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Abstract: Students' development of reasoning through algebraic symbols is a crucial component of most mathematics courses. This paper reports the journey of one second-semester calculus student, Cedric, as he attempted to reason about and create algebraic representations for arbitrary partial sums and infinite series through two exploratory teaching interviews. We report Cedric's symbolizing activity in terms of Eckman's (2023) expression framework, focusing on Cedric's development and attribution of meaning to a personal expression template for denoting partial sums and series. Specifically, we describe how Cedric leveraged his initial meanings for partial sums to create personal expressions to reason about infinite series. We then describe Cedric's construction of a general personal expression template for partial sums and series, including cognitive and physical modifications he made to his personal expression template as he symbolized various series. Our results show potentially profitable task sequences and lines of questioning that instructors might utilize to help students be more successful in constructing productive meanings for algebraic symbols.

Keywords: Symbolizing activity, algebraic representations, personal expression, teaching interventions, infinite series

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Introduction

Over the last 500 years, humans have developed a vast system of algebraic symbols to record, organize, and reason about various mathematical topics (Cajori, 1993). For certain topics, such as infinite series, algebraic representations are privileged, with few textbooks representing series using visual or graphical methods (González-Martín et al., 2011). One element that can limit students' understanding of algebraic representations is when they do not recognize the potential to communicate a mathematical process and object through the same expression (which Gray and Tall (1994) described as a standard mathematical practice). For example, a mathematician might use the expression $\sum_{i=1}^{10} i^2$ to consider either (1) the process of adding the first ten perfect squares or (2) the sum of the first ten perfect squares. However, students may impute various meanings to their symbolic expressions, even using similar notations to say pointedly different things (Thompson, 2002). These claims beg the following questions: (1) *what meanings do students attribute to symbolic representations?* and (2) *what instructional interventions might teachers utilize to assist students in constructing notations to which they can impute both process and concept meanings for mathematical topics?*

Our previous work (Eckman et al., 2023; Eckman & Roh, 2022a, 2022b) has focused on the first question. Specifically, we conducted a *collective case study* (Stake, 2003) to investigate individual students' construction of algebraic representations (in the context of infinite series) and their attribution of meaning to these symbols. Thus far, we have described one student's attribution of largely normative meanings for mathematical topics to novel symbols to convey their ideas (Eckman & Roh, 2022a, in revisions). We have also described how two students' individually attributed meanings to instructor-provided notation impacted their symbolization of sets and set relationships (Eckman et al., 2023). These results highlight the importance of describing instructional interventions that might help students (1) construct appropriate meanings for mathematical topics and (2) accept (or become prepared to accept) conventional notations as a method to convey their meanings.

The purpose of this paper is to provide an initial answer to the question *what instructional interventions might teachers utilize to assist students in constructing notations to which they can impute both process and concept meanings for mathematical topics?* Specifically, we describe an *instrumental case study* (Stake, 2003) of a single student, Cedric, in which our tasks and lines of questioning allowed him to reflect on the concepts he was attempting to symbolize. The purpose of an instrumental case study is to deeply investigate a particular phenomenon to gain insight into a general topic. In this report, Cedric is our case, the phenomenon is Cedric's symbolization of topics related to partial sums, and the general topic is students' development of viable notation to express their thinking. For this case study, Cedric participated in two exploratory teaching interviews (Castillo-Garsow, 2010; Moore, 2010; Sellers, 2020) that we designed to explore his process of symbolizing ideas related to infinite series convergence, particularly the notion of arbitrary partial sums.

We share this case study for two reasons. First, we wish to show examples of how an instructor might ask students to engage in symbolizing practices within their classrooms. Second, we wish to show evidence of how a student who appears to easily adopt conventional notation for partial sums might struggle to symbolize more complex examples (e.g., when the general term of summation is unknown). We recognize that the results from a single student are difficult to generalize. Still, we anticipate that educators might gain insight into addressing student symbolization by examining our interview methodology and recognize from our results that students who easily symbolize introductory examples for a topic may struggle to attribute properties of more complex examples to the same notation. We provide the following research questions to guide our discussion:

1) *In what ways might instructional activities facilitate a student's reflective discussion on symbolizing mathematical ideas (such as arbitrary partial sums)?*
2) *What lines of questioning might extend a student's initial reflections on mathematical ideas (such as arbitrary partial sums) toward constructing symbols as instantiations of their thinking?*

We begin this paper by introducing our theoretical perspective on the nature of symbols and students' symbolization. We then share examples from the research literature to contextualize the types of meanings for infinite series convergence that Cedric exhibited in the interview. After briefly reviewing the methodology of our study, we share how Cedric's thinking and symbolization of infinite series and partial sums evolved over the task sequence. We conclude by discussing the implications of Cedric's experience for individual instructors and researchers.

Theoretical Perspective

In this paper, we employ a radical constructivist (Glasersfeld, 1995) perspective on student learning of mathematical ideas. Radical constructivism centers on the idea that individuals construct and maintain cognitive structures through their experiences. These cognitive structures, which Piaget (1970) called *schemes*, enable individuals to consider their current experience as analogous to something they have previously experienced (Glasersfeld, 1995). An individual organizing his current experience in terms of his cognitive structures, or what Piaget (1970) called *assimilation*, entails constructing a *meaning in the moment* (Thompson et al., 2014) for his experience. In the context of representations, we consider a student's *meaning in the moment* (for a representation) to consist of the activated schemes, potential actions, and implications evoked in the student's mind when they create or interpret the representation. Glasersfeld (1995) introduced the term *re-presentation* to describe the cognitive entities (i.e., components of previous experience) an individual uses as a lens to make sense of current experience (i.e., constructing a *meaning in the moment*). In other words, when a student creates a representation, they imbue that representation with a meaning, which they re-present to themselves in later experience when they re-create that representation (or are exposed to the representation by another individual).

We use the terms in the previous paragraph to motivate our definition of symbol. Specifically, we adopt Glasersfeld's (1995) definition of this term: "a word will be considered a symbol, only when it brings forth in the user an abstracted re-presentation" (p.99). In our definition, we only consider a student's representations to constitute symbols when to the researcher, the student has "something to say through them" (Thompson & Sfard, 1994, p. 6).

In this paper, we also adopt Eckman's (2023) constructs to describe students' creation of representations to organize or express their thinking (which we have also described in Eckman and Roh, in revision). We define *symbolizing activity* as a process of mental activities that entails students' creation or interpretation of a perceptible artifact (writing, drawing, gesture, verbalization) to organize, synthesize, or communicate their thinking. Additionally, we define *symbolization* as the status of completing the symbolizing activity. The terminology "symbolizing activity" has been employed by several researchers in mathematics education, and our definition of this term aligns with other definitions in several aspects. But there is still some difference in our definition for this construct. For example, Tillema (2007) employed the terminology "symbolizing activity" to describe the representations students utilized while communicating with each other (which we consider an instance of sharing *personal expressions* to create a shared *communicative expression*). Zandieh et al. (2017) used the terminology "symbolizing activity" to describe students' collective creation of group symbols to describe concepts in linear algebra (which we would term a *communicative expression*). In contrast with these two authors, we use the term *symbolizing activity* as a general construct encompassing individuals' creation of symbols to reflect on their thinking (i.e., *personal expression*), communicate with others or establish group symbolizing norms (i.e., *communicative expressions*), or interpret *conventional expressions* an authority presents.

Although we present a broad definition for symbolizing activity, we differentiate between the types of symbols students create during their symbolization. Specifically, we adopt Eckman and Roh's (in revision) three classifications for symbol types, which they categorize as expressions. Although we present all three types of symbols to contextualize our expressions framework, the focus of this paper will be related to Cedric's personal expression creation.

1) A *personal expression* describes students' investment of meaning to a self-generated form of representation and consists of two components: (a) a meaning and (b) a perceptible artifact
2) A *communicative expression* describes an expression whose meaning users collaboratively negotiate for the purpose of communicative discourse, and the users of a communicative expression may not be the creator of the expression
3) A *conventional expression* is an element from the lexicon of normative representations upheld by mathematicians as conventional modes of communication about a mathematical topic.

Each expression type corresponds with a different method of communication (e.g., individual reflection, collaborative discourse, presenting notational conventions to novices in a discourse). While an individual may experience the same perceptible artifact in each situation (e.g., writing the variable x in their scratchwork, proposing x for symbolizing a variable during group work, attempting to interpret their instructor's use of x in her board work) the entity which maintains the meaning for each perceptible artifact is distinct (e.g., the individual, the group, the mathematical community). In this regard, we consider the three categories of *personal, communicative,* and *conventional* expressions to denote the result of distinct symbolizing activities. However, we acknowledge that the same perceptible artifact might be used for each type of expression.

The expressions we have described refer to complete notational entities, which may or may not include sub-symbols within the expression. We use the term *inscription* to refer to sub-symbols within an expression. Similar to a personal expression, we consider an inscription to be comprised of both a *meaning* and a *perceptible artifact*. For instance, a student might construct the personal expression $\Sigma_{n=1}^{5} n$ to re-present the sum of the numbers 1 to 5. In this case, we would consider the symbol $\Sigma_{n=1}^{5} n$ to be the student's *personal expression* but the symbols Σ, n, $=$, 1, 5, and n to be inscriptions to which the student can attribute distinct meanings. We consider the definition of the terms *expression* and *inscription* synonymous and use the terms primarily for hierarchically discussing the syntactic organization of a particular symbol. Consequently, we might use the term *inscription* to refer to the symbol $\Sigma_{n=1}^{5} n$ in the context of a more complex *expression* such as $\Sigma_{n=1}^{5} n + \Sigma_{n=6}^{10} n = \Sigma_{n=1}^{10} n$.

