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ABSTRACT: Undergraduate research in tertiary education offers mathematics students a pathway to engage in high-impact practices. Using a simple sum equals product identity from number theory as a motivator, we build a series of inclusion-exclusion identities for convex polygons using the symmetry inherent in the tangent function. The techniques used are simple and accessible, illuminating and generalizable, in a manner that rejects a singular line of inquiry in favor of a plurality of mathematical ideas.

Keywords: Sum equals product, tangent function, inclusion-exclusion identities

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Introduction

Mathematics educators have long recognized the importance of introducing the right blend of research opportunities for students by creating lines of inquiry that connect different branches of mathematics. Utilizing small teaching techniques, the necessary scaffolding can be built to connect diverse areas of mathematics. The critical interconnections that get developed between different mathematical concepts can be used to create pathways to uncover the richness of mathematics, broaden access, and promote equity. This article explores connections between two areas of mathematics to develop a line of inquiry that leads to an inclusion-exclusion identity which explores the symmetry inherent in the tangent function. In Section 1 we introduce a tangent identity for convex polygons, motivated by a result in number theory, and develop two illuminating Lemmas for odd and even cases. The inclusion-exclusion identities are developed in Section 2. The ideas presented in this article should be accessible to upper level undergraduates in mathematics who have completed courses in trigonometry and discrete mathematics.

1 A tangent identity for polygons

It is known that the *sum equals product* equation $a_1 + a_2 + \cdots + a_n = a_1 a_2 \cdots a_n$ has positive integer solutions. For example,

$$a_i = 1 \text{ for } 1 \leq i \leq n-2, a_{n-1} = 2, a_n = n,$$

is a solution to the equation [BR, GU]. It has been shown that $n = 2, 3, 4, 6, 24, 114, 174$, and 444 are the only numbers $n < 1000$ for which the above is the only solution (up to permutations) [MI]. This result has been extended to $n \leq 10^{11}$ [WE]. To see that a case such as $n = 5$ admits other positive integer solutions, observe that by setting $a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 2, a_5 = 2$ we obtain a solution different from the first set. Positive integer solutions can be constructed by padding as many 1's as necessary, since these do not alter the product [EC]. The notion of exceptional values of the positive integer solutions have also been studied [EC]. Of course, if we allow for non-integer solutions, there are other interesting solutions one can consider. For example, $a_i = n^{\frac{1}{n-1}}$ for $i = 1, \cdots, n$ is a solution to the sum equals product equation.

The *sum equals product* identity has provided the author viable avenues to inspire undergraduate research projects [KA]. Undergraduate research helps promote excellence in tertiary education by offering opportunities for students in the upper end of the mathematics spectrum to engage in high-impact practices [PR] . In this article, we attempt to juxtapose a line of inquiry from one branch of mathematics (number theory) with that of another branch of mathematics (trigonometry), to challenge ourselves to discover a pathway to a generalized result. The path of inquiry into the *sum equals product* equation becomes engrossing and illuminating if trigonometry enters the landscape.

Tangent identities in trigonometry have elegant structure. For the $n = 3$ case, a simple trigonometric identity for a plane triangle provides yet another non-integer solution to the *sum equals product* identity. Let the interior angles of the triangle be α_1, α_2, and α_3. The following identity connects the sum of tangents to their product. Namely,

$$\sum_{i=1}^{3} \tan \alpha_i = \tan \alpha_1 \tan \alpha_2 \tan \alpha_3. \tag{1.1}$$

A proof uses the simple sum formula for tangent,

$$\tan(\alpha_1 + \alpha_2) = \frac{\tan \alpha_1 + \tan \alpha_2}{1 - \tan \alpha_1 \cdot \tan \alpha_2}.$$

Indeed, using $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$ as the domain of tangent, an elementary proof can be established, starting with the interior angle formula for a triangle: $\alpha_1 + \alpha_2 + \alpha_3 = \pi$,

$$\begin{aligned}
\tan(\alpha_1 + \alpha_2 + \alpha_3) &= 0, \\
\frac{\tan(\alpha_1 + \alpha_2) + \tan \alpha_3}{1 - \tan(\alpha_1 + \alpha_2) \tan \alpha_3} &= 0, \\
\frac{\frac{\tan \alpha_1 + \tan \alpha_2}{1 - \tan \alpha_1 \tan \alpha_2} + \tan \alpha_3}{1 - \frac{\tan \alpha_1 + \tan \alpha_2}{1 - \tan \alpha_1 \tan \alpha_2} \tan \alpha_3} &= 0, \\
\frac{\sum_{i=1}^{3} \tan \alpha_i - \tan \alpha_1 \tan \alpha_2 \tan \alpha_3}{1 - \sum_{i_1=1}^{2} \sum_{i_2>i_1}^{3} \tan \alpha_{i_1} \tan \alpha_{i_2}} &= 0.
\end{aligned}$$

