we will see below that *some* open covers have finite subcovers, but not all do. Therefore, S is not compact.
   (a) Let $\mathscr{F}$ be the collection of intervals $\{(-n, n) : n \in \mathbb{N}\}$. Explain why $\mathscr{F}$ is an open cover for $S = (0, 1)$.
   (b) Explain why $\mathscr{F}$ has a finite subcover. That is, find a finite collection $\mathscr{G} \subseteq \mathscr{F}$ such that the union of all of the open intervals in $\mathscr{G}$ still covers S. (You don't have to look too hard.)
   (c) Now let's look at a different open cover of S. Define $\mathscr{F}$ to be the collection of intervals $\left\{\left(0, 1 - \dfrac{1}{n}\right) : n \in \mathbb{N}\right\}$. Explain why this choice of $\mathscr{F}$ is also an open cover for S.
   (d) Explain why this open cover $\mathscr{F}$ for S has no finite subcover.
2. Show that the interval $[0, \infty)$ is not compact by finding an open cover of $[0, \infty)$ that has no finite subcover.

The students were at first quiet as they read the questions. Then gradually they began to compare notes, ask each other questions, look at me inquiringly, go to the board to write things out, etc. The students' initial answers were "diamonds in the rough." For example, the "finite subcovers" called for in questions 1(b) and (d) above are not just sets, but *collections* of sets, introducing a new level of subtlety of notation. Likewise, in addressing question 2, it took a while for the students to understand just what they were supposed to find, namely, a collection of open sets in $\mathbb{R}$ which, *as a collection*, had a certain property. However, in the end, everything came together: at least at the level of these questions, the class came to an understanding of the open cover definition of compact set.

Whereas these worksheet questions are straightforward, they are the fruit of careful thought about the definition of compactness and how my students could be led step-by-step to apply it. The crucial issue, as noted above, is whether *every* open cover of the given set has a finite subcover. To put this in bold relief I needed to place an open cover that does have a finite subcover alongside one that does not. The format and wording of the questions were likewise important; I felt a great responsibility to express things well and to use notation carefully. Not only did I want to set an example of quality mathematical exposition in all of the course materials, but I also wanted to make sure that the only difficulties the class encountered were mathematical, not the result of poor presentation.

4.2 Tests

For the tests I borrowed a page from the pandemic and used a take-home rather than in-class format. This allowed me to ask questions that I would likely not have posed on a traditional test, for example:

- Let S denote the set of all infinite sequences of 0's and 1's, such as $\{1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, \ldots\}$, for example. Prove that S is uncountable.
- Prove that $\operatorname{cl}(S \cap T) \subseteq (\operatorname{cl} S) \cap (\operatorname{cl} T)$ and find an example to show that equality need not hold.
- Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x$ if x is rational and $f(x) = 0$ if x is irrational. Prove that f has a limit at c iff $c = 0$.

With the class being small ($n = 5$) and the students being required to sign an honor pledge, I had little concern about security. Also I accepted questions about the test, and indeed these conversations were the occasion, I believe, of some of the deepest learning that took place all term. Yes, this was a test, but it was also another opportunity to help my students grow as mathematicians, one that I eagerly seized.

4.3 Final presentations

The small class size also allowed for another innovation, that is, to devote the final exam period to student presentations rather than to a traditional exam. This would require the students to learn and publicaly exposit a topic in real analysis. (As it turned out, the "public" included departmental colleagues, who enlivened the proceedings with friendly yet challenging questions for the presenters!) The quality of the presentations led me to declare the experiment a success. However, this gambit was not without cost: I had to develop a reasonably full menu of potential topics and supporting reference materials for this assignment. These topics had to be carefully chosen so as to be accessible while articulating well with the course material.

My general specifications for the talks were basic:

- The topic of the presentation can draw from any aspect of real analysis, as long as you include statements and proofs of significant results.
- Please use Beamer to create slides supporting your presentation.
- Aim for a duration of half an hour.

Next, here are a few of the ten sample topics that I provided:

- **Metric spaces.** A *metric* on a set is a function satisfying certain assumptions that gives the "distance" between any two elements of the set. The set $\mathbb{R}$ of real numbers with the usual metric $d(x, y) = |x - y|$ is probably the most familiar metric space, but there are many others. We have defined topological notions such as neighborhood, open set, etc. in $\mathbb{R}$ using distance, and therefore we can do the same in metric spaces.