During their symbolizing activity, a student may also develop a symbolic template to re-present a class of personal expressions they perceive as analogous in terms of meaning and syntax. In these situations, we use the term *personal expression template* to describe the general syntactic structure of the expression whose inscriptions the student modifies (according to their needs) to symbolize situations they perceive to have analogous structures or properties. For example, a student might repeatedly use personal expressions of the form $\sum_{n=a}^{b} s_i$ to denote various partial sums (e.g., $\Sigma_{n=1}^{5} n$, $\Sigma_{j=0}^{\infty} \binom{2}{j}$, $\Sigma_{i=9}^{11} \left(\frac{i^2-3i+2}{(i-4)^2}\right)$), recognizing that the inscriptions they write in place of $n, a, b,$ and s_i may vary across instantiations of the expression template. A personal expression template is different from a personal expression in that a student re-presents their *meaning* at that moment for a particular situation through a personal expression and their meaning for the structure of the class of the analogous (to them) examples through the personal expression template. In other words, a student's instantiation of his personal expression template is his personal expression. However, suppose a student has not experienced (or reflected on) situations similar to his current experience. In that case, he may (at best) be capable of creating an ad hoc personal expression. In this instance, we do not consider the student's personal expression to be an instantiation of a personal expression template, but rather the student's best attempt to re-present their meanings through the inscriptions that make the most sense to them (in that moment).

As a final comment, we note that although students might consider their constructed personal expression template sufficient to support their symbolizing activity in the moment of creation, they may update their template as their meanings for a topic evolve. In the results section, we describe the personal expression template Cedric created to organize his thinking about infinite series and his modification of his template as he progressed through our activities.

Literature Review

The purpose of the literature section is to review the various meanings for sequences, infinite series, and convergence that are reported within the literature. We present this review to contextualize Cedric's meanings during his symbolizing activity. Before presenting the literature, we briefly review what we consider a normative meaning and corresponding symbolization for infinite series convergence.

A Normative Meaning and Symbolization for Infinite Series Convergence

The conventional expression used by many calculus textbooks (e.g., Larson & Edwards, 2015; Stewart, 2012) to represent an infinite series is summation (Σ) notation. For example, a student would likely see their instructor represent a geometric series such as $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ through the conventional expression $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$. In this expression, there are several inscriptions, each with their own conventional meanings:

1) the inscription Σ can refer to an additive process or a sum,
2) the variable n refers to the position of a summand in the series,
3) the inscription "1" in the lower index refers to the position of the first summand computed by the sum,
4) the inscription = is used to show a relationship between the indexing variable, n, and the lower limit of summation "1,"
5) the general summand $\frac{1}{2^{n-1}}$ represents an explicit, closed-form rule for determining the value of the nth summand in the series,
6) and the inscription ∞ is the upper index, or the position of the final summand, computed by the sum (in this case, the inscription ∞ is used to denote that the entire infinite series is being considered).

In general, the symbolic expression for an infinite series is $\sum_{n=1}^{\infty} a_n$. A mathematician can use this expression to indicate (1) the process of summing consecutive terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ and (2) the metaphorical "result" of this process (c.f., Lakoff & Núñez, 2000). A mathematician might determine the limit of the corresponding sequence of partial sums to determine whether an infinite series converges. Two common conventional expressions for the limit of the sequence of partial sums are (1) $\lim_{n\to\infty} S_n$, where $S_n = \sum_{i=1}^{n} a_i$, or (2) $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$. The notation related to infinite series utilizes various topics that researchers have shown to be difficult for students to comprehend, including sequences (e.g., Martínez-Planell et al., 2012; McDonald et al., 2000), limits and convergence (e.g., Roh, 2008, 2010; Sierpińska, 1987; Swinyard & Larsen, 2012; Tall & Vinner, 1981), and summation (Barahmand, 2017, 2021; Eckman & Roh, 2022b). Researchers have also reported that summation notation can be problematic for students to interpret or instructors to convey information about additive processes (particularly the indices; Katz, 1986; Strand et al., 2012; Strand & Larsen, 2013). Finally, summation notation is one of many notations that mathematicians use fluidly to describe either a process (i.e., adding summands) or a result (i.e., sum) according to their needs (Gray & Tall, 1994). We now address the various meanings that students might possess for sequences, limit, and infinite series convergence in more detail.

Students' Meanings for Sequences

McDonald et al. (2000) described two meanings students exhibited for a sequence: sequence as a list and sequence as a function. Students who considered a sequence a list believed that a discernible pattern (e.g., recursive rule, explicit rule) was necessary to define a sequence (Przenioslo, 2006). Such students also considered a sequence a well-ordered set of values (Sierpińska, 1987). Additionally, these students constructed the general term of a sequence but used it as a formula for a particular summand rather than an arbitrary term capable of generating a sequence (Eckman & Roh, in revision). In contrast, students who considered a sequence as a function defined a correspondence between the values of an indexing variable and the terms of the sequence (McDonald et al., 2000). There is evidence that students who envisioned the indexing variable in the sequence's general term simultaneously denoting the process of computing a partial sum value and the result considered a sequence as a function (Eckman & Roh, in revision).

Students' Meanings for Limit of a Sequence

There is a significant amount of literature on the concept of limit, particularly to limits of sequences (e.g., Oehrtman et al., 2014; Roh, 2008, 2010; Sierpińska, 1987) and functions (e.g., Cornu, 1991; Cottrill et al., 1996; Swinyard & Larsen, 2012; Tall & Vinner, 1981; Williams, 1991). Roh (2008) summarized three meanings students possessed for limit of a sequence: asymptote, cluster point, and limit point. Students who believed that a limit of a sequence is akin to an asymptote often imagined a limit as a dynamic process where the value of a sequence approaches a particular value but cannot achieve that value. Students who believed that a limit of a sequence is like a cluster point envisioned that the sequence values could approach or achieve a particular value as the value of the sequence index increases without bound. In some cases, students leveraged their cluster point meaning to identify multiple limits for conventionally divergent sequences whose values clustered around more than one value. In contrast, students who maintained a limit point meaning for sequence convergence considered the limit of a sequence to be a unique value such that, for any error bound, a finite number of sequence terms exist outside of that error bound (when the error bound is centered at the limit point).

Students' Meanings for Infinite Series Convergence

The research literature related to students' meanings for infinite series is diverse, including framing students' interpretations of series convergence in visual or non-visual terms (Alcock & Simpson, 2004, 2005), investigating students' comparison of finite and infinite series (Barahmand, 2017, 2021), comparing novice and expert conventions of Taylor series (Martin, 2013), and investigating students' meanings for series convergence (Eckman & Roh, 2022b; Martin et al., 2011). Eckman and Roh (2022b) reported that students often believed that an infinite series converges if they (the student) believed that the running total (i.e., a dynamic sum created by perpetually adding consecutive summands in the series) approaches an asymptotic value. They also reported three implications for the *asymptotic running total* meaning. First, students exhibiting *decreasing summands convergence* believed that if the summands in a series tend toward zero, the series converges. Second, students exhibiting *monotone running total divergence* believed that because the running total will perpetually increase (or decrease) in a non-alternating series, then the running total will eventually surpass every upper (or lower) bound. Finally, students exhibiting *running total recreation through grouping* constructed a monotone series from an alternating series by grouping pairs of consecutive summands and then made inferences about the convergence of the original series by examining the newly grouped series.

Methodology

We conducted two remote 90-minute individual exploratory teaching interviews (Castillo-Garsow, 2010; Moore, 2010; Sellers, 2020) during the Summer 2021 session with second-semester calculus students at a large university in the United States. The student participants were selected by responding to an email invitation to participate in research interviews we sent to second-semester calculus students enrolled in the summer session. We chose to conduct exploratory teaching interviews because this methodology affords flexible questioning and spontaneous task revision, two features we considered necessary for investigating students' nascent personal expressions (particularly if these expressions were novel). In these exploratory teaching interviews, the first author functioned as a teacher-researcher, and the second author served as a witness to the interview. The role of the teacher-researcher was to present the tasks to the student, clarify the student's thinking and responses, and introduce spontaneous tasks or questions (if necessary) to determine the boundaries that a student's thinking afforded in a situation. The role of the witness was to provide an objective check on the teacher-researcher's actions, both in the moment of the interview (e.g., hypothesize about student thinking, suggest questions or tasks) and during analysis (e.g., verify or challenge the teacher-researcher's models of student thinking).