The last statement establishes identity (1.1). Note that if one of the interior angles equals $\frac{\pi}{2}$, then (1.1) holds true in the limit sense.

At the same time, we obtain the following useful identity for $\tan\left(\sum_{i=1}^{3} \alpha_i\right)$.

$$\tan\left(\sum_{i=1}^{3} \alpha_i\right) = \frac{\sum_{i=1}^{3} \tan \alpha_i - \tan \alpha_1 \tan \alpha_2 \tan \alpha_3}{1 - \sum_{i_1=1}^{2} \sum_{i_2>i_1}^{3} \tan \alpha_{i_1} \tan \alpha_{i_2}} \tag{1.2}$$

Next let us also explore the case $n = 4$. A convex quadrilateral in a plane has the property that all of its diagonals lie entirely inside of it. When we consider a plane convex quadrilateral, with interior angles $\alpha_1, \alpha_2, \alpha_3$, and α_4, the identity that connects the sum of tangents to their product slightly changes its form. Namely,

$$\begin{aligned}
\sum_{i=1}^{4} \tan \alpha_i &= \sum_{i_1=1}^{2} \sum_{i_2>i_1}^{3} \sum_{i_3>i_2}^{4} \tan \alpha_{i_1} \tan \alpha_{i_2} \tan \alpha_{i_3} \qquad (1.3) \\
&= \tan \alpha_1 \tan \alpha_2 \tan \alpha_3 + \tan \alpha_1 \tan \alpha_2 \tan \alpha_4 + \tan \alpha_2 \tan \alpha_3 \tan \alpha_4
\end{aligned}$$

Using $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$ as the domain of tangent, the proof follows a similar pattern as before, starting with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2\pi$,

$$\begin{aligned}
\tan(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) &= 0, \\
\frac{\tan(\alpha_1 + \alpha_2) + \tan(\alpha_3 + \alpha_4)}{1 - \tan(\alpha_1 + \alpha_2) \tan(\alpha_3 + \alpha_4)} &= 0, \\
\frac{\frac{\tan \alpha_1 + \tan \alpha_2}{1 - \tan \alpha_1 \tan \alpha_2} + \frac{\tan \alpha_3 + \tan \alpha_4}{1 - \tan \alpha_3 \tan \alpha_4}}{1 - \frac{\tan \alpha_1 + \tan \alpha_2}{1 - \tan \alpha_1 \tan \alpha_2} \cdot \frac{\tan \alpha_3 + \tan \alpha_4}{1 - \tan \alpha_3 \tan \alpha_4}} &= 0, \\
\frac{\sum_{i=1}^{4} \tan \alpha_i - \sum_{i_1=1}^{2} \sum_{i_2>i_1}^{3} \sum_{i_3>i_2}^{4} \tan \alpha_{i_1} \tan \alpha_{i_2} \tan \alpha_{i_3}}{1 - \sum_{i_1=1}^{3} \sum_{i_2>i_1}^{4} \tan \alpha_{i_1} \tan \alpha_{i_2} + \tan \alpha_1 \tan \alpha_2 \tan \alpha_3 \tan \alpha_4} &= 0.
\end{aligned}$$

The last statement establishes identity (1.3).