  Goals for this presentation:

  - Define metric space and give examples.
  - Define topological terms such as neighborhood, open set, boundary, etc. for metric spaces.
  - Prove that any neighborhood of a point of a metric space is an open set.
  - Give an example to show that in a metric space, a set can be closed and bounded without being compact.

- **The Peano Axioms.** The Peano axioms represent an early, but important, attempt to axiomatize the natural numbers, and from there the rational and real number systems.

  Goals for this presentation:

  - Set forth the Peano axioms and explain what each one says.
  - Explain the definition of addition.
  - Answer the questions in Exercise 33, Section 3.1 of our text. (Note the hints and commentary in the back of the book.)
  - Explain the definition of multiplication.
  - Prove that multiplication is commutative.

- **The Cantor Set.** The Cantor set, a subset of the closed interval $[0, 1]$, is celebrated for the counterintuitive phenomena that it manifests. As just one example, the Cantor set is uncountable, and yet its complement consists of intervals the sum of whose lengths is 1. This set arose from Cantor's efforts to determine the set of points for which Fourier series converge.

  Goals for this presentation:

- Define the Cantor set.
- Use drawings to illustrate as best you can how the set is constructed.
- Complete parts (a)-(e) of Exercise 11 following Section 3.5 of our text.
- If you have time, then also prove that the Cantor set is a *perfect set*, that is, it contains all of its accumulation points.

- **The Riemann Rearrangement Theorem.** Convergent series of constants divide into two very different types: absolutely convergent and conditionally convergent. One striking difference between the two is that an absolutely convergent series converges to the same sum even if the terms are *rearranged*, that is, taken in a different order, whereas really the very opposite is true for a conditionally convergent series: given any real number as a target, we can always rearrange the terms of a conditionally convergent series so as to converge to that number. This latter fact is known as the Riemann rearrangement theorem, and its proof is outlined in Exercises 15-17(a) following Section 8.2 of the text.

  Goals for this presentation:

  - Review the definitions of absolute and conditional convergence of series.
  - Carry out Exercises 15-17(a) following Section 8.2 of our text.

The students all adopted one of the suggested topics, even if in theory they were free to scour the literature and come up with ideas of their own. They more than fulfilled my expectations by putting great thought and care into their presentations and taking full ownership of their talks. One student, for example, chose the Cantor set C as his topic, covering the subject admirably and creating a graphic in support of his proof that C contains no interval (Figure 1). He argued that if c is any positive number, then no matter how small c is, we can find a positive integer k with $1/3^k < c$; it follows from the "middle-thirds" definition of C that C cannot contain an interval of length c.

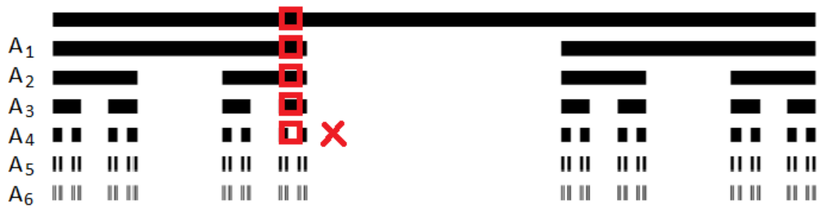

Figure 1: The Cantor set C. The red boxes represent an interval alleged to lie in C.

Although a traditional final exam would of course have been a perfectly acceptable option, the sight of my young mathematicians taking the stage and demonstrating mastery of their chosen topics left me with no regrets over how we used the final exam period.

5 Conclusion: Was it a success?

From my perspective, this course revision brought a level of classroom engagement that I had rarely seen using traditional instruction. In the past I taught and hoped that good things would happen; now I had the privilege of *watching* good things happen, class after class, as my students wrestled with the subject of real analysis. In fact, I now use this slides-and-worksheets approach in some of my other courses as well, and have been similarly pleased with the results.

But what did the students think about all this? What were the course evaluations like this time around? Well, I am grateful to say that they were all favorable. However, I am going to give the last word to one class member in particular, who in three sentences captured and affirmed everything that I had hoped that this transformed real analysis course would accomplish. In many ways, this evaluation made it all worth it:

> The structure of group-work made the class incredibly interesting. We each had to consistently work hard to both understand the material, and be able to clearly explain it to others. The assignments were challenging, but being able to discuss them in class when we were having difficulties made it a very rewarding challenge.

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