Cedric was one of two students who participated in four phases of tasks over the two interviews (a third student participated in the first interview but did not complete the second interview). These tasks aimed to help the students develop a personal expression template to symbolize various partial sums and series. The first three phases of our task were inspired by Radford's (2000) task sequence for middle school students' development of an algebraic rule for a sequence. Radford's (2000) tasks included (1) students reasoning about the nature of a sequence in groups, (2) developing a written rule to describe calculating the value of an arbitrary term in the sequence, and (3) constructing an algebraic rule for the general term of the sequence (using their written rule as a reference). Our modifications of these three tasks in the context of individual students reasoning about partial sums for the Day 1 interview included (1) verbally discussing how to compute individual summands and partial sums for six series, (Phase 1), (2) creating a written note describing how to determine an arbitrary partial sum (Phase 2), and constructing a personal expression template to describe an arbitrary partial sum (Phase 3). During the Day 2 interview, we presented tasks related to a fourth and final phase, which we created after reviewing Zazkis and Hazzan's (1998) recommendation to reverse the order of task questioning. Specifically, we asked the students to symbolize arbitrary partial sums and series for the six series from Day 1 using their personal expression template. In the results section, we report Cedric's efforts symbolize four of the six series we presented in our interview, which we portray in Table 1.

Table 1

The Four Series Cedric Symbolized that we Report in this Paper

Series	Expanded Form	Series type	Sequence of Partial Sums	Converge
1. $\sum_{n=1}^{\infty}\frac{1}{7(n)}$	$\frac{1}{7}+\frac{1}{14}+\frac{1}{21}+\cdots$	p-series ($p=1$)	Monotone increasing	No
3. $\sum_{n=0}^{\infty}(.001)\cdot(-1)^n$	$.001-.001+.001-\cdots$	Alternating series (Grandi's)	Oscillating	No
4. $\sum_{n=1}^{\infty}\sum_{i=1}^{99}[10^{-2n-1}-10^{-2(n+1)-1}i]$ $=\sum_{k=0}^{\infty}\frac{495}{10000}\left(\frac{1}{100}\right)^k$	$\frac{99}{10^3}+\frac{98}{10^3}+\cdots+\frac{1}{10^3}+$ $\frac{99}{10^5}+\cdots+\frac{1}{10^5}+$ $\frac{99}{10^7}+\cdots+\frac{1}{10^7}+\cdots$	Geometric	Monotone increasing convergent	Yes
6. Random generation of terms with no (easily) discernable rule	$\frac{1}{3}+\frac{1}{8}+\frac{1}{13}+\frac{1}{7}+\frac{1}{35}+\cdots$	N/A	Monotone increasing	N/A

Cedric was an aerospace engineering major who had taken second-semester calculus for part of the previous semester but became sick with COVID-19 and missed a significant portion of the semester (prompting him to retake the course during the summer). We confirmed that Cedric missed the sequences and series unit during his illness and considered him little more than a novice at the topic of infinite series convergence. Still, Cedric generally seemed comfortable with mathematical tasks, calculations, and conventional notation. He also expressed an affinity for computer programming at various times throughout the interview. In summary, Cedric exuded a sense of confidence in his actions that afforded (in our minds) the opportunity to report the affordances and obstacles to symbolization that a student might experience who appeared (to us) to be comfortable with academic mathematics.

We conducted our analysis in the spirit of grounded theory (Strauss & Corbin, 1998), which consisted of three major components. First, we created detailed field notes for each interview, recording Cedric's meanings for the series and his use of symbols in each task. Second, we analyzed how Cedric's meanings for series convergence changed throughout the first interview and mapped his meanings to the various inscriptions in his personal expression template using open coding techniques. Similarly, we examined the evolution of Cedric's personal expression template through the second interview, isolating key moments when our interview tasks evoked cognitive conflict that required him to alter his meaning or template. During the axial coding stage, we analyzed our open codes describing Cedric's symbolization during the two interviews to construct an overarching narrative and rationale for his behavior. Through this analysis, we determined that Cedric's meanings for partial sums were largely normative, and Cedric justified his meanings through both his symbolization and use of a looping metaphor. In the results section, we contextualize Cedric's case by describing his progression through reasoning about series, constructing a written note, and creating a personal expression template. We then describe how his personal expression template changed throughout the two interviews.

Results

We separate our results into two sections. In the first section, we describe (1) Cedric's development of meaning for infinite series as he progressed through our initial Day 1 task sequence (Phase 1) and (2) his construction of a written artifact and initial personal expression template to describe (his image of) his meaning for arbitrary partial sums (Phases 2 and 3). In the second section, we describe how Cedric modified his personal expression template on Day 2 to convey notions of series convergence that he did not consider (or, at least, did not symbolize) during the first interview (Phase 4).

Results Part 1: Cedric's Meanings for Partial Sums and Initial Personal Expression Template

Phase 1: Reasoning about Summands and Partial Sums of Individual Series

In the following subsections, we briefly summarize how Cedric's meanings for determining individual summands and partial sums evolved as he considered four different series during the Phase 1 tasks of the first interview. During this task, we asked Cedric to describe how he would compute the values of (1) the 37^{th} summand in a series and (2) the 37^{th} partial sum in a series. We also address Cedric's nascent use of summation notation as a preliminary personal expression template to symbolize his thinking about summands and partial sums.

Series 1: Initial Meanings and Symbolization for Series. When we asked Cedric to determine the 37th summand of Series 1, $\frac{1}{7}+\frac{1}{14}+\frac{1}{21}+\frac{1}{28}+\frac{1}{35}+\frac{1}{42}+\cdots$, his initial reaction was to symbolize the series with the expression $\sum_{n=1}^{\infty}\frac{1}{7n}$ (see Figure 1). Cedric then proposed a "computer loop" analogy to contextualize his expression for Series 1.

Interviewer: How is what you've written so far (*referring to Cedric's personal expression* $\sum_{n=1}^{\infty}\frac{1}{7n}$) correlated to this idea of coming up with the 37th summand?

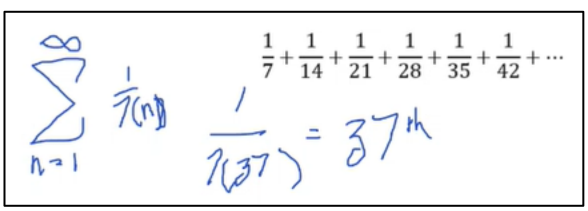
Cedric: […] I just sort of did this (*referring to personal expression*) because it reminds me of doing it in my coding.

Interviewer: Okay.

Cedric: So this is like a function, like a loop, where it loops continuously and each time it loops, it increases the value of n by one (*indicates lower index* $n = 1$ *with cursor*). And then this (*indicates general term* $\frac{1}{7n}$) is what's within the function and it has to stop at when one over seven to the power of like n (*writes* $\frac{1}{7(n)}$), n, 37 (*erases* n*, leaving* $\frac{1}{7}$). So it would be the seven and then 37 (*writes* $\frac{1}{7(37)}$). […] it should be the 37th one (*i.e., summand*).

Figure 1

Cedric's Personal Expressions for Series 1



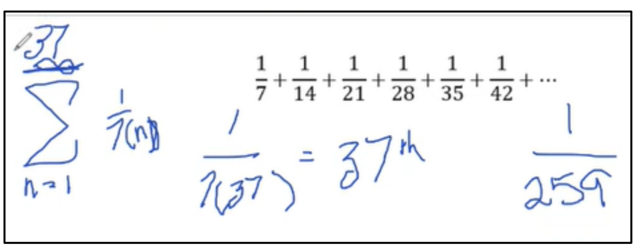
In the transcript above, Cedric stated that he imagined a computer loop while creating his expression $\sum_{n=1}^{\infty}\frac{1}{7n}$. What Cedric referred to as a "loop" was an iterative operation where the "function" $\frac{1}{7n}$ was evaluated for each value of the index (beginning with $n = 1$ and ending with $n = 37$). Cedric then used his iterative "loop" process to reason that the 37th summand of the series would occur when he evaluated his function $\frac{1}{7n}$ with $n = 37$, which resulted in his normative response that $\frac{1}{7(37)}$ is the 37th summand of Series 1.

After Cedric introduced his computer loop metaphor, we wondered whether he would apply this same thinking to reason about partial sums. Consequently, we asked Cedric to describe how he would determine the 37th partial sum of Series 1. After reflecting on our question, Cedric spontaneously changed the upper index in his personal expression for Series 1 from ∞ to 37 (i.e., $\sum_{n=1}^{37} \frac{1}{7n}$; see Figure 2). When we asked Cedric to clarify his change to his expression, Cedric shared the following:

> Cedric: For the second [question] where you want to find the [37th partial] sum, I know that this equation (*i.e.,* $\sum_{n=1}^{37} \frac{1}{7n}$) is doing the sort of loop thing where you add each one [i.e., summand] while increasing n by one. […] At first […] it's infinity since it keeps going (*referring to the upper index ∞ above Σ*). But then I realize that if it asked for 37 (*i.e., the 37th partial sum*), then there should be some sort of upper bound as to where n (*indicates upper index 37 above Σ*). And I don't know if this is breaking sort of the laws of math right now with this equation, and I might be committing an act of heresy. But this is what I think is most logical to me, both incorporating my math knowledge as well as coding knowledge.

Figure 2

Cedric's Revised Personal Expression for the 37th Partial Sum of Series 1



In the transcript above, Cedric used the inscription ∞ in the upper index to convey the process of generating summands in the series. However, his reflection on his looping metaphor (which we had elicited from Cedric through our questioning) allowed him to modify his personal expression to convey his idea of a partial sum (instead of an entire infinite series). Cedric also expressed that he was unsure whether his creation of a non-infinite upper index constituted an act of "mathematical heresy," but ultimately decided he would continue to use non-infinite upper index values as needed to symbolize partial sums because the symbolic notation made sense to him.