At the same time, we obtain the following identity for $\tan\left(\sum_{i=1}^{4}\alpha_i\right)$.

$$\tan\left(\sum_{i=1}^{4}\alpha_i\right) = \frac{\sum_{i=1}^{4}\tan\alpha_i - \sum_{i_1=1}^{2}\sum_{i_2>i_1}^{3}\sum_{i_3>i_2}^{4}\tan\alpha_{i_1}\tan\alpha_{i_2}\tan\alpha_{i_3}}{1-\sum_{i_1=1}^{3}\sum_{i_2>i_1}^{4}\tan\alpha_{i_1}\tan\alpha_{i_2}+\tan\alpha_1\tan\alpha_2\tan\alpha_3\tan\alpha_4} \tag{1.4}$$

If we want to explore the generalization of identity (1.1) and identity (1.3) for other closed convex polygons, it is desirable to separate the odd and even cases. A closer inspection of identity (1.2) and identity (1.4) reveals an inclusion-exclusion principle for the tangent products. We begin with two results that generalize identities (1.2) and (1.4). The proofs of these generalized results require mathematical induction, the details of which present opportunities to explore nuanced techniques, in much the same way as we can study fruit in a bowl. Before we do so, let us introduce new notation to simplify several results to follow.

For convenience we define a n-tuple notation to capture products of tangents. Indeed, define the products as follows.

$$\begin{aligned}(\alpha_i) &= \tan\alpha_i\\ (\alpha_i,\alpha_j) &= \tan\alpha_i\tan\alpha_j\\ &\vdots\\ (\alpha_1,\alpha_2,\cdots,\alpha_n) &= \tan\alpha_1\tan\alpha_2\cdots\tan\alpha_n.\end{aligned}$$

Lemma 1. *For any **odd** set of angles $\alpha_1,\alpha_2,\cdots,\alpha_{2n-1}$, with $n\geq 2$, the following identity is true.*

$$\tan\left(\sum_{i=1}^{2n-1}\alpha_i\right) = \frac{\sum_{i=1}^{2n-1}(\alpha_i) - \sum_{i_1=1}^{2n-3}\sum_{i_2>i_1}^{2n-2}\sum_{i_3>i_2}^{2n-1}(\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}) + \sum_{i_1=1}^{2n-5}\sum_{i_2>i_1}^{2n-4}\sum_{i_3>i_2}^{2n-3}\sum_{i_4>i_3}^{2n-2}\sum_{i_5>i_4}^{2n-1}(\alpha_{i_1},\alpha_{i_2},\alpha_{i_3},\alpha_{i_4},\alpha_{i_5}) - \cdots + (-1)^{n-1}(\alpha_1,\alpha_2,\cdots,\alpha_{2n-2},\alpha_{2n-1})}{1-\sum_{i_1=1}^{2n-2}\sum_{i_2>i_1}^{2n-1}(\alpha_{i_1},\alpha_{i_2}) + \sum_{i_1=1}^{2n-4}\sum_{i_2>i_1}^{2n-3}\sum_{i_3>i_2}^{2n-2}\sum_{i_4>i_3}^{2n-1}(\alpha_{i_1},\alpha_{i_2},\alpha_{i_3},\alpha_{i_4}) - \cdots + (-1)^{n-1}\sum_{i_1=1}^{2}\sum_{i_2>i_1}^{3}\cdots\sum_{i_{2n-3}>i_{2n-4}}^{2n-2}\sum_{i_{2n-2}>i_{2n-3}}^{2n-1}(\alpha_{i_1},\alpha_{i_2},\cdots,\alpha_{i_{2n-3}},\alpha_{i_{2n-2}})}$$

Proof. By induction.

The case $n=2$, holds true from (1.2). Assume the result for $2n-1$. Then,

$$\begin{aligned}&\tan\left(\sum_{i=1}^{2n+1}\alpha_i\right)\\ =\ &\tan\left[\left(\sum_{i=1}^{2n-1}\alpha_i\right)+\alpha_{2n}+\alpha_{2n+1}\right]\end{aligned}$$

Simplify the above by using the sum of tangents for two angles.

$$\tan\left(\sum_{i=1}^{2n+1}\alpha_i\right)$$

$$= \frac{\tan\left(\sum_{i=1}^{2n-1}\alpha_i\right)+\tan(\alpha_{2n}+\alpha_{2n+1})}{1-\tan\left(\sum_{i=1}^{2n-1}\alpha_i\right)\tan(\alpha_{2n}+\alpha_{2n+1})}$$

$$= \frac{\boxed{\tan\left(\sum_{i=1}^{2n-1}\alpha_i\right)}+\frac{\tan\alpha_{2n}+\tan\alpha_{2n+1}}{1-\tan\alpha_{2n}\tan\alpha_{2n+1}}}{1-\boxed{\tan\left(\sum_{i=1}^{2n-1}\alpha_i\right)}\frac{\tan\alpha_{2n}+\tan\alpha_{2n+1}}{1-\tan\alpha_{2n}\tan\alpha_{2n+1}}}$$