We consider Cedric's modification of his personal expression $\Sigma_{n=1}^{\infty}\frac{1}{7n}$ to $\Sigma_{n=1}^{37}\frac{1}{7n}$ evidence that his initial personal expression had evolved into a personal expression template. Specifically, Cedric's explanation that he could denote the notion of an "upper bound" for a partial sum through the upper limit of his expression implies that he was capable (in that moment) or re-presenting (at the least) (1) an infinite series or (2) the 37[th] partial sum through his variation of summation notation. In other words, Cedric appeared (to us) to have attributed an "upper bound of summation" meaning to the "upper index" inscription of his summation-oriented expression, which he might then modify to denote various partial sums or series (according to his needs). We discuss other components of Cedric's personal expression template while describing his symbolization for Phase 3 and 4 activities.

After we discerned that Cedric had constructed a viable personal expression template to re-present partial sums, we wished to investigate the meanings he attributed to his personal expression for the 37[th] partial sum of Series 1. To do this, we asked Cedric whether he needed to set his expression $\sum_{n=1}^{37}\frac{1}{7n}$ equal to a value to determine if he could re-present the value of a partial sum independently of the corresponding additive process. Cedric's reflection on our question highlights that he was capable of re-presenting both a process and result meaning for a specific partial sum through his expression $\sum_{n=1}^{37}\frac{1}{7n}$.

> Cedric: I sort of don't know what the missing piece is, which is what it (*i.e.,* $\sum_{n=1}^{37}\frac{1}{7n}$) should equal in the end. […] Maybe this is the final product because in calculus, in math before there have been cases where there is like, there's a point where you don't have to solve it (*i.e., an expression*) entirely. You can just leave it as is, because it's sort of the shorthand way of expressing the answer rather than writing it fully out. […] I could go and do $\frac{1}{7}+\frac{1}{14}+\frac{1}{21}$ and stay here for 90 minutes doing that math. […] this is sort of a shorthand answer (*indicating personal expression* $\sum_{n=1}^{37}\frac{1}{7n}$).

In this excerpt, Cedric indicated that (to him) mathematical expressions can function as "shorthand answers" for complicated calculations. We take this statement and Cedric's description of how to calculate the value of the 37th partial sum by hand as evidence that he could re-present both a mathematical process (i.e., add together 37 summands) and a mathematical object (i.e., the 37th partial sum) through his expression $\sum_{n=1}^{37} \frac{1}{7n}$.

In summary, during Cedric's reasoning about Series 1, he constructed a personal expression template $\sum_{n=1}^{\square} \frac{1}{7n}$ (see Figure 3) to symbolize partial sums (we place the blank box in the place of the upper index to indicate Cedric's ability to utilize various inscriptions in this syntactic position of his expression). Cedric also exhibited an ability to re-present either an additive process or the concept of a sum through the instantiation of his expression template $\sum_{n=1}^{37} \frac{1}{7n}$.

Figure 3

The First Iteration of Cedric's Personal Expression Template

$$\sum_{n=1}^{\square} \frac{1}{7n}$$

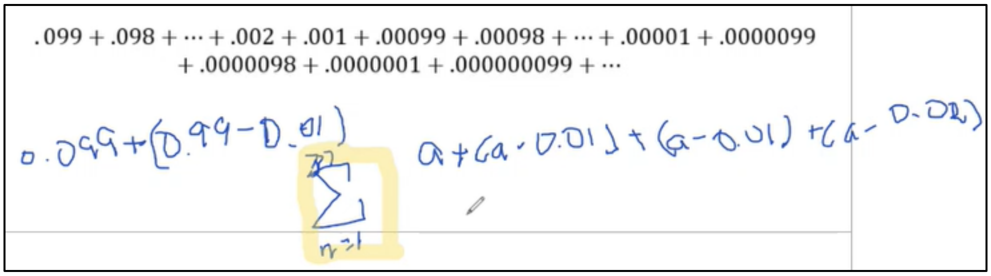
Series 4 and 6: Reasoning about Series without Knowing the General Term of Summation. We purposefully chose the remaining series in the Phase 1 tasks to exhibit more complicated general terms (i.e., Series 4) or no readily discernible general term (i.e., Series 6). After Cedric's spontaneous introduction of summation notation and his computer loop analogy for Series 1, we wondered how providing him with a series for which he could not easily describe the general term might impact the inscriptions in his personal expression template and the meanings he attributed to these inscriptions across different series.

We first asked Cedric to reason about Series 4, a geometric series whose terms we had expanded into finite arithmetic series to mask its geometric nature. The expanded form of Series 4 that we presented to Cedric was $.099 + .098 + \cdots + 0.002 + .001 + .00099 + .00098 + \cdots .00001 + .0000099 + .0000098 + \cdots + .000000099 + \cdots$. Cedric noticed the common difference between the first 99 summands and attempted to describe the series symbolically by writing each summand as $a - 0.01$ (where a referred to the initial summand in the series and

-0.01 was the common difference[1]; see Figure 4). Cedric then claimed that Series 4 was much more difficult to interpret than Series 1 and stated that if he were to use summation notation to describe Series 4, he would only be able to write $\Sigma_{n=1}^{37}$.

Figure 4

Cedric's Written Work while Reasoning about Series 4



We interpreted Cedric's comment that he could only construct a personal expression containing a Σ and indices as an indication he considered the general term inscription to be a variable portion of his personal expression template (but was unsure what inscription to write for his symbolization of Series 4). After confirming that Cedric wanted to write something (e.g., a general term) to the right of the Σ, we asked Cedric what he would need to know to symbolize Series 4 using his notation.

> Cedric:[…] this is a very specific example of how to do things (*indicates expanded form of series written numerically*). And then this is more like a general example (*indicates rule a + (a-0.01) + (a - 0.01) + (a - 0.02)*). I would say this is sort of the expression (*indicates symbolic expression*) [...], and this is the expression filled in with numbers (*indicates expanded numerical form of series*). I would need to know a simpler method, like the most basic method on how to solve these types of equations (*i.e., model the summands of Series 4*). […] It's, much more difficult […] I am just thoroughly stuck.

Cedric's behavior in this moment implies that to him, the inscriptions he had previously utilized (or imagined utilizing) for the general term inscription of his personal expression template for partial sums were insufficient to convey the properties he perceived about Series 4.

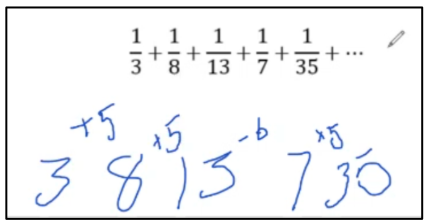
Sensing Cedric's uncertainty regarding how to symbolize his meaning for Series 4 through his personal expression template, we presented Cedric with the expanded form of Series

[1] The actual common difference was -0.001, but we did not point out this error and it did not seem to impact Cedric's reasoning about the series.

6, $\frac{1}{3}+\frac{1}{8}+\frac{1}{13}+\frac{1}{7}+\frac{1}{35}+\cdots$, which we had constructed by randomly writing fractional summands without regard for a potential general term. After a brief reflection, Cedric provided a set of recursive rules by which he could generate the visible terms of the series (see Figure 5). We then asked whether Cedric could create a personal expression for the sum of the first 37 terms of Series 6. Cedric said (similar to his response for Series 4) that he could write $\Sigma_{n=1}^{37}$ to describe adding together the first 37 terms of the series. However, he could not complete his personal expression because he did not know an explicit rule for the general summand of Series 6.

Figure 5

Cedric's Recursive Rules for Generating the First 5 Summands of Series 6



In summary, our purpose in presenting Cedric with Series 4 and 6 was to determine whether Cedric could extend his personal expression template and computer loop analogy to situations where he could not easily construct an explicit rule for the general summand of a series. We found that although Cedric constructed a recursive rule to generate summands for each series, he may not have considered his personal expression template $\Sigma_{n=1}^{\square}\square$ (see Figure 6) fully capable of conveying his meaning because (to him) the "general term" inscription of his personal expression template required an explicit, closed form function rule. In other words, Cedric considered his template an inappropriate symbolic medium through which to denote his thinking about Series 4 and Series 6 because he could only model the summands of each series with a recursive rule.

Figure 6

The Second Iteration of Cedric's Personal Expression Template

$$\sum_{n=1}^{\square}\square$$

Phase 2: Constructing a Written Note Describing an Arbitrary Partial Sum

The purpose of the Phase 1 tasks was not for students to develop personal expressions, which Cedric spontaneously created, but to develop the ability to verbally reason through examples of partial sum calculation. Since Cedric successfully described a process for computing individual partial sums, we decided to introduce the Phase 2 tasks, where we asked him to construct a written note to describe his meanings for an arbitrary partial sum. Cedric originally wrote a written note related to determining the general summand of a series (see Figure 7). When we asked Cedric to explain his rule, he returned to Series 1 and described how he constructed the general term $\frac{1}{7n}$. This general term, which he called a "pattern," seemed to function as the central component of his computer loop analogy. Throughout the remainder of this paper, we will use Cedric's term "pattern" to refer to the general summand of a series. When we asked Cedric how he would implement his written note in a case where there was no "pattern," Cedric stated that (to him) infinite series required a pattern. In retrospect, we acknowledge that a better question for Cedric would have been, "How would you use your written note when unsure how to describe the pattern of terms in the series?" However, Cedric's response revealed that (to him) a series must be generated from some rule. This distinction in Cedric's thinking about the role of a general term in a series would inform one evolution of his personal expression template during the Day 2 interview.

Figure 7

Cedric's Initial Written Note

Written note:

Start by looking through the series to see if
there is a noticeable pattern.
if there is a pattern, try to form
an expression in terms of a variable n that is
a simplified version of the pattern

We then decided to prompt Cedric to alter his written note to reflect the notion of an arbitrary partial sum. We appended the statement "If there is a pattern and you form an expression, then…" and asked Cedric to complete the written note by describing an arbitrary partial sum. Cedric completed our statement (see Figure 8) by defining an algorithmic process to generate the summands of the series necessary to compute the value of a partial sum.

Generally speaking, Cedric's written note described a set of processes related to computing the value of a partial sum (i.e., the processes of constructing a general term or recursive rule for the summands of a series, generating summands, and computing the values of partial sums). We also consider Cedric's written note to reflect his image of a dynamic computer loop generating summands and printing a resulting sum.

Figure 8

Cedric's Written Note about an Arbitrary Partial Sum

> If there is a pattern and you form an expression, then…
>
> Come up with a method that the pattern can be continuously repeated until the nth term and provide an answer

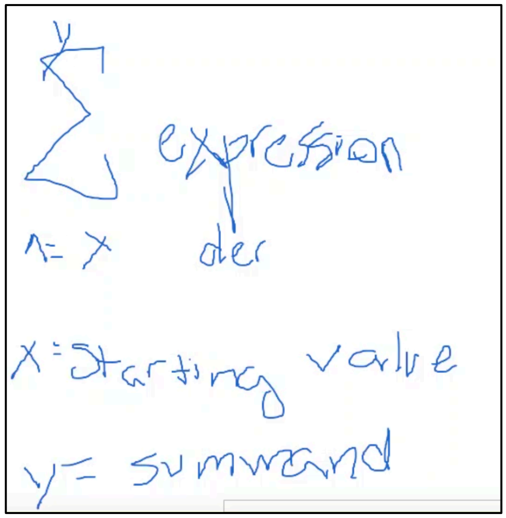
Phase 3: Developing a Personal Expression for an Arbitrary Partial Sum

After Cedric successfully created his written note to describe an arbitrary partial sum, we asked him to propose a formal personal expression template to symbolize his written note. We allowed Cedric to utilize notations with which he was already familiar or propose new inscriptions. However, we told Cedric that if he proposed a novel inscription, we wanted him to describe the information he intended to convey with his inscription in a glossary that we provided.

Cedric initially wrote a large loop symbol (which we will symbolize in this paper using the inscription ℚ) to denote his meaning for an arbitrary partial sum. However, Cedric quickly erased his inscription ℚ and proposed a form of summation notation instead, $\Sigma_{n=x}^{y}$expression der(ived from a pattern) (see Figure 9). When we asked Cedric to describe the inscriptions comprising his expression template, he stated that x referred to the starting value of a partial sum and y referred to the ending value. Based on Cedric's responses, we consider x and y to constitute arbitrary inscriptions in his personal expression template, which he envisioned replacing with a numerical value when creating an instantiation of the template (i.e., a personal expression for a particular partial sum or series).

Figure 9

Cedric's Personal Expression Template for an Arbitrary Partial Sum



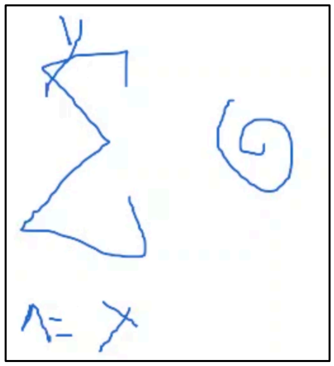
When Cedric began writing "expression derived from a pattern" as his inscription for the general term, we asked whether he could construct a single inscription to convey his written idea of the general summand in a series. Cedric then proposed the inscription ℚ for the general term of summation, again leveraging his computer loop analogy to contextualize his personal expression template (we show Cedric's final personal expression template for Day 1 in Figure 10).

> Cedric: To me it's (i.e., his personal expression template $\sum_{n=x}^{y}$ ℚ) like a loop [...] I just keep thinking of the coding thing where it keeps looping and [...] there's commands within that loop that keep on repeating. And each time it repeats, it incrementally increases a certain variable by one, which allows the rest of the commands within the loop to change very slightly in order to continue the pattern, and then at the end [...] print out the results of either the value or an expression that is formed from the loop.

In our follow up questions to Cedric, we learned two additional properties of Cedric's novel inscription ℚ. First, Cedric envisioned replacing the inscription ℚ with a function rule for the general summand. Second, Cedric stated that he could not meaningfully use his expression if he did not know a series' "pattern." From Cedric's first comment, we consider ℚ to be an arbitrary inscription he envisioned replacing with a closed-form, explicit function rule when creating an instantiation of the template. From Cedric's second comment, we infer that Cedric did not consider his inscription @ capable of conveying an unknown pattern for a series (at least in this moment). These comments further contextualize Cedric's ability to quickly symbolize Series 1 and uncertainty regarding how to symbolize Series 4 and 6.

Figure 10

Cedric's Final Personal Expression Template for Day 1



While Cedric did not explicitly describe the role of the inscription Σ, we inferred that Cedric attributed the additive process involved in computing a partial sum to this inscription. We did not explicitly confirm Cedric's meaning for n at the end of the Day 1 interview. However, in the tasks for which he successfully constructed a closed-form explicit rule for the general summand of a series (e.g., Series 1), Cedric appeared to attribute the position of the summands in a series to this inscription.

Results Part 2: Cedric's Use and Modification of his Personal Expression Template

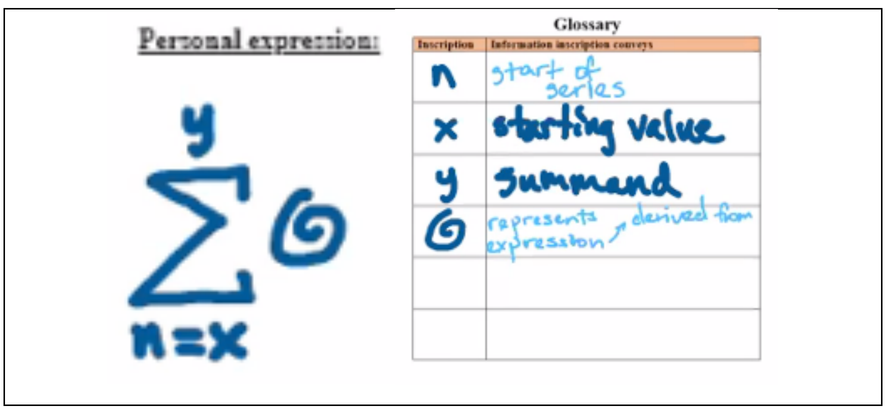
The purpose of our activities related to Phase 1 (reason about specific summands and partial sums), Phase 2 (construct a written note to describe an arbitrary partial sum), and Phase 3 (construct a personal expression to symbolize an arbitrary partial sum) was to help Cedric construct a personal expression template for an arbitrary partial sum. In the Phase 4 tasks (which occurred during the Day 2 interview), we asked Cedric to create instantiations of his personal expression template to describe the arbitrary partial sums and the series he encountered during the Day 1 interview. We will focus our discussion on Cedric's creation of personal expressions for Series 1, his generalization of the inscription @ for Series 3, and his modification of his personal expression template for Series 6.

Symbolizing Series 1: Stabilizing the Meanings for each Inscription in the Personal Expression

Between the two interviews, we reconstructed Cedric's personal expression template and created a glossary to record the inscriptions Cedric utilized and the ideas he wished to convey with each inscription. At the beginning of the Day 2 interview, we presented these items to Cedric in a glossary (see Figure 11) and asked him to review his inscriptions from the Day 1 interview. We purposefully left the definition cell in the glossary blank for n and ɷ so that Cedric could describe his meanings for these inscriptions. Consequently, Cedric defined n as the "start of series" and ɷ as "represents expression derived from [pattern]" (see light blue writing in the glossary for Figure 9). While Cedric's meaning for the inscription ɷ was analogous to his descriptions from Day 1, Cedric's description of n on Day 2 (i.e., start of series) differed (in our minds) from his description of n on Day 1 (i.e., position of summands in the series). We did not attempt to further investigate Cedric's thinking about his inscription n during the inscription review portion of the interview but determined to revisit Cedric's apparent change in meaning for the inscription n during his attempt to symbolize Series 1.

Figure 11

Our Recreation of Cedric's Expression Template and Glossary from Day 1



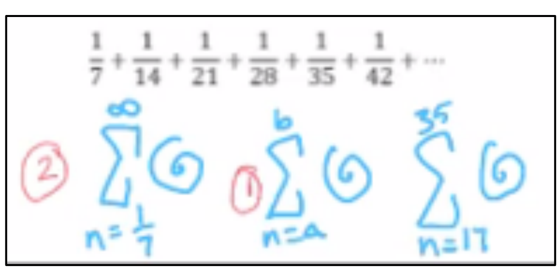
To introduce the Phase 4 tasks, we told Cedric that we would like him to use his personal expression template to answer two questions for each series we presented him: (1) *How would you represent the sum for any number of terms in a series (i.e., an arbitrary partial sum)?*, and (2) *How would you represent the entire series (e.g., Series 1)?* We then asked Cedric to create personal expressions for Series 1 to answer the two questions and further investigate his meaning for his inscription n.