By substituting for the term in the "box" using the inductive assumption and after a considerable amount of careful simplifications, we obtain the following:

$$\tan\left(\sum_{i=1}^{2n+1}\alpha_i\right)$$

$$= \frac{\sum_{i=1}^{2n+1}(\alpha_i)-\sum_{i_1=1}^{2n-1}\sum_{i_2>i_1}^{2n}\sum_{i_3>i_2}^{2n+1}(\alpha_{i_1},\alpha_{i_2},\alpha_{i_3})+\sum_{i_1=1}^{2n-3}\sum_{i_2>i_1}^{2n-2}\sum_{i_3>i_2}^{2n-1}\sum_{i_4>i_3}^{2n}\sum_{i_5>i_4}^{2n+1}(\alpha_{i_1},\alpha_{i_2},\alpha_{i_3},\alpha_{i_4},\alpha_{i_5}) - \cdots + (-1)^n(\alpha_1,\alpha_2,\cdots,\alpha_{2n},\alpha_{2n+1})}{1-\sum_{i_1=1}^{2n}\sum_{i_2>i_1}^{2n+1}(\alpha_{i_1},\alpha_{i_2})+\sum_{i_1=1}^{2n-2}\sum_{i_2>i_1}^{2n-1}\sum_{i_3>i_2}^{2n}\sum_{i_4>i_3}^{2n+1}(\alpha_{i_1},\alpha_{i_2},\alpha_{i_3},\alpha_{i_4})-\cdots+\cdots+(-1)^n\sum_{i_1=1}^{2}\sum_{i_2>i_1}^{3}\cdots\sum_{i_{2n-2}>i_{2n-3}}^{2n-1}\sum_{i_{2n-1}>i_{2n-2}}^{2n}(\alpha_{i_1},\alpha_{i_2},\cdots,\alpha_{i_{2n-2}},\alpha_{i_{2n-1}})}$$

This establishes the inductive step. □

Lemma 2. *For any **even** set of angles $\alpha_1, \alpha_2, \cdots, \alpha_{2n}$, with $n \geq 2$, the following identity is true.*

$$\tan\left(\sum_{i=1}^{2n}\alpha_i\right)$$

$$= \frac{\sum_{i=1}^{2n}(\alpha_i)-\sum_{i_1=1}^{2n-2}\sum_{i_2>i_1}^{2n-1}\sum_{i_3>i_2}^{2n}(\alpha_{i_1},\alpha_{i_2},\alpha_{i_3})+\sum_{i_1=1}^{2n-4}\sum_{i_2>i_1}^{2n-3}\sum_{i_3>i_2}^{2n-2}\sum_{i_4>i_3}^{2n-1}\sum_{i_5>i_4}^{2n}(\alpha_{i_1},\alpha_{i_2},\alpha_{i_3},\alpha_{i_4},\alpha_{i_5}) - \cdots + (-1)^{n-1}\sum_{i_1=1}^{2}\sum_{i_2>i_1}^{3}\cdots\sum_{i_{2n-2}>i_{2n-3}}^{2n-1}\sum_{i_{2n-1}>i_{2n-2}}^{2n}(\alpha_{i_1},\alpha_{i_2},\cdots,\alpha_{i_{2n-2}},\alpha_{i_{2n-1}})}{1-\sum_{i_1=1}^{2n-1}\sum_{i_2>i_1}^{2n}(\alpha_{i_1},\alpha_{i_2})+\sum_{i_1=1}^{2n-3}\sum_{i_2>i_1}^{2n-2}\sum_{i_3>i_2}^{2n-1}\sum_{i_4>i_3}^{2n}(\alpha_{i_1},\alpha_{i_2},\alpha_{i_3},\alpha_{i_4}) - \cdots + (-1)^{n-1}(\alpha_1,\alpha_2,\cdots,\alpha_{2n-1},\alpha_{2n})}$$

Proof. By induction.