Cedric's initial personal expression that he created to describe an arbitrary partial sum for Series 1 was $\sum_{n=\frac{1}{7}}^{\infty}$ @. After pausing for a moment, Cedric stated that his expression $\sum_{n=\frac{1}{7}}^{\infty}$ @ referred to the infinite series (i.e., Question 2) and proposed the expression $\sum_{n=a}^{b}$ @ to describe an arbitrary partial sum (i.e., Question 1). We found several components of Cedric's initial symbolization interesting. First, we noted that Cedric had set the lower index of summation in his first personal expression equal to $\frac{1}{7}$. We immediately conjectured that Cedric's current definition of n as denoting the start of the series (i.e., first position of a summand) and his consequent description of x as the value of the summand (in that starting position) influenced his decision to write $n = \frac{1}{7}$ as the lower index. Second, we found it interesting that Cedric's personal expression for an arbitrary partial sum in Series 1 utilized the inscriptions a and b instead of x and y (which Cedric included in his personal expression template). Unfortunately, Cedric's change of inscription seemed less important (to us) at the time of the interview than his use of a summand value as a lower limit, so we have no further data to contextualize Cedric's use of a and b.

To determine whether Cedric's variation in symbolization was persistent across examples, we asked Cedric to create a personal expression for a specific partial sum for Series 1. Cedric quickly proposed the sum of the 17th to 35th summand and wrote the personal expression $\sum_{n=17}^{35}$ @ to convey this idea (see Figure 12). Cedric also spontaneously explained that in each personal expression he had constructed, he could use the inscription @ to stand in for the explicit pattern of Series 1. Cedric's description of the inscription @ as capable of standing in for an explicit rule also marked an evolution from his claim during Day 1 that @ was a placeholder that needed to be replaced by an explicit rule for his expression to function.

Figure 12

Cedric's Three Personal Expressions for Components of Series 1



We recognized that Cedric was writing inscriptions that (to us) appeared to represent vastly different ideas. On the one hand, Cedric used the value of the first summand in his symbolization of Series 1 (see Figure 10, left expression). On the other hand, Cedric used the positions of the summands to represent the partial sum comprising the 17th to 35th summands in Series 1 (see Figure 10, right expression). To see whether Cedric could problematize his difference in symbolization, we highlighted the lower index in each of Cedric's personal expressions and asked him to describe what each lower index conveyed. For his expressions $\sum_{n=\frac{1}{7}}^{\infty}$ Ⓠ and $\sum_{n=a}^{b}$ Ⓠ, Cedric claimed that the lower indices $\frac{1}{7}$ and a referred to the value of the starting summand in a series or partial sum (respectively). However, when Cedric began to describe the lower index for his expression $\sum_{n=17}^{35}$ Ⓠ, he paused. After a brief reflection, Cedric said that his use of $\frac{1}{7}$ was a mistake, and he wanted the lower index to refer to the "starting place value" (i.e., position) of the series and not the value of the first summand. From this moment to the end of the interview, Cedric consistently used the upper and lower indices in his expressions to denote positions of summands in a series (and not summand values).

In summary, although the syntactic components of Cedric's personal expression template (e.g., lower index position, upper index position) did not change while symbolizing Series 1, the types of inscriptions he wrote in his template and the meanings he attributed to the positions varied. We enacted two instructional interventions to help Cedric problematize and reconcile his thinking. First, we asked Cedric to construct additional personal expressions to determine whether he would provide the same definition for the lower limits of an arbitrary and specific partial sum as he had for the series. Second, we asked Cedric to compare the instantiations of his personal expression template to aid his identification of inconsistencies in his expressions. Our intervention (1) resulted in Cedric's recommittal to using the lower limit to denote the position of summands and (2) appeared to play a role in his decision to use Ⓠ representationally for a known rule.

Symbolizing Series 3: Cognitively Adapting a Personal Expression

After Cedric's response to our interventions, we conjectured that his newly attributed meaning to Ⓠ might be profitable for constructing expressions for series where he could not discern a rule for the general summand. Consequently, we asked Cedric to symbolize Series 3, $.001 - .001 + .001 - .001 + .001 - \cdots$, a variation of Grandi's series for which Cedric could not determine an explicit rule for the general summand during Day 1. We anticipated that as Cedric considered how to symbolize the general summand of Series 3, he would either (1) cognitively modify his personal expression template by attributing new meanings to one or more inscriptions in his personal expression template or (2) physically alter his template by changing or introducing a new inscription. As Cedric reasoned about Series 3, he quickly determined that (to him) his personal expression template $\sum_{n=a}^{b}$ Ꝏ was capable of describing an arbitrary partial sum and $\sum_{n=1}^{\infty}$ Ꝏ was capable of denoting Series 3. When we asked Cedric why he could use the same personal expressions for Series 1 and Series 3, Cedric claimed that his current symbolization method (i.e., personal expression template) focused on three components of a series: the starting term, the ending term, and the pattern (i.e., rule) needed to generate the terms of a partial sum or series. When we asked Cedric why he was able to use the loop inscription Ꝏ in each situation, he said the following:

> Cedric: The loop (*i.e., the inscription* Ꝏ) is the key component […] it's sort of a template. […] the only thing that you're changing around is what the loop represents. […] that's something that sort of came to me. I think subconsciously […] that's how you refer to it, because I've seen so many math formulas and so many express[ions], and so many theorems where it's like the only thing that changes is a small portion of it. And it makes sense because you want a rule or a theorem or an equation that can be used to cover a possibly infinite number of problems, but account for the uniqueness of each problem within the equation, theorem or rule.

Cedric's comments on the nature of symbolic mathematical expressions were much more general than we anticipated (in that he explicitly presented the purpose of a mathematical inscription to "cover a possibly infinite number of problems, but account for the uniqueness of each problem"). From his comment, we concluded that Cedric's cognitive meaning he attributed to his inscription Ꝏ had evolved throughout the Day 2 interview. Although the syntactic structure of his personal expression $\sum_{n=a}^{b}$ Ꝏ remained unchanged from Day 1 (apart from replacing the inscriptions x, y with a, b), we perceived a significant evolution in the meanings Cedric imputed to the inscriptions in his expression template. Specifically, Cedric now appeared (to us) to be capable of re-presenting known explicit or recursive rules for the general term of a series through his inscription Ꝏ. In contrast, during Day 1, he appeared (to us) to only be capable of re-presenting explicit closed-form rules through this inscription. As a result, Cedric developed a productive representation system (i.e., personal expression template) to symbolize an arbitrary partial sum for Series 3 on Day 2. In contrast, he could not symbolize or apply his looping metaphor to Series 3 on Day 1.

Symbolizing Series 6: Physically Adapting an Expression Template for New Classes of Series

At this point, Cedric had constructed two personal expression templates: (1) the template $\sum_{n=a}^{b}$ ꝏ for an arbitrary partial sum and (2) the template $\sum_{n=1}^{\infty}$ ꝏ for an infinite series. However, we were still unsure whether Cedric could re-present *any* partial sum or *any* infinite series through his expression templates. To further test Cedric's symbolizing abilities, we asked him to use his personal expressions to describe Series 6, for which he had previously struggled to construct either an explicit or recursive rule for the general summand.

Cedric's initial reaction to Series 6 was that he could not use his personal expression templates (e.g., $\sum_{n=a}^{b}$ ꝏ) to describe the series. However, Cedric quickly noted that his thinking about series had evolved, and he now believed that some series might exist whose patterns cannot be described. We then asked Cedric whether he could construct a personal expression (generally speaking) to describe an arbitrary partial sum for Series 6. Cedric responded that if he were able to construct a pattern for the first five (visible) summands in Series 6, then he would be able to use his personal expression template $\sum_{n=a}^{b}$ ꝏ. Cedric also stated that without a pattern for Series 6, his expression would "fall apart" because he could not define what the loop inscription ꝏ would represent.

Our decision to have Cedric symbolize Series 6 provided further insights into the meanings he attributed to the inscription ꝏ. Specifically, we confirmed that Cedric thought he could only use the inscription ꝏ to denote series for which he knew an explicit or recursive rule. When confronted with a situation where he could not construct either rule (i.e., Series 6), Cedric felt incapable of conveying the series through his personal expression templates. At this moment of the interview, we decided to push Cedric in his symbolizing activity to see whether he could find another way to symbolize Series 6 (and how he might do so).

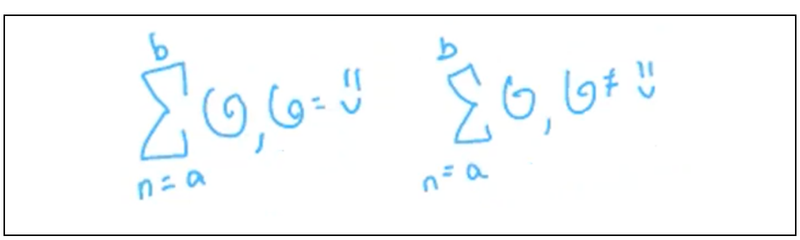
We subsequently asked Cedric to describe how he might modify his personal expression template or create a new one to symbolize a series with "no pattern." After a brief reflection, Cedric introduced a new inscription: ☺. When we asked Cedric to describe what his inscription ☺ conveyed, he provided the following definition:

Cedric: Smiley face (*i.e.,* ☺) represents that it (*i.e., the loop inscription* ℚ) does equal a pattern […] represented by the symbol, the loop. But if there isn't one [pattern], then […] you would write that underneath it, saying that there is no pattern that can be identified. […] There is a plausible pattern sort of thing. […] Like for this one (*Series 6*), you could come up with some sort of pattern using only the terms that you're given and just assuming that all of the terms are within one repetition of the pattern. But if there is just no way that you can do a pattern […] Maybe it's just like there's some letters in there without any numbers […] Then I say that it (*i.e., the loop inscription* ℚ) would not equal the smiley face, and there isn't a pattern.

In this excerpt, Cedric proposed using two inscriptions, ℚ and ☺, to denote (1) the algebraic pattern for calculating the nth summand in the series (ℚ) and (2) whether or not this algebraic rule was capable of being described. (☺) Using these two properties and corresponding inscriptions, he presented (and symbolized) two scenarios. First, Cedric presented a situation in which he was not sure of the pattern for the general summand but thought it possible to identify given sufficient time to investigate. He proposed the expression ℚ = ☺ to identify (in his mind) a series exhibiting the relationship between ℚ and ☺ that he imagined for the first scenario. Second, Cedric presented a situation in which he did not think it possible to describe the pattern for the general summand. He proposed the expression ℚ ≠ ☺ to identify (in his mind) a series exhibiting the relationship between ℚ and ☺ that he imagined for the second scenario. When we asked Cedric why he picked ☺ for an inscription to denote his (perceived) ability to define the general summand of a series, he stated that when he finds the pattern for a series, he feels happy inside. We then asked Cedric where he would write an expression such as ℚ = ☺ in relation to his personal expression template $\sum_{n=a}^{b}$ ℚ. In response, Cedric quickly provided two examples, which we show in Figure 13 below.

Figure 13

Cedric's Personal Expression Templates for Series for which he has not Determined a Pattern

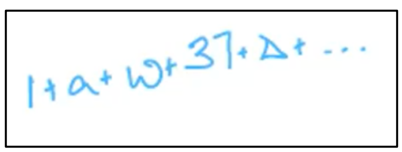


In this moment, we considered Cedric's personal expression template for a partial sum to have evolved from the form $\Sigma_{n=a}^{b} \mathbb{Q}$ to the form $\Sigma_{n=a}^{b} \mathbb{Q}, \mathbb{Q} \square$ ☺. In Cedric's newly proposed template, we considered that (to Cedric) (1) the box could be filled with the inscription $=$ or the inscription $\neq$ and (2) the comma and second condition were optional and only needed if the explicit, closed form rule for a particular general summand could not be described (in the moment of symbolization).

Since Cedric proposed the symbolization $\sum_{n=a}^{b} \mathbb{Q}, \mathbb{Q} =$ ☺ to denote an arbitrary partial sum for Series 6, we decided to ask Cedric to define a series for which he would use the expression $\sum_{n=a}^{b} \mathbb{Q}, \mathbb{Q} \neq$ ☺ to symbolize. In response to our question, Cedric wrote the following series: $1 + a + w + 37 + \Delta + \cdots$ (see Figure 14). Cedric then stated that for this series, there was no possible way (in his mind) to construct an algebraic formula to generate the next summand in the series. Cedric also saw little practical value for a series of the form he had created (e.g., summands comprised of randomly generated alphanumerical characters or strings). Still, he noted that he imagined being tested on his ability to symbolize this type of series in a mathematics course.

Figure 14

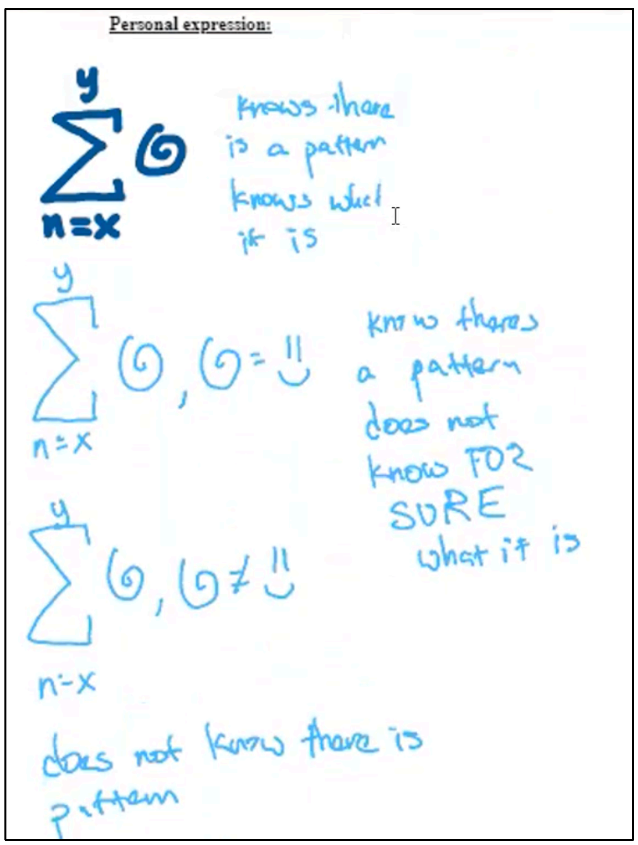
Cedric's Example of a Series which he would Describe using the Expression $\sum_{n=a}^{b} \mathbb{Q}, \mathbb{Q} \neq$ ☺



After describing his randomly generated series, Cedric spontaneously summarized the three classes of series he imagined portraying through his updated personal expression template (see Figure 15). In the first case, Cedric used personal expressions of the form $\sum_{n=a}^{b} \mathbb{Q}$ to describe a series for which he successfully determined the general summand of the series. In series that fit this case, we observed that Cedric used either the inscription $\mathbb{Q}$ or the explicit closed-form rule for the general summand in his personal expressions. In the second case, Cedric used personal expressions of the form $\sum_{n=a}^{b} \mathbb{Q}, \mathbb{Q} =$ ☺ to denote series for which he imagined that he could (given sufficient time) determine a rule for the general summand of a series. We found it interesting that Cedric classified Series 6, which we constructed to not exhibit any discernible general summand, as a series fitting case 2. We conjecture that Cedric believed that we had created Series 6 from a complicated algebraic expression and that he could determine this expression if given sufficient time. Finally, Cedric used personal expressions of the form $\sum_{n=a}^{b}$ ɷ , ɷ ≠ ☺ to portray series for which he imagined that there was no possible way to describe the pattern of a series. Cedric's example series for this case included a list of what appeared (to us) to be randomly generated strings of alphanumeric characters. In our follow-up questions to Cedric's presentation of the three forms of his personal expression template, Cedric repeatedly proposed Series 6 as an instance of case 2 (i.e., unknown but knowable pattern) and his spontaneously constructed series as an instance of case 3 (i.e., no pattern). Although we did not ask Cedric to confirm that Series 1, 3, and 4 constituted series of case 1 (i.e., known pattern), we consider Cedric's earlier work to indicate he believed this to be the case.

Figure 15

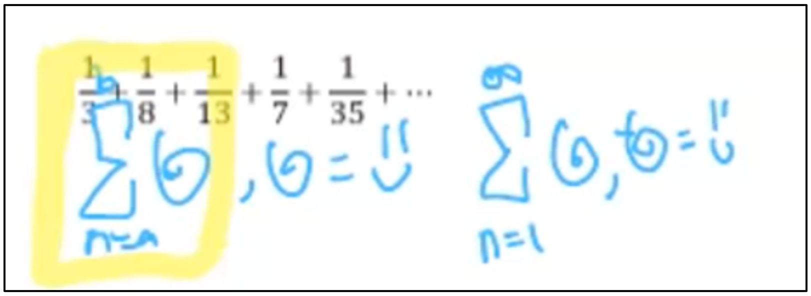
Cedric's Three Personal Expression Templates for Arbitrary Partial Sums



After Cedric described the three forms of his updated personal expression template, we returned to Series 6 and asked him to create a personal expression for the series. Cedric quickly and confidently wrote the expression $\sum_{n=a}^{b}$ ɷ , ɷ = ☺ for an arbitrary partial sum and $\sum_{n=1}^{\infty}$ ɷ , ɷ = ☺ for Series 6 (see Figure 16). We consider Cedric's actions at this moment as additional evidence that his meanings for the inscriptions in his personal expression template (and for the template itself) had evolved due to our questioning. In particular, we consider Cedric's confident symbolization of Series 6 at the end of Day 2 using (mostly) inscriptions that he had introduced during Day 1 (when he was unable to attribute his meanings for Series 6 to his expression template) a significant advancement in the types of meaning he could re-present through his symbols.

Figure 16

Cedric's New Personal Expressions for Series 6



From a teaching intervention standpoint, our decision to include Series 6 in our symbolizing activity task sequence provided Cedric with the opportunity to reflect on the types of series that he could re-present with his personal expression template $\sum_{n=a}^{b}$ ©. As a result of his reflections, Cedric created a new inscription, ☺, to describe whether he could construct a rule for the general summand of a series. Our subsequent decisions to ask Cedric to produce or reference series that fit his updated personal expression template helped him synthesize the series he imagined symbolizing into three distinct categories and create three forms of his personal expression template (one for each category).