The case $n = 2$, holds true from (1.4). Assume the result for $2n$. Then,

$$\begin{aligned} &\tan\left(\sum_{i=1}^{2n+2}\alpha_i\right) \\ = \quad &\tan\left[\left(\sum_{i=1}^{2n}\alpha_i\right) + \alpha_{2n+1} + \alpha_{2n+2}\right] \end{aligned}$$

Simplify the above by using the sum of tangents for two angles.

$$\begin{aligned} &\tan\left(\sum_{i=1}^{2n+2}\alpha_i\right) \\ = \quad &\frac{\tan\left(\sum_{i=1}^{2n}\alpha_i\right) + \tan(\alpha_{2n+1}+\alpha_{2n+2})}{1 - \tan\left(\sum_{i=1}^{2n}\alpha_i\right)\tan(\alpha_{2n+1}+\alpha_{2n+2})} \\ = \quad &\frac{\boxed{\tan\left(\sum_{i=1}^{2n}\alpha_i\right)} + \frac{\tan\alpha_{2n+1}+\tan\alpha_{2n+2}}{1-\tan\alpha_{2n+1}\tan\alpha_{2n+2}}}{1 - \boxed{\tan\left(\sum_{i=1}^{2n}\alpha_i\right)}\frac{\tan\alpha_{2n+1}+\tan\alpha_{2n+2}}{1-\tan\alpha_{2n+1}\tan\alpha_{2n+2}}} \end{aligned}$$

By substituting for the term in the "box" using the inductive assumption and, as before, after careful simplifications, we obtain the following:

$$\begin{aligned} &\tan\left(\sum_{i=1}^{2n+2}\alpha_i\right) \\ = \; &\frac{\begin{array}{c}\sum_{i=1}^{2n+2}(\alpha_i) - \sum_{i_1=1}^{2n}\sum_{i_2>i_1}^{2n+1}\sum_{i_3>i_2}^{2n+2}(\alpha_{i_1},\alpha_{i_2},\alpha_{i_3}) + \sum_{i_1=1}^{2n-2}\sum_{i_2>i_1}^{2n-1}\sum_{i_3>i_2}^{2n}\sum_{i_4>i_3}^{2n+1}\sum_{i_5>i_4}^{2n+2}(\alpha_{i_1},\alpha_{i_2},\alpha_{i_3},\alpha_{i_4},\alpha_{i_5}) \\ - \cdots + (-1)^n\sum_{i_1=1}^{2}\sum_{i_2>i_1}^{3}\cdots\sum_{i_{2n}>i_{2n-1}}^{2n+1}\sum_{i_{2n+1}>i_{2n}}^{2n+2}(\alpha_{i_1},\alpha_{i_2},\cdots,\alpha_{i_{2n}},\alpha_{i_{2n+1}})\end{array}}{\begin{array}{c}1 - \sum_{i_1=1}^{2n+1}\sum_{i_2>i_1}^{2n+2}(\alpha_{i_1},\alpha_{i_2}) + \sum_{i_1=1}^{2n-1}\sum_{i_2>i_1}^{2n}\sum_{i_3>i_2}^{2n+1}\sum_{i_4>i_3}^{2n+2}(\alpha_{i_1},\alpha_{i_2},\alpha_{i_3},\alpha_{i_4}) - \cdots + \\ \cdots + (-1)^{n-1}(\alpha_1,\alpha_2,\cdots,\alpha_{2n+1},\alpha_{2n+2})\end{array}} \end{aligned}$$

This establishes the inductive step. □

2 Inclusion-exclusion identities

The next two propositions generalize the identities in (1.1) and (1.3) for odd and even convex polygons respectively. These two results provide an inclusion-exclusion identity for the sum of tangents of the interior angles of a convex polygon, using odd powers of products of tangents. Before we establish these two propositions, it is worth noting how the product terms of tangent on the right-hand-side are arranged in Lemma 1 and Lemma 2. In both odd and even convex polygons the product of tangent terms in the numerator have an odd number of products in each summand, while tangent terms in the denominator have an even number of products in each summand. We have summarized the types of product terms in Table 1 by referring to them as single, double, triple, ..., $n-1$ terms, n terms etc.