Discussion and Conclusion

At the beginning of this paper, we presented two general questions related to (1) the meanings that students attribute to their personal expressions and (2) the types of instructional interventions that might help students to attribute productive meanings to their symbols in the context of partial sums. Although we introduced three types of symbols (i.e., *personal, communicative,* and *conventional* expressions) in the theory section, we have chosen in this paper to present an *instrumental case study* (Stake, 2003) in which we focus on Cedric's development of *personal expressions* and *personal expression templates* to which he could attribute his evolving meanings for partial sums and series. Our analysis of Cedric's case can provide deep insights into his meanings and symbolization for infinite series and possible teaching interventions that instructors might utilize to help students viably symbolize mathematical topics.

In the context of Cedric's meanings for infinite series, we have shown that our interview tasks allowed Cedric to develop productive (to him) meaning for partial sums and series and construct personal expressions as re-presentational mediums for his meanings. Cedric's symbolization of the series we presented to him was largely successful. In particular, Cedric's use of his computer loop metaphor proved critical to constructing his initial personal expression template $\sum_{n=a}^{b}$ Ꝏ at the end of the Day 1 interview. As Cedric began creating instantiations of this expression template to symbolize various partial sums and series on Day 2, he made cognitive and syntactical modifications to the inscriptions in the template. Cognitively, Cedric modified his inscription Ꝏ to portray explicit and recursive rules, whether or not he knew a closed form, an explicit rule for the general summand (in that moment). Syntactically, Cedric introduced a new inscription, ☺, to denote whether he thought it possible to create a rule for the general summand of a series. Cedric did not describe why he introduced the inscription ☺ to his template rather than cognitively attributing various classes of series to the inscription Ꝏ. We conjecture that (in his mind) Cedric's computer loop analogy required a defined rule for the "code" to function, which would have prohibited him from attributing an unknown or unknowable general term to the inscription Ꝏ. Instead, Cedric likely chose to categorize the series he encountered into those that could be programmed fully ($\sum_{n=a}^{b}$ Ꝏ), those whose programming was complete save for constructing and inputting the rule for the general summand ($\sum_{n=a}^{b}$ Ꝏ, Ꝏ = ☺), and those whose programming could be theoretically described but not physically completed ($\sum_{n=a}^{b}$ Ꝏ, Ꝏ ≠ ☺).. We acknowledge that Cedric's responses to our tasks indicate a familiarity with mathematics that not all students might possess. We consider Cedric's relative strength at symbolizing our initial examples and subsequent struggles to symbolize later examples as indicative of the importance of making conversations about symbolization a more explicit component of mathematical instruction.

Cedric also exhibited non-normative meanings for the inscriptions in his template (compared with the analogous inscriptions in the conventional expressions for partial sums). For instance, one conventional expression for the pth partial sum of a series whose summands are defined by the sequence $(a_n)_{n=1}^{\infty}$ is $\Sigma_{n=1}^{p} a_n$. The purpose of the subscript n in a_n (in the conventional sense) is to indicate that the summand a_n is described in terms of the indexing variable n. Had Cedric's symbolization "perfectly" matched convention, we would have anticipated a subscript of n on his inscription Ꝙ (i.e., $\sum_{n=a}^{b}$ Ꝙ$_n$).

Similarly, Cedric's focus on finding the closed-form rule for the summands of a series resulted in his decision to introduce the three personal expression templates we described in the previous paragraph. In contrast, the traditional interpretation of the inscription a_n in the conventional expression $\Sigma_{n=1}^{p} a_n$ is that a_n denotes an arbitrary sequence rule, which may (or may not) be capable of being explicitly defined. In this conventional sense, a mathematician would encapsulate Cedric's three personal expressions templates to differentiate between situations into the conventional expression $\Sigma_{n=1}^{p} a_n$. We conjecture that Cedric's repeated use of his computer "looping" metaphor, while initially productive in his symbolizing activity, played a role in his decision to create three templates. We anticipate that Cedric would need to (at least momentarily) abandon his metaphor to reconcile his three personal expression templates with the conventional expression $\Sigma_{n=1}^{p} a_n$. Our ability to construct such a fine-grained hypothesis regarding the next step in Cedric's symbolic development (and our attribution of his non-normative symbolization to his meanings for a single inscription) also provides significant evidence for the efficacy of our tasks as an instructional intervention for assessing or improving student symbolization.

The specific research questions we proposed for this study were related to teaching interventions that might help support Cedric in his symbolizing activity. The first research question for this study was *in what ways might instructional activities facilitate reflective discussions on symbolizing mathematical ideas (such as arbitrary partial sums)?* While our data consists of one student studying a single topic, we anticipate that instructors might modify our instructional design to support their students' symbolizing activity for various concepts. For instance, we purposefully structured the interview tasks to follow Radford's (2000) three-phase approach to symbolization, leading Cedric to construct meaning for partial sums and then symbolize his meanings in natural language and a personal expression. We also added a fourth phase, where Cedric formed instantiations of his personal expression templates to help him refine his symbol to convey the properties of the various series he encountered. We consider this four-phase approach to symbolization broad enough that instructors at all levels of mathematics might employ these phases to introduce new forms of conventional symbols or test students' abilities to symbolize with familiar forms of notation. We have also shown that providing Cedric with opportunities to reason about various infinite series, particularly those for which he could not discern a general summand, inspired productive reflections on his meanings and symbolization for series. Our ordering of the infinite series in our tasks (i.e., series with simple explicit rules followed by series with simple recursive rules followed by series with no discernable closed-form rule) also provides an example of distinct types of examples that instructors might ask their students to symbolize, as well as a potentially productive order in which to present these examples. Finally, we showed potentially productive symbolization tools, such as a glossary in which students can record (and modify) their inscriptions and personal expression templates as their thinking about a particular topic evolves.

Our second research question was *what lines of questioning might extend students' initial reflections on mathematical ideas (such as arbitrary partial sums) toward constructing symbols as instantiations of their thinking?* We have shown that asking Cedric to create multiple personal expressions for components of Series 1 and then comparing these expressions for consistency helped Cedric solidify his first personal expression template $\sum_{n=a}^{b} \text{Ꝏ}$. We also found that asking Cedric to symbolize Series 6 while simultaneously giving him the freedom to introduce new inscriptions led to the emergence of two other personal expressions, $\sum_{n=a}^{b} \text{Ꝏ}, \text{Ꝏ} = \text{☺}$ and $\sum_{n=a}^{b} \text{Ꝏ}, \text{Ꝏ} \neq \text{☺}$. Each of these lines of questioning—probing for consistency across examples and providing freedom to create new inscriptions—allowed Cedric to progress in constructing personal expressions to which he could confidently attribute his meanings for infinite series. More generally, we consider allowing students the freedom to create new inscriptions (possibly as group work) during an initial activity related to a new topic might (1) help students identify important ideas related to a topic and (2) provide a springboard for instructors to motivate conventional notations (see Zandieh et al. (2017) for an example of this type of activity). We also consider providing students with the opportunity to compare their symbolization across multiple instances of similar examples (in the teacher's mind) to be a potentially fruitful activity to help students solidify their meanings for the inscriptions within their personal expression templates.

Our findings expand the findings of previous researchers and provide instructional insights for facilitating productive student symbolizing activity. For example, we have introduced the notion of *personal expression templates*, which extends Eckman and Roh's (in revision) theoretical framework related to personal, communicative, and conventional expressions. Additionally, we have reinforced findings from previous researchers (e.g., Eckman & Roh, 2022a; Radford, 2000) that students struggle to conceive of the general term of a sequence or series but can construct these notions through targeted activities. We have also provided a set of potentially productive instructional interventions to help student symbolization, including specific instructional tasks and lines of questioning that proved profitable for our students. We encourage all instructors to provide opportunities for students to reflect on the meanings they attribute to their personal expressions.

In line with the special issue theme, our study showcases three connections students might make among representations.

1) First, we showed how Cedric connected his naturalistic meanings and linguistic representations for computing partial sums to a symbolic personal expression template for series.
2) Second, we showed how Cedric connected his representations for series across moments by introducing novel modifications to his personal expression template to denote various series types.
3) Finally, we showed how Cedric leveraged his analogy of a computer loop to construct his personal expression template and the affordances (or obstacles) that this analogy played in his subsequent symbolizing activity.

Although we only present a case of one student's symbolization in this paper, we consider our findings to contribute to the field of mathematics education in the theoretical and practical sense. Theoretically speaking, our study provides potential insights insight into students' symbolizing practices in mathematics, including the coevolution of their meanings and symbolization over time. From a practical standpoint, our study highlights certain instructional interventions that might support students' construction of productive symbols (and meanings) for mathematical topics, such as specific methodological practices for introducing, assessing, and extending students' symbolization of mathematical topics. We anticipate that instructors might use these interventions at all levels of mathematics that require students to construct (and attribute meaning to) symbolic expressions (e.g., fractions, algebraic formulas, set-builder notation). Finally, our results suggest that instructors who utilize the methods we have described in this paper will find opportunities to (1) more easily introduce new symbolic notations to their students and (2) intervene more effectively when their students struggle to symbolize appropriately.

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