# of angles	Numerator (types of tan term)	Denominator (types of tan term)
3	1 (single), 3 (triple product)	2 (double product)
4	$1, 3$	$2, 4$
5	$1, 3, 5$	$2, 4$
6	$1, 3, 5$	$2, 4, 6$
7	$1, 3, 5, 7$	$2, 4, 6$
8	$1, 3, 5, 7$	$2, 4, 6, 8$
9	$1, 3, 5, 7, 9$	$2, 4, 6, 8$
10	$1, 3, 5, 7, 9$	$2, 4, 6, 8, 10$
$\vdots$	$\vdots$	$\vdots$
$2n-1$	$1, 3, \cdots, 2n-1$ (n terms)	$2, 4, \cdots, 2n-2$ ($n-1$ terms)
$2n$	$1, 3, \cdots, 2n-1$ (n terms)	$2, 4, \cdots, 2n-2, 2n$ (n terms)

Table 1: Arrangement of tangent products in Lemma 1 & 2

Proposition 3. *For any* **odd** *sided convex polygon with interior angles $\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}$, with $n \geq 2$, the following inclusion-exclusion identity holds true.*

$$\begin{aligned}&\sum_{i=1}^{2n-1} \tan\alpha_i \\ = \;&\sum_{i_1=1}^{2n-3}\sum_{i_2>i_1}^{2n-2}\sum_{i_3>i_2}^{2n-1}(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}) - \sum_{i_1=1}^{2n-5}\sum_{i_2>i_1}^{2n-4}\sum_{i_3>i_2}^{2n-3}\sum_{i_4>i_3}^{2n-2}\sum_{i_5>i_4}^{2n-1}(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}, \alpha_{i_4}, \alpha_{i_5}) \\ &+ \cdots + (-1)^n(\alpha_1, \alpha_2, \cdots, \alpha_{2n-2}, \alpha_{2n-1}).\end{aligned}$$

Proof. The interior angles $\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}$, with $n \geq 2$ satisfy $\sum_{i=1}^{2n-1} \alpha_i = (2n-3)\pi$. Applying the tangent function to both sides, with domain as $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$ for each interior angle, we have $\tan\left(\sum_{i=1}^{2n-1} \alpha_i\right) = 0$. The inclusion-exclusion identity for an odd sided convex polygon follows from Lemma 1. □

Proposition 4. *For any* **even** *sided convex polygon with interior angles $\alpha_1, \alpha_2, \cdots, \alpha_{2n}$, with $n \geq 2$, the following inclusion-exclusion identity holds true.*

$$\begin{aligned}&\sum_{i=1}^{2n} \tan\alpha_i \\ &= \sum_{i_1=1}^{2n-2}\sum_{i_2>i_1}^{2n-1}\sum_{i_3>i_2}^{2n}(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}) - \sum_{i_1=1}^{2n-4}\sum_{i_2>i_1}^{2n-3}\sum_{i_3>i_2}^{2n-2}\sum_{i_4>i_3}^{2n-1}\sum_{i_5>i_4}^{2n}(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}, \alpha_{i_4}, \alpha_{i_5}) \\ &\quad + \cdots + (-1)^n \sum_{i_1=1}^{2}\sum_{i_2>i_1}^{3} \cdots \sum_{i_{2n-2}>i_{2n-3}}^{2n-1}\;\sum_{i_{2n-1}>i_{2n-2}}^{2n}(\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_{2n-2}}, \alpha_{i_{2n-1}}).\end{aligned}$$

Proof. The interior angles $\alpha_1, \alpha_2, \cdots, \alpha_{2n}$, with $n \geq 2$ satisfy $\sum_{i=1}^{2n} \alpha_i = (2n-2)\pi$. Applying the tangent function to both sides, with domain as $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$ for each interior angle, we have $\tan\left(\sum_{i=1}^{2n} \alpha_i\right) = 0$. The inclusion-exclusion identity for an even sided convex polygon follows from Lemma 2. □

3 Conclusion and Remark

Motivated by a result in number theory, using as a prototype a simple sum equals product identity for triangles involving the tangent function, we established a more general inclusion-exclusion identity for higher order convex polygons. The techniques used are interesting, simple, and highlight the symmetry inherent in the tangent function. It would be interesting to investigate the inclusion-exclusion results for regular convex polygons, with interior angle α, and construct algebraic equations satisfied by $\tan\alpha$. We hope the reader found motivation through this article to uncover similar number theoretic results to break open new lines of inquiry that reach other branches of mathematics. One such idea suitable for graduate students is to extend the sum equals product identity to include Gaussian integers. Earnest interrogation of diverse mathematical concepts can lead to new discoveries. The point is, perhaps above all, about new mathematical tools that will become available as a result.